Towards a nonet of hybrid mesons: strong and radiative decays

Vanamali Shastry
Jan Kochanowski University in Kielce, Poland

In collaboration with: Francesco Giacosa (UJK & GUF), Enrico Trotti (UJK), Anna Okopińska (UJK), and Christian S. Fischer (JLU)

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Contents

Introduction

- QCD demands that the physically observable states must be color singlets.
  - Mesons ($\bar{q}q$), and baryons (qqq) are the most common combinations.
- Are other combinations allowed? Yes!
  - Hybrids ($\bar{q}qg$), tetra/pentaquarks, glueballs, molecular states, etc.
- Quark models, effective QFT models, sum rules, BSE/DSE, etc.
- But below $\sim$ 2 GeV???
  - NRQM says one hybrid nonet, Lattice shows one hybrid isovector.
- Experimental evidence?
  - Two known states: $\pi_1(1600)$ (known since $\sim$ 30 years) and the recently observed $\eta_1(1855)$

Light hybrids

- The $\pi_1(1600)$:
  - Broad consensus of being a hybrid state.
  - Originally, two light resonances: $\pi_1(1600)$ and the lighter $\pi_1(1400)$
  - Same quantum numbers: $J^{PC} = 1^{-+}$; Complimentary decay channels.
  - JPAC analysis of COMPASS data shows only one pole (JPAC, 2019), Lattice simulations by HadSpec agree (Hadron Spectrum, 2020).
  - Possible final state interactions (Bass & Marco, 2002)
- The $\eta_1(1855)$:
  - Recently seen by BESIII in the $\eta\eta'$ channel; $J^{PC} = 1^{-+}$; nature unknown. (BESIII, 2022)
  - Mass ($1855 \pm 9^{+6}_{-1}$ MeV) and total width ($188 \pm 18^{+3}_{-8}$) known; partial width unknown.
  - Molecular state? Hybrid (we say so)? Tetraquark? Models-galore!
- Where are the other hybrids and other exotics?

Techniques

How to effectively study meson decays?

- Light hadrons are non-perturbative systems.
- No “effective” way to study them!
- Models, models, and more models.
- Past attempts - only LO interactions
- New(-ish) application: Partial wave analysis (VS, E. Trotti, F. Giacosa, Phys. Rev. D105 (2022) 5, 054022)
- New parameterisation for loop function: the Sill distribution (F. Giacosa, A Okopińska, VS, Eur. Phys. J A57 (2021) 12, 336.)

Partial Wave Analysis

Scattering cross-section can be decomposed into (infinitely many) partial waves:

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)\right|^2 \tag{1}$$

The angular information is lost when calculating the decay width.

- Decay width is a number, scattering cross-section is a function of angles and momenta.
- Decays *do* proceed through partial waves (number of $\ell-$channels are finite, restricted by the J^P values of the parent and decays products).
- Decompose the *amplitude*.
- Helicity formalism (Jacob & Wick, 1959); Tensor formalism (Zemach, 1965); Covariant helicity formalism (Chung, 1993 & 1997).

Covariant helicity formalism

- Helicity coupling amplitudes are functions of E/m
- The amplitude for $|J, J_3\rangle \rightarrow |s, m_s\rangle + |\sigma, m_\sigma\rangle$ decay is given by,

$$\mathcal{M}^J(\theta, \phi; J_3) \propto \sum_{S_3, m_\sigma} D^{J*}_{J_3\delta}(\phi, \theta, 0) F^J_{m_s m_\sigma} \tag{2}$$

- Important: $|S, S_3\rangle = |s, m_s\rangle \otimes |\sigma, m_\sigma\rangle$; $\delta = m_s - m_\sigma$
- In general,

$$F^J_{m_s m_\sigma} \propto \langle J J_3 m_s m_\sigma | \mathcal{M} | J J_3 \rangle \tag{3}$$

- The helicity coupling amplitude is related to $F^J_{m_s m_\sigma}$ via,

$$F^J_{m_s m_\sigma} = \sum_{\ell S} \sqrt{\frac{2\ell + 1}{2J + 1}} \langle \ell 0 S \delta | J \delta \rangle \langle s m_s \sigma m_\sigma | S \delta \rangle G^J_{\ell S} \tag{4}$$

- $G^J_{\ell S}$ is the helicity coupling amplitude.

Available data

$a_1(1260)$ $I^G(J^{PC}) = 1^-(1^{++})$

See also our review under the $a_1(1260)$ in PDG 2006, Journal of Physics G33 1 (2006).

