Garden of Singularities

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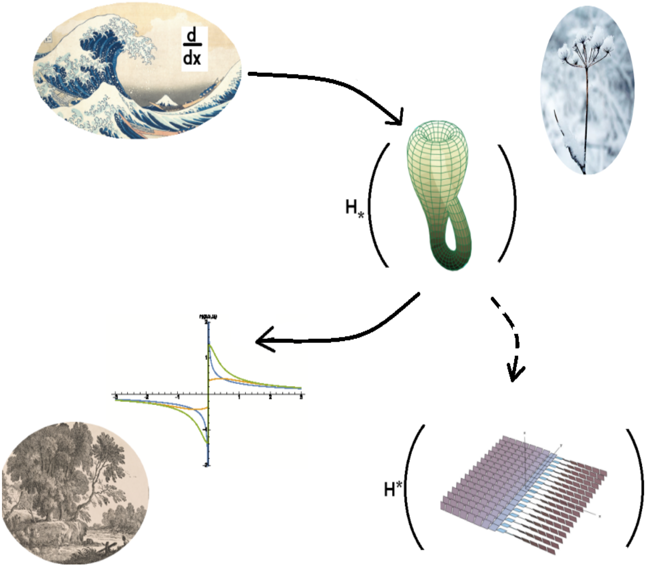
Map of Garden



Outline

Differential equations

Function

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A function is an application of a set of objects into a set of images which applies given object onto one **and only one** image point.

- Function is single valued prescription - there is no 'multivalued functions',
- $f : x \to e^x$ - a function,
- $f : x \to \sqrt{x}$ - not a function if defined on the whole $\mathbb{C}$.

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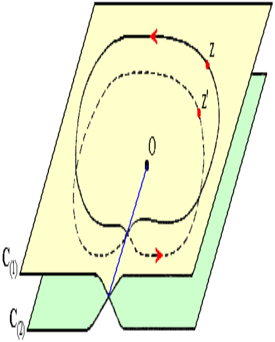
Critical point

A critical point of an application of the Riemann sphere $\mathbb{C}P^1$ (complex plane with ∞ point) onto itself is any singular point, isolated or not, around which at least two determinations are permuted. Such a points is an obstacle for an application to be a function.



$$z \to \sqrt{z}$$

Singularities

Singular points($x = 0$):

- Pole $\frac{1}{x^a}$, $a \in \mathbb{N} \setminus \{0\}$ (noncritical),
- Branch point x^a, a - noninteger; (critical),
- Essential singularity:
  - $e^{1/x}$ (noncritical),
  - $\tan(\log(x))$ (critical and nonisolated).

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Uniformization

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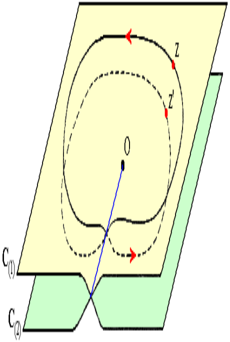
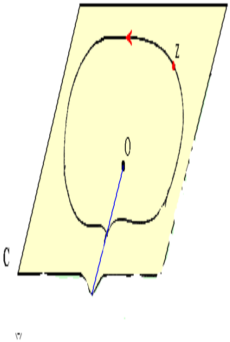
Consider a multivalued application of the Riemann sphere onto itself. There exists two classical methods, called uniformizations, to define from it a single valued application, i.e., a function.

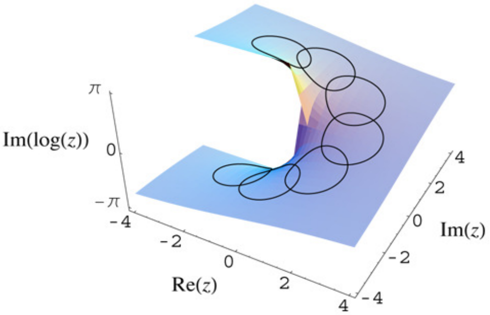
Uniformization is possible only when the locations of singularities are know!





Solution of ODEs

'Classical'/Cauchy version:
Find local solution in terms of a power series in some neighbourhood of an expansion point and extend it by analytic continuation.



Solution of ODEs

Painlevé version

To integrate an ODE is to find for the general solution a **finite expression**, possibly multivalued, in a **finite** number of functions, valid in the whole domain of definition.