$a_1(1260)$ MASS	1230 ± 40 MeV
$a_1(1260)$ WIDTH	250 to 600 MeV
D-wave/S-wave AMPLITUDE RATIO IN DECAY OF $a_1(1260) \to \rho\pi$	-0.062 ± 0.020 (S = 2.3)

$b_1(1235)$ $I^G(J^{PC}) = 1^+(1^{+-})$

$b_1(1235)$ MASS	1229.5 ± 3.2 MeV (S = 1.6)
$b_1(1235)$ WIDTH	142 ± 9 MeV (S = 1.2)
$b_1(1235)$ D-wave/S-wave AMPLITUDE RATIO IN DECAY OF $b_1(1235) \to \omega\pi$	0.277 ± 0.027 (S = 2.4)
$b_1(1235)$ D-wave/S-wave AMPLITUDE PHASE DIFFERENCE IN DECAY OF $b_1(1235) \to \omega\pi$	$10 \pm 5\,^\circ$

$K_1(1400)$ $I(J^P) = 1/2(1^+)$

$K_1(1400)$ MASS	1403 ± 7 MeV
$K_1(1400)$ WIDTH	174 ± 13 MeV (S = 1.6)
D-wave/S-wave RATIO FOR $K_1(1400) \to K^*(892)\pi$	0.04 ± 0.01

$\pi_2(1670)$ $I^G(J^{PC}) = 1^-(2^{-+})$

$\pi_2(1670)$ MASS	$1670.6^{+2.9}_{-1.2}$ MeV (S = 1.3)
$\pi_2(1670)$ WIDTH	258^{+8}_{-9} MeV (S = 1.2)
D-wave/S-wave RATIO FOR $\pi_2(1670) \to f_2(1270)\pi$	-0.18 ± 0.06
F-wave/P-wave RATIO FOR $\pi_2(1670) \to \rho\pi$	-0.72 ± 0.16

- Also available for some decays of $K_1(1270)$ and $\pi_1(1600)$.

$$i\mathcal{M} = i g_c\, \epsilon_\mu \epsilon^{\mu *} = -i\, g_c \begin{cases} 1 & J_3 = S_3 = \pm 1 \\ \gamma & J_3 = S_3 = 0 \end{cases} \Rightarrow \begin{matrix} G_2 = \sqrt{\frac{2}{3}} g_c \left(1 - \frac{E}{M}\right) \\ G_0 = \frac{1}{\sqrt{3}} g_c \left(2 + \frac{E}{M}\right) \end{matrix} \tag{5}$$

Some observations

- The decay amplitude involves polarization tensors and an interaction kernel - the latter is to be modelled.

$$i\mathcal{M} = \epsilon_{\mu_1\mu_2\ldots}\Gamma^{\mu_1\mu_2\ldots\nu_1\nu_2\ldots}\epsilon^*_{\nu_1\nu_2\ldots}$$

- The contribution to a given ℓ-channel can come from two sources: PT and interaction kernel. Ergo, contact interactions can produce higher partial waves!
- The ratio of the PWAs can be used to fix the ratio of the coupling constants.
- Large ratio $\Rightarrow$ large contribution from the higher order interactions $\Rightarrow$ Higher order interactions are as important as the leading order interactions to explain the partial wave amplitudes of meson decays.
- The larger the 3-momentum of the decay products, the larger the ratio of the PWAs. ($a_1(1260) \to \rho\pi$ decay has $D/S = -0.062 \pm 0.02$ - a violation)
- $g_{LO}/g_{NLO} \sim \# M_p^2$, where $\# = \mathcal{O}(1)$ (valid for all decays that we have studied).
- More details: VS, E Trotti, and F Giacosa, Phys.Rev.D 105 (2022) 5, 054022

Spectral Integration

- Mesons are mostly unstable states.
- Large contribution to the self-energy.
- Modeling the self-energy $\longrightarrow$ crux of the distribution.
- Finite width $\Longrightarrow$ distribution of mass. Which value to use??
- Integrate over the spectral distribution function.

$$G_s(s) = \frac{1}{s - M^2 + \Pi(s) + i\varepsilon}; \quad d_s(s) = -\frac{1}{\pi}\mathrm{Im}\left[G_s(s)\right] \tag{6}$$

- Normalization: $\int ds d_s(s) = 1$
- Thresholds: $\theta(s - s_{\mathrm{th}})$

The challenge is to model the imaginary part of the loop function (Im $[\Pi(s)]$).

The Sill distribution

A novel distribution that

1. is normalised
2. takes care of thresholds inherently

$$\Pi(s) = i\sqrt{s - s_{th}}\,\tilde{\Gamma}$$
$$= i\sqrt{s - s_{th}}\,\frac{M\Gamma}{\sqrt{M^2 - s_{th}}} \qquad (7)$$

$$d_s(s) = \frac{1}{\pi}\frac{\sqrt{s - s_{th}}\,\tilde{\Gamma}}{(s - M^2)^2 + (\sqrt{s - s_{th}}\,\tilde{\Gamma})^2}$$

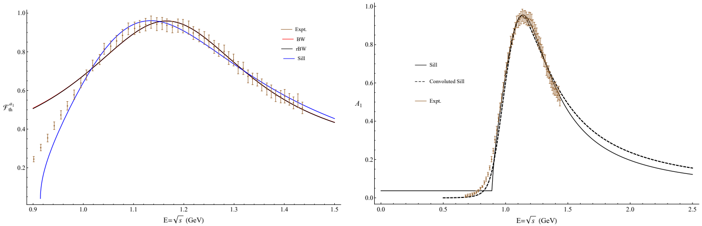

Figure 1: Example spectral functions: The naive $a_1\,(1260)$ (left) and convoluted $a_1\,(1260)$ (right). For further details, see F. Giacosa, A Okopińska, VS, Eur. Phys. J A57 (2021) 12, 336.

Lagrangian involving the hybrids

1. Two-body strong decay (VS, C. S. Fischer, F. Giacosa, Phys. Lett. B834 (2022) 137478):
   - $SU(3)$ flavor symmetry
   - Parity and charge conjugation
   - Axial anomaly terms
2. Radiative production from J/ψ decay (VS, F. Giacosa, under preparation):
   - $SU(3)$ flavor symmetry
   - Parity and charge conjugation
3. Radiative decays of the hybrids (VS, F. Giacosa, under preparation):
   - $SU(3)$ flavor symmetry
   - Parity and charge conjugation
   - Vector meson dominance: $V_{\mu\nu} \to V_{\mu\nu} + \frac{e_0}{g_\rho} Q F_{\mu\nu}$

The Lagrangians

2-body strong decay:

$$\begin{aligned}\mathcal{L}^{\pi}_{hyb} &= g^{c}_{b_1\pi}\mathrm{Tr}\left[\Pi^{hyb}_{\mu}[P,B^{\mu}]\right]+g^{d}_{b_1\pi}\mathrm{Tr}\left[\Pi^{hyb}_{\mu\nu}[P,B^{\mu\nu}]\right]\\ &-g_{\eta\pi}\eta_N\mathrm{Tr}\left[\Pi^{hyb}_{\mu}\partial^{\mu}P\right]+g_{\rho\pi}\mathrm{Tr}\left[\tilde{\Pi}_{\mu\nu}[P,V^{\mu\nu}]\right]\\ &+g_{\rho\omega}\mathrm{Tr}\left[\Pi^{hyb}_{\mu}\{V^{\mu\nu},V_{\nu}\}\right]+g_{f_1\pi}\mathrm{Tr}\left[\Pi^{hyb}_{\mu}\{A^{\mu\nu},\partial_{\nu}P\}\right]\end{aligned} \tag{8}$$

Radiative production:

$$\mathcal{L}_{RP} = g_{\gamma\eta_1}J_{\mu}F^{\mu\nu}\mathrm{Tr}[\{Q,\Pi_{1,\nu}\}] \tag{9}$$

$$= g_{\gamma\eta_1}C_{\eta_L}J_{\mu}\eta^{L}_{1,\nu}F^{\mu\nu}+g_{\gamma\eta_1}C_{\eta_H}J_{\mu}\eta^{H}_{1,\nu}F^{\mu\nu} \tag{10}$$

Radiative decay:

$$\mathcal{L}_{RD} = g_{\rho\pi}\frac{e_0}{g_\rho}\mathrm{Tr}\left[\tilde{\Pi}_{1,\mu\nu}[P,Q]\right]F^{\mu\nu}+g_{\rho\omega}\frac{e_0}{g_\rho}\mathrm{Tr}\left[\Pi_{1,\mu}\{Q,V_{\nu}\}\right]F^{\mu\nu} \tag{11}$$

Parameters

- The parameters for two-body strong decays were fitted using experimental and lattice data. (VS, C. S. Fischer, F. Giacosa, Phys. Lett. B834 (2022) 137478).
- Two sets of parameters since the sign of the D/S-ratio for the $\pi_1(1600) \to b_1\pi$ decay is not known.
- Two possible isoscalar mixing angles: $\theta_h = 0^\circ,\ 15^\circ$
- Production process: parameter ($g_{\gamma\eta_1}$) represents the coupling of J/ψ to $\gamma\eta_1^{(\prime)}$ final states. The values fitted from the $BR(J/\psi \to \gamma\eta_1(1855) \to \gamma\eta\eta') = (2.7 \pm 0.76) \times 10^{-6}$ (BESIII,2022).