For $u' + u^2 = 0$:

- $u = u_0 \sum_{j=0}^{\infty}[-(x - x_0)u_0]^j$ - nonintegrated (no finite form, locally defined),
- if radius of convergence is known, summation performed, we get the meromorphic function $(x - x_1)^{-1}$ that is a solution.

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- 'Double interest' in differential equations:
  - Sources of new functions (since 1614, Lord Napier ($u' = u$ Galileo) 'Mirifici Logarithmorum Canonis Descriptio' [1],
  - Class of equations that can be integrated with existing functions available.

[1] 'A Description of the Wonderful Law of Logarithms'

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Paul Painlevé

- French Prime Minister (1917), (1925),
- ODEs: P. property, P. transcendents,
- ODEs: In 1908, he became Wilbur Wright's first airplane passenger in France and in 1909 created the first university course in aeronautics,
- Painlevé conjecture: Among the solutions to the n-body problem: there are noncollision singularities for $n \geq 4$,
- General Relativity: Gullstrand–Painlevé coordinates for Schwarzschild metric.

Linear equations

Linear ODE

The general solution of a linear ODE is uniformizable.

$$\sum_{k=0}^{N} a_k(x) \frac{d^k u}{dx^k} = 0, a_N(x) \neq 0. \tag{1}$$

The only solutions of these equations are singularities of the coefficients $a_k(x)$ (Frobenius method).

Fixed singularities

The singularities of the equation coefficients are called **fixed singularities**.

Linear equations have only fixed singularities.

- Linear ODEs define functions (Airy, Bessel, etc.),
- Nonlinear ODE is integrable if it is linearisable,
- To extend a list of known functions it is necessary to consider nonlinear ODEs.

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Nonlinear ODEs posses two types of singularities:

- fixed - singularities of the coefficients of ODE,
- movable - the singularities of solutions; position depends on initial data; not present in linear ODEs.

Up to now no general methods exist that allows to determine the positions of movable singularities.

(Trivial) example [Goriely]

The equation

$$\dot{x} = x^3, \quad x(t_0) = x_0$$

has the solutions

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Painlevé property

Painlevé property (refolmulated)

One calls Painlevé property of an ODE the absence of movable critical singularities in its general solution.
General(i.e. not singular) solutions are considered.

- Movable critical singularites are obstacles in uniformization.
- One have to know where start cuts and how to do it.
- Only noncritical movable singularities are tractable.

Class of abstractions of the equations with PP

The PP of an ODE is invariant under an arbitrary homographic transformation of the dependent variable and an arbitrary holomorphic change of independent variable:

$$(x,u) \to (X,U) : u = \frac{a(x)U + b(x)}{c(x)U + d(x)}, \quad X = e(x), \quad ad - bc \neq 0 \tag{2}$$

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Determine all the **algebraic** differential equations of first order, then second order, then third order, etc., whose general solution has no movable critical points.

- Current state of classification:
  - 1st order - Riccati and Weierstrass equation,
  - 2nd order - 53 canonical equations, 47 integrable in terms of known functions,
  - 3rd order - nothing interesting...
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6 algebraic ODEs which solutions has only noncritical movable singularities. They cannot be integrated in terms of known transcendent functions, they define new functions.

- (PI) $x'' = 6x^2 + \lambda t$,
- (PII) $x'' = 2x^3 + tx + \mu$,
- $(PIII)$ $txx'' = tx'^2 - xx' + at + bx + cx^3 + dtx^4$,
- (PIV) $txx'' = tx'^2 - xx' + at + bx + cx^3 + dtx^4$,
- (PV) very long equation,
- (PVI) very long equation.

$(PIII)$ can be used to construct correlation function for 2D Ising model: T.T. Wu, B.M McCoy, C.A.Tracy and E. Barouch (1976), *Spin-spin correlation functions for the two dimensional Ising model: exact theory in the scaling region*, Physical Review B13, 316-374

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The Painlevé property/Painlevé test

- Deduce global structure of solution (types of singularities) from the local behaviour around some points in the complex plane. Only sufficient conditions $\rightarrow$ by the contraposition - it gives a result when it fails.
- Consists of two steps:
  - (local study) necessary conditions for absence of critical points,
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Substitute formal power series into an equation and do the following steps:

- Expansion around 'singular solutions' - finding all dominant balances.
- Compute eigenvalues of the variational equation (the Kovalevskaya exponents).
- If the exponents are positive integer/rational (resonance conditions) then check additional conditions.