θ_h	$g_{\gamma\eta_1}$	
	Set-1	Set-2
0°	0.015 ± 0.002	0.014 ± 0.002
15°	0.013 ± 0.002	0.011 ± 0.002

- VMD parameter is $g_{\text{VMD}} = e_0/g_\rho = 0.0181 \pm 0.0001$. Extracted from the $\rho^0 \to e^+e^-$ width.

Strong decays: $\pi_1(1600)$

For details: VS, C. S. Fischer, F. Giacosa, Phys. Lett. B834 (2022) 137478

- The fit reproduces the mass and width of the isovector as given in PDG
- The dominant decay channel is the $b_1(1235)\pi$ channel.
- Also decays to $\rho\pi$, K^*K, $f_1^{(\prime)}\pi$, $\eta^{(\prime)}\pi$, and $\rho\omega$ states.
- The $\rho\omega$ and $\rho\pi$ decay channels also give rise to radiative decays.
- Partial widths in the range predicted by lattice studies (HadSpec).
- The pseudoscalar decay channel is purely due to axial anomaly term.
- $\eta\pi$ partial width is ~ 10 times suppressed compared to $\eta'\pi$.

Other members of the nonet

- The masses of the various members of a nonet are related to the mass of the isovector at the tree-level.
- The following relations hold true:

$$m^2_{\eta_{1,N}} = m^2_{\pi_1} \tag{12}$$

$$m^2_{K_1} = m^2_{\pi_1} + \delta_s^{hyb} \tag{13}$$

$$m^2_{\eta_{1,S}} = m^2_{\pi_1} + 2\delta_S^{hyb} \tag{14}$$

  where, δ_S^{hyb} is the strangeness contribution.
- Isoscalars mix - Mixing angle (θ_h) depends on the behavior of the nonet under chiral transformation
- Two unknowns (δ_S^{hyb}, θ_h) but only one data point (mass of $\eta_1(1855)$) $\Rightarrow$ the mass of the kaon depends on the mixing angle (*data, not physics!*)

The scenarios

State	Scenario-1 ($\theta_h = 36.7^\circ$)		Scenario-2 ($\theta_h = 0^\circ$)		Scenario-3 ($\theta_h = 15^\circ$)		Parameter Set
	M (MeV)	Γ (MeV)	M (MeV)	Γ (MeV)	M (MeV)	Γ (MeV)	
K_1^{hyb}	1706	140 ± 45	**1761**	$\mathbf{312 \pm 97}$	1754	286 ± 88	1
		73 ± 29		$\mathbf{170 \pm 65}$		155 ± 59	2
η_1^L	1543	21 ± 4	**1661**	$\mathbf{81 \pm 15}$	1646	69 ± 13	1
		22 ± 4		$\mathbf{83 \pm 16}$		71 ± 13	2
η_1^H	1855	607 ± 159	**1855**	$\mathbf{259 \pm 92}$	1855	411 ± 130	1
		249 ± 80		$\mathbf{157 \pm 68}$		192 ± 80	2

Table 1: The masses and widths of the kaons and the isoscalars in the three scenarios discussed earlier.

(Table from VS, C. S. Fischer, F. Giacosa, Phys. Lett. B834 (2022) 137478)

The hybrid kaon: $K_1^{hyb}(1750)$

- Mass $\in 1.7 - 1.8$ GeV; Width ~ 300 MeV.
- Possible mixing with the 1^- $K^*(1410)$ and $K^*(1680)$ not considered.
- Width sensitive to $h_1 - h_1'$ mixing angle.
- Dominant decay channels: $K_1(1270/1400)\pi$, $a_1(1260)K$, $b_1(1235)K$
- Also decays to $(\rho/\omega/\phi)K$, $K^*(\pi/\eta)$, $\eta^{(\prime)}K$, $(\rho/\omega)K^*$.
- The $(\rho/\omega/\phi)K$ and $(\rho/\omega)K^*$ channels also lead to radiative decays.

The isoscalars

- One the isoscalar ($\eta_1(1855)$) recently observed by BESIII; mass = 1855 GeV, width = 188 MeV
- We take $\eta_1(1855)$ as the heavy isoscalar: only way to explain the large mass
- Dominant decay channel: $K_1(1270)K$; Also decays to K^*K, $\eta'\eta$ (observed); K^*K^*, $f_1\eta$, and $\omega\phi$.
- The light isoscalar is a puzzle: mass similar to the mass of the $\pi_1(1600)$; width much smaller (*Caveat!* Tree-level result)
- Dominant decay channel is $a_1(1260)\pi$; also decays to K^*K, $\eta'\eta$, and $\rho\rho$
- Mixing angle is not known – homo-chiral nonet $\Rightarrow$ small mixing expected.

Radiative production from J/ψ decays

$$J/\psi \to \gamma \mathcal{R}^*(1^{-+}) \to \gamma \phi_1 \phi_2$$

Production Channel ($\phi_1 \phi_2$)	Branching ratio (10^{-4})			
	Set-1		Set-2	
	$\theta_h = 0^\circ$	$\theta_h = 15^\circ$	$\theta_h = 0^\circ$	$\theta_h = 15^\circ$
$\eta_1(1660)$				
$a_1 \pi$	0.63 ± 0.18	0.13 ± 0.04	0.57 ± 0.18	0.10 ± 0.04
$K^* K$	$(2.20 \pm 0.67) \times 10^{-3}$	$(1.86 \pm 0.56) \times 10^{-3}$	$(1.91 \pm 0.58) \times 10^{-3}$	$(1.34 \pm 0.40) \times 10^{-3}$
$\eta' \eta$	$(2.40 \pm 0.72) \times 10^{-3}$	$(8.58 \pm 2.58) \times 10^{-3}$	$(2.09 \pm 0.62) \times 10^{-3}$	$(6.56 \pm 1.96) \times 10^{-3}$
$\rho\rho$	$(4.04 \pm 1.22) \times 10^{-4}$	$(7.01 \pm 2.10) \times 10^{-4}$	$(3.54 \pm 1.06) \times 10^{-4}$	$(5.45 \pm 1.64) \times 10^{-4}$
$\eta_1'(1855)$				
$K_1(1270) K$	1.80 ± 0.54	2.18 ± 0.67	1.03 ± 0.30	0.90 ± 0.28
$K^* K$	$(1.77 \pm 0.54) \times 10^{-2}$	$(1.79 \pm 0.54) \times 10^{-2}$	$(1.77 \pm 0.54) \times 10^{-2}$	$(4) \times 10^{-2}$
$K^* K^*$	$(4.56 \pm 1.36) \times 10^{-4}$	$(5.37 \pm 1.60) \times 10^{-4}$	$(4.44 \pm 1.34) \times 10^{-4}$	$(5.10 \pm 1.54) \times 10^{-4}$
$f_1(1285) \eta$	$(1.01 \pm 0.30) \times 10^{-2}$	$(6.96 \pm 2.08) \times 10^{-4}$	$(9.16 \pm 2.74) \times 10^{-2}$	$(6.09 \pm 1.82) \times 10^{-4}$
$\eta \eta'$	$(2.70 \pm 0.76) \times 10^{-2}$ (BESIII, 2022)			

Table 2: The branching ratios of the production of the hybrid isoscalars in the radiative process.

- The BR for the radiative production has the same nature as that of two-body strong decays.

Radiative decays

Decay Channel	Width (keV)		Decay	Width (keV)	
	$\theta = 15^\circ$	$\theta = 0^\circ$		$\theta = 15^\circ$	$\theta = 0^\circ$
$\pi_1(1600)$			$\eta_1(1660)$		
$\pi\gamma$	4.73 ± 1.35		$\rho\gamma$	0.15 ± 0.03	0.16 ± 0.03
$\rho\gamma$	$(5.45 \pm 2.06) \times 10^{-2}$		$\omega\gamma$	$(1.67 \pm 0.32) \times 10^{-2}$	$(1.76 \pm 0.33) \times 10^{-2}$
$\omega\gamma$	0.16 ± 0.06		$\phi\gamma$	$(1.23 \pm 0.23) \times 10^{-3}$	$(1.64 \pm 0.31) \times 10^{-5}$
$\phi\gamma$	$(1.48 \pm 0.56) \times 10^{-4}$		$\eta_1'(1855)$		
$K_1^{hyb}(1750)$			$\rho\gamma$	$(1.73 \pm 0.33) \times 10^{-2}$	0.
$K\gamma$	4.43 ± 1.27	4.49 ± 1.28	$\omega\gamma$	$(6.91 \pm 1.31) \times 10^{-4}$	$(3.06 \pm 0.58) \times 10^{-4}$
$K^*\gamma$	0.14 ± 0.05	0.14 ± 0.05	$\phi\gamma$	$(4.18 \pm 0.79) \times 10^{-2}$	$(4.42 \pm 0.84) \times 10^{-2}$

Table 3: Radiative decay widths of the 1^{-+} hybrids for two values of mixing angle.