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Results

Numerical scanning of complex plane

Problem statement [Kycia-Numerics]

- ODE as a system of first order DE

$$\frac{d\vec{y}(x)}{dx} = \vec{f}(\vec{y}; x), \quad \vec{y}(x) : x \in \mathbb{C} \to \mathbb{C}^n. \tag{3}$$

- Initial value $\vec{y}(x_0) = \vec{y}_0$.
- Path, e.g., $(t \in \mathbb{R}^+)$
  - Semiline $x(t) = x_0 + (t + shift) \cdot e^{i\phi}$
  - Spiral $x(t) = (x_0 + (at + b)e^{i \cdot dir \cdot t})e^{i\phi}$
- Domain - path connected region (ideally connected by paths along which integration is performed).
- Condition for singularity proximity - the crude estimation $||\vec{y}|| < \mathrm{Large\ \ const}$. Not the state of art, but it can be improved.

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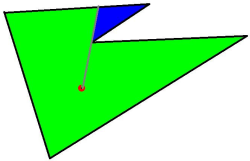

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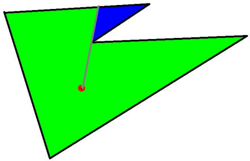
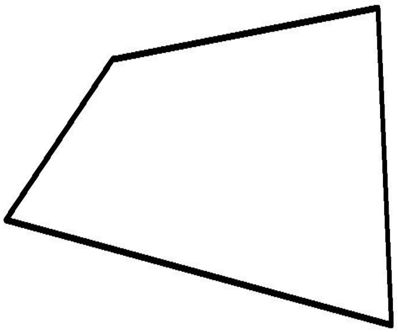
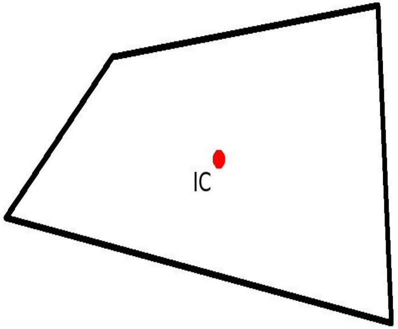


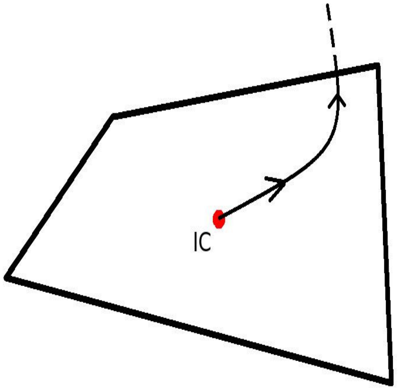
Domain



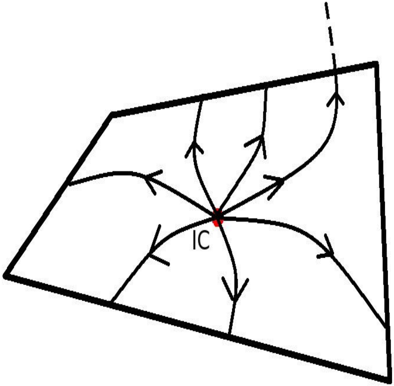
Initial Conditions



Integration along path



Full integration



Lane-Emden and Isothermal Sphere equations

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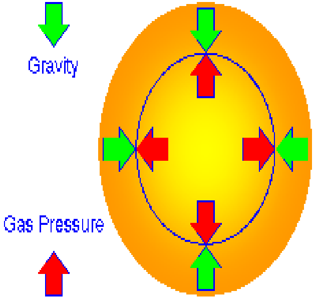

Equivalence of two opposite forces:

- Gravity
- Polytropic gas pressure

LE equation model molecular cloud cores [Yu-Qing Lou, Esimbek].

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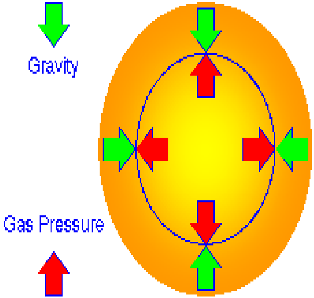

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Equations [Kycia, Filipuk]

The Emden-Fowler equation

$$\frac{d^2u(x)}{dx^2} + \frac{\alpha}{x}\frac{du(x)}{dx} + x^n u(x)^p = 0, \quad u(0) = 1. \tag{4}$$

Generalized Isothermal Sphere equation

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Location of singularities [Kycia, Filipuk]

A nonzero analytic solutions of the Generalized Emden-Fowler and Isothermal Sphere equations have $n+2$ singularities located symmetrically with respect to the origin on the rays connecting the origin with all $(n+2)$ roots of -1 in the complex plane.