- The $\pi_1(1600)$ and the $K_1^{hyb}(1750)$ decay dominantly to the $\pi\gamma$ and $K\gamma$.
- The neutral isovector and kaon cannot decay to pseudoscalars at the tree-level.
- The decay of the $\eta_1'(1855)$ to $(\rho/\omega)\gamma$ is very sensitive to the mixing angle.
- The $\eta_1(1660)$ couples strongly to $\rho\gamma$ while the $\eta_1'(1855)$ couples dominantly to $\phi\gamma$

Summary

- The Sill distribution:
  1. Novel parameterization of the loop function
  2. Is normalised; threshold is built-in
- Partial wave analysis:
  1. Higher order interactions play a large role in the decays of mesons.
  2. Partial widths are sensitive to the nature of interference between the partial waves.
- Light Hybrids:
  1. The $\eta_1(1855)$ is (almost) purely $\bar{s}sg$.
  2. The light isoscalar is possibly the narrowest.
  3. The kaon can be as broad as the $\pi_1(1600)$.
  4. The dominant modes of $\pi_1(1600)$ and the kaon radiative decays are to the pseudoscalars.
  5. The isoscalar mixing angle can be inferred from radiative decay of the $\eta_1(1855)$ to vector states.

We look forward for data from the EIC (and other future colliders).

Thank you!

Helicity amplitudes - construction

- In general,

$$F^J_{m_s m_\sigma} \propto \langle JJ_3 m_s m_\sigma | \mathcal{M} | JJ_3 \rangle \tag{15}$$

- The helicity coupling amplitude is related to $F^J_{m_s m_\sigma}$ via,

$$F^J_{m_s m_\sigma} = \sum_{\ell S} \sqrt{\frac{2\ell+1}{2J+1}} \langle \ell 0 S\delta | J\delta \rangle \langle s m_s \sigma m_\sigma | S\delta \rangle G^J_{\ell S} \tag{16}$$

- $G^J_{\ell S}$ is the helicity coupling amplitude.
- To construct $F^J_{m_s m_\sigma}$, we need the polarization tensors. (rather, polarization 4-vectors and invariants constructed from them)
- The basic objects are the spin-1 polarization 4- vectors, metric tensor, 4-momenta.
- From the above construct the rank-ℓ tensors that represent the partial wave in the ℓ-channel.
- These are “pure" spin waves, and have the required q^ℓ behavior.

Spin waves and partial waves

- Start with polarization vectors (PVs). These PVs transform as,

$$\phi^{\mu}(m) \rightarrow \sum_{m'} \phi^{\mu}(m') D^{J}_{mm'}(\phi, \theta, \psi) \tag{17}$$

- The pure-spin tensors can be constructed from $\tilde{\omega}.\epsilon$, where . means contraction, $\tilde{\omega}_{\alpha} = \tilde{g}_{\alpha\beta}\omega^{\beta}$, ω_{β} and ϵ_{β} are momentum space wave functions, and $\tilde{g}_{\alpha\beta} = -g_{\alpha\beta} + \frac{p_{\alpha}p_{\beta}}{p^2}$.
- Sample tensors:

$$\psi^{(0)}_{\alpha\beta} = (\tilde{\omega}.\epsilon)\tilde{g}_{\alpha\beta} \tag{18}$$

- Partial waves can be constructed from $\tilde{q}.q$
- Some partial wave tensors:

$$P\text{-wave: } \tilde{q}_{\alpha}\,; \quad D\text{-wave: } \tilde{q}_{\alpha}\tilde{q}_{\beta} - \frac{1}{3}(q.\tilde{q})\tilde{g}_{\alpha\beta} \tag{19}$$

- Note: $q = |\vec{p}_B| - |\vec{p}_C|$; $\vec{p}_{B(C)}$ are the 3-momenta of the decay products

Application to 2-body decays

The amplitude, ($J^P \longrightarrow s^\pi 0^-$ decay)

$$i\mathcal{M}^J(\theta,\phi;J_3) \propto D^{J*}_{J_3 S_3}(\phi,\theta,0)F^J_{S_3 0}; \tag{20}$$

In the frame of reference where momenta: $k_{0,\mu} = (M_p, \vec{0})$ (parent), $k_{1,\mu} = (E_{d,1}, 0, 0, -k)$, and $k_{2,\mu} = (E_{d,2}, 0, 0, k)$ (daughters) ($\theta = \phi = 0$),

$$i\mathcal{M}^J(0,0;J_3) \propto F^J_{S_3 0} \tag{21}$$

The helicity amplitudes ($F^J_{S_3 0}$) are related to the ℓS coupling amplitudes ($G^J_{\ell S}$) as,

$$F^J_{S_3 0} = \sum_{\ell S} \sqrt{\frac{2\ell+1}{2J+1}} \langle \ell 0 S S_3 | J S_3 \rangle \langle S S_3 0 0 | S S_3 \rangle G^J_{\ell S} \tag{22}$$

$$F^J_{m_S m_\sigma} = \sum_{\ell S} \sqrt{\frac{2\ell+1}{2J+1}} \langle \ell 0 S \delta | J \delta \rangle \langle s m_S \sigma m_\sigma | S \delta \rangle G^J_{\ell S} \tag{23}$$

Decay width:

$$\Gamma_{A\rightarrow BC} = \# \frac{k}{8\pi M_A} \sum_{spin} |i\mathcal{M}|^2 = \frac{k}{8\pi M_A} \sum_{\ell S} |G^J_{\ell S}|^2 \tag{24}$$

Which ℓ values are valid?

$$A(J^P, k_0) \rightarrow B(s^\pi, k_1) + C(\sigma^\kappa, k_2)$$

Spin states: $|J, M\rangle$ (parent), $|s, m_s\rangle$, and $|\sigma, m_\sigma\rangle$ (decay products).
Parity: P (parent), π, κ (decay products)
Relative angular momentum: $|\ell, m_\ell\rangle$

$$|J, M\rangle = |\ell, m_\ell\rangle \oplus |S, \delta\rangle, \quad |S, \delta\rangle = |s, m_s\rangle \oplus |\sigma, m_\sigma\rangle, \quad \delta = m_s - m_\sigma \tag{25}$$

Thus, $\ell \in [|J - S|, J + S]$. But, are all these values allowed?

$$P = \pi \otimes \kappa \otimes (-1)^\ell \tag{26}$$

Ex: $a_1(1260) \rightarrow \rho\pi$: $J^P = 1^+$, $s^\pi = 1^-$, $\sigma^\kappa = 0^-$. So, $\ell \in [0, 2]$. But,
$+1 = (-1)(-1)(-1)^\ell \Rightarrow \ell \in$ even

Fitted Parameters

Parameter	Value	
	Set−1 ($D/S > 0$)	Set−2 ($D/S < 0$)
m_{π_1}	1.663 ± 0.01 GeV	1.662 ± 0.01
$g^c_{b_1\pi}$	88 ± 23 GeV	$-(119 \pm 22)$
$g^d_{b_1\pi}$	$-(23.3 \pm 5.60)$ GeV^{-1}	26.7 ± 5.3
$g_{\rho\pi}$	$0.35 \pm 0.05)$ GeV	0.35 ± 0.05
$g_{f_1\pi}$	8.02 ± 0.83 GeV	8.12 ± 0.83
$g_{\rho\omega}$	$-(0.37 \pm 0.07)$	$-(0.38 \pm 0.07)$
$g_{\eta\pi}$	4.91 ± 0.56	4.94 ± 0.55
$\chi^2/d.o.f$	0.35	0.28

Table 4: The values of the mass of π_1 and the coupling constants along with the uncertainties when the D/S−ratio for the $b_1\pi$ decay channel is positive, and negative .

The $\pi_1(1600)$

Channel	Width (MeV)	Channel	Width (MeV)
$\Gamma_{b_1\pi}$	220 ± 34	$\Gamma_{f_1\pi}$	16.2 ± 3.1
$\Gamma_{\rho\pi}$	7.1 ± 1.8	$\Gamma_{f_1'\pi}$	0.83 ± 0.16
Γ_{K^*K}	1.2 ± 0.3	$\Gamma_{\eta\pi}$	0.37 ± 0.08
$\Gamma_{\rho\omega}$	0.08 ± 0.03	$\Gamma_{\eta'\pi}$	4.6 ± 1.0
		Γ_{tot}	250 ± 34

Table 5: The partial widths and branching ratios of various decay channels and the total width (parameter Set-1; see text for discussion).

Predictions for other nonets

	m_{K_1} (GeV)	$m_{\eta_1^L}$ (GeV)	$m_{\eta_1^H}$ (GeV)	θ_h	δ_S^{hyb} (GeV^2)
Scenario-1	1.707	1.542	1.855	36.7°	0.151
Scenario-2	1.761	1.661	1.855	0°	0.341
Scenario-3	1.754	1.646	1.855	15°	0.317

Table 6: The masses of the kaons and the isoscalars, and isoscalar mixing angle for the three scenarios discussed in the text.