Asymptotics and stability [Kycia, Filipuk]

There are singular solutions which are asymptotically stable for $x \to \infty$.

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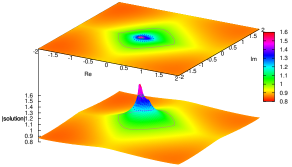
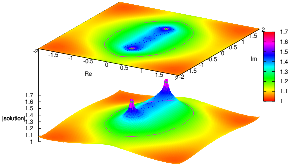
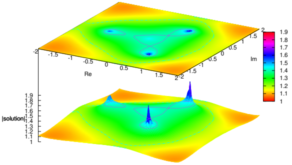


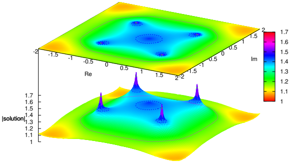
(a) $n = -1$



(b) $n = 0$



(c) $n = 1$



(d) $n = 2$

Figure: $p = 5$ and $u(0) = 1.5$, the Generalized Emden-Fowler solution.

Generalized isothermal sphere equations [Kycia, Filipuk]

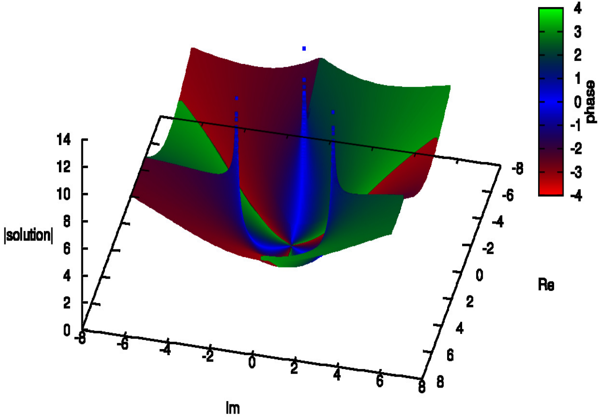


Figure: $u(0) = 0$, $n = 1$

Field theory and mathematical physics

Semilinear wave equations

Based mainly on [Kycia].

Semilinear wave equation

$$\Box U(t,x) - U(t,x)^p = 0, \qquad \Box = \partial_{tt} - \triangle,$$

where $x \in R^n$, $n \geq 3$, p-even to preserve reflection symmetry or $U^p \to |U|^{p-1}U$.

Spherical symmetry

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r - U^p = 0,$$

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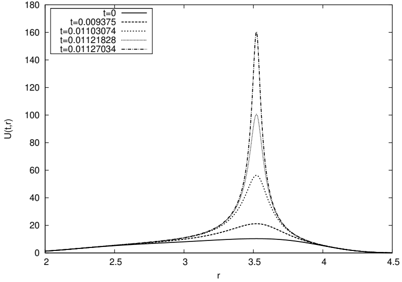
There exist 'smooth' initial data that develop singularity when $t \to T < \infty$.



- Example of nonglobal existence.
- Common behavior for many nonlinear PDEs, see [Eggers, Fontelos].
- Blowup dynamics is governed by self-similar solutions.
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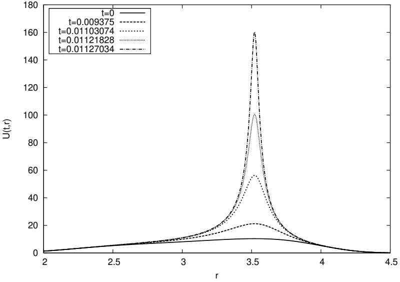

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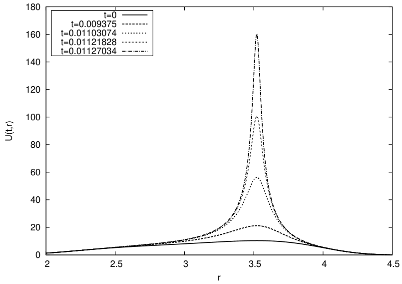

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Self-similar solutions

$$U(t,r) = \frac{u(\rho)}{(T-t)^{\alpha}}, \qquad \rho = \frac{r}{T-t}, \qquad \alpha = \frac{2}{p-1} (>0).$$

Self-similar solution

Self-similar profile

1. $b_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}$
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Semilinear profiles

ODE for self-similar profiles

$$(1-\rho^2)u'' + \left(\frac{n-1}{\rho} - \frac{2(p+1)}{p-1}\rho\right)u' - \frac{2(p+1)}{(p-1)^2}u + u^p = 0,$$

where $' = \frac{d}{d\rho}$.

- Fixed singularities at 0, ± 1, ∞.
- Question: Is there a global (on $[0;1]$) analytic solution ?
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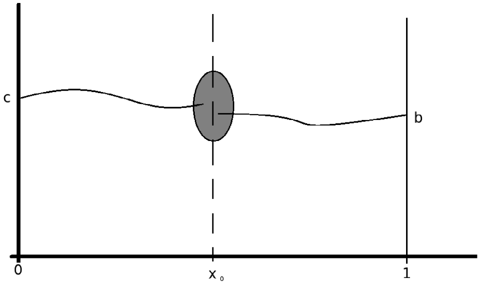

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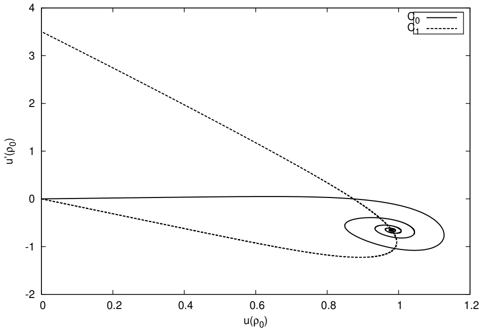

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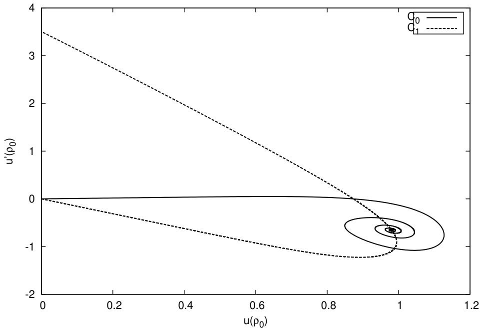

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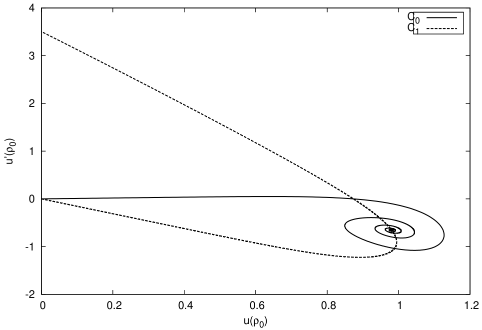

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Perturbed Lane-Emden equation [Kycia-PerturbedLE]

Make more general (perturbed) Lane-Emden equation:

$$\begin{gathered} p(x)\frac{d^2u(x)}{dx^2} + q(x)\frac{du(x)}{dx} + r(x)u(x) + \delta u(x)^p = 0, \\ p(x) = (1 + a_{-1}x + a_0x^2 + \ldots + a_nx^{n+2}), \\ q(x) = \left(\frac{\alpha}{x} + b_{-1} + b_0x + \ldots + b_nx^{n+1}\right), \\ r(x) = \left(c_{-1}\frac{1}{x} + c_0 + \ldots + c_nx^n\right), \end{gathered} \tag{6}$$

Applications in:

- Perturbed profiles of field theory models.
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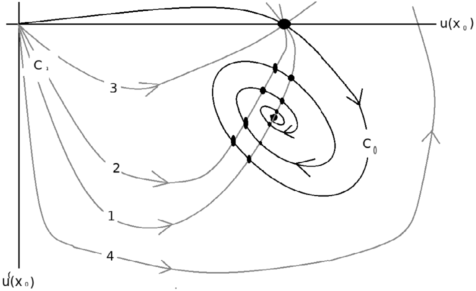

Main result of [Kycia-PerturbedLE]:

- Infinite number of analytic solutions exist when u_∞-type solution exists.
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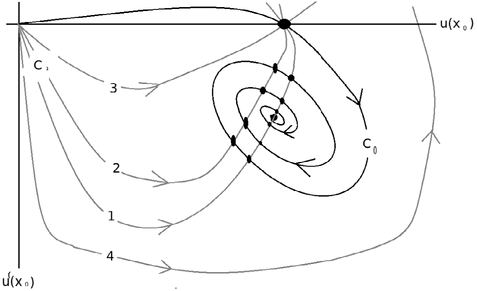

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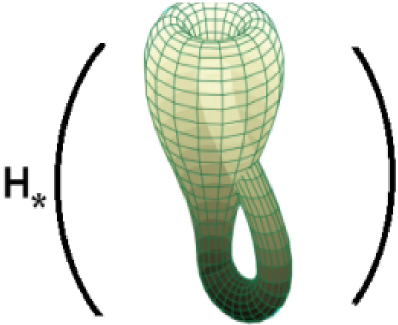
Topology



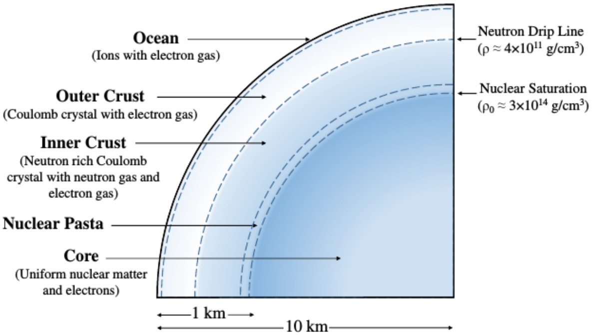
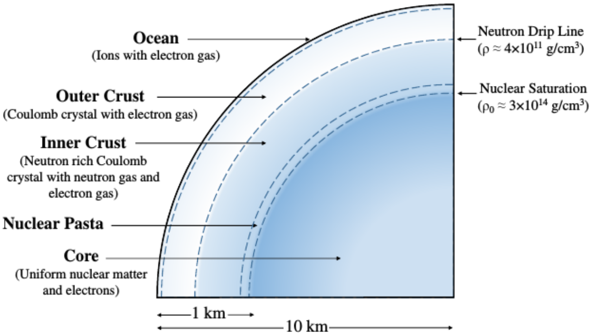


Figure: Core of neutron star from [Caplan, Horowitz].

We focus on pasta phases.

Introduction



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Protons(p), and neutrons(n) interacts by:

$$\begin{aligned} V_{np}(r) &= ae^{-r^2/\Lambda} + (b-c)e^{-r^2/2\Lambda}, \\ V_{nn}(r) &= ae^{-r^2/\Lambda} + (b+c)e^{-r^2/2\Lambda}, \\ V_{pp}(r) &= ae^{-r^2/\Lambda} + (b-c)e^{-r^2/2\Lambda} + \frac{\alpha}{r}e^{-e/\lambda}. \end{aligned} \tag{7}$$

Electron gas interaction is included in λ factor.

Pasta phases

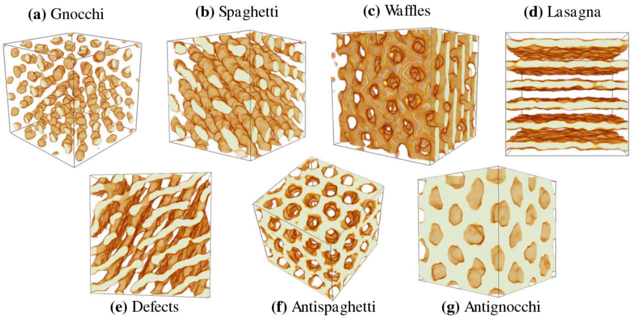

Figure: Pasta phases from [Caplan, Horowitz].

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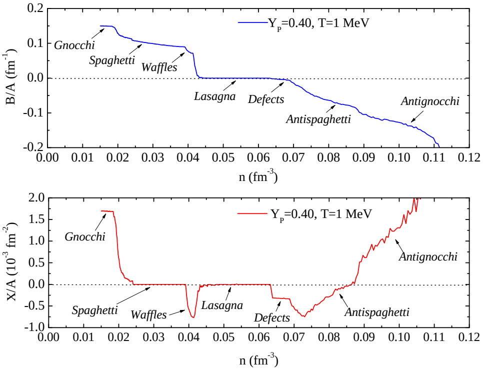


Figure: Pasta phases from [Caplan, Horowitz].

V - Volume; A - Area; $