The $K_1(1750)$

Channel	Width (MeV)		Channel	Width (MeV)	
	Set−1	Set−2		Set−1	Set−2
$\Gamma_{K_1(1270)\pi}$	125 ± 42	48 ± 25	$\Gamma_{\rho K}$	2.18 ± 0.56	2.19 ± 0.57
$\Gamma_{K_1(1400)\pi}$	103 ± 45	98 ± 43	$\Gamma_{\omega K}$	0.82 ± 0.21	0.82 ± 0.21
$\Gamma_{h_1(1170)K}$	1.53 ± 0.28	1.37 ± 0.24	$\Gamma_{\phi K}$	0.49 ± 0.12	0.49 ± 0.13
$\Gamma_{\eta K}$	0.29 ± 0.07	0.29 ± 0.07	$\Gamma_{K^*\pi}$	0.67 ± 0.17	0.67 ± 0.17
$\Gamma_{\eta' K}$	2.77 ± 0.62	2.81 ± 0.62	$\Gamma_{K^*\eta}$	0.30 ± 0.08	0.30 ± 0.08
$\Gamma_{\rho K^*}$	0.045 ± 0.016	0.047 ± 0.016	$\Gamma_{\omega K^*}$	0.011 ± 0.004	0.012 ± 0.004
$\Gamma_{a_1 K}$	11.0 ± 2.32	11.3 ± 2.35	$\Gamma_{b_1 K}$	64 ± 14	3.11 ± 2.88
			Γ_{tot}	312 ± 97	170 ± 65

Table 7: The partial widths and branching ratios of various decay channels and the total width for the hybrid kaon $K_1^{hyb}(1750)$. We have assumed the mass of the state to be 1761 MeV.

The isoscalars

Channel	Width (MeV)		Channel	Width (MeV)	
	Set−1	Set−2		Set−1	Set−2
$\Gamma_{a_1\pi}$	80 ± 15	82 ± 16	$\Gamma_{K_1(1270)K}$	253 ± 92	151 ± 67
Γ_{K^*K}	0.29 ± 0.075	0.29 ± 0.075	Γ_{K^*K}	1.45 ± 0.37	1.46 ± 0.38
$\Gamma_{\eta'\eta}$	0.41 ± 0.09	0.41 ± 0.09	$\Gamma_{\eta'\eta}$	2.28 ± 0.51	2.31 ± 0.51
$\Gamma_{K_1(1270)K}$	0	0	$\Gamma_{a_1\pi}$	0	0
$\Gamma_{\rho\rho}$	0.081 ± 0.028	0.082 ± 0.029	$\Gamma_{\rho\rho}$	0	0
$\Gamma_{K^*K^*}$	0	0	$\Gamma_{K^*K^*}$	0.075 ± 0.027	0.077 ± 0.028
$\Gamma_{\omega\phi}$	0	0	$\Gamma_{\omega\phi}$	$\sim 10^{-4}$	$\sim 10^{-4}$
$\Gamma_{f_1\eta}$	0	0	$\Gamma_{f_1\eta}$	2.15 ± 0.56	2.21 ± 0.57
Γ_{tot}	81 ± 15	83 ± 16	Γ_{tot}	259 ± 92	157 ± 68

Table 8: The partial widths and branching ratios of various decay channels and the total width of the η_1^L (left) and the $\eta_1(1855)$ (right) for $\theta_h = 15^\circ$. This corresponds to the “Scenario-3" discussed in the text.

The “hybrid" Kaons: $K_1(1750)$

- “Scenario-2": $M = 1760$ MeV.
- Possible mixing with the 1^- $K^*(1410)$ and $K^*(1680)$ not considered. (Significant difference in decay channels)
- Width sensitive to $h_1 - h_1'$ mixing angle.
- The D/S-ratio for the decay into axial kaons is > 1 for Set-2 and < 1 for Set-1.

	$K_1(1270)\pi$		$K_1(1400)\pi$	
	Set-1	Set-2	Set-1	Set-2
Scenario-1	0.58 ± 0.09	19 ± 82	0.16 ± 0.01	0.25 ± 0.02
Scenario-2	0.57 ± 0.07	1.76 ± 0.47	0.19 ± 0.01	0.28 ± 0.02
Scenario-3	0.57 ± 0.07	1.93 ± 0.59	0.18 ± 0.01	0.28 ± 0.02

Table 9: D/S-ratios of the decay of the hybrid kaon into $K_1(1270)\pi$ and $k_1(1400)\pi$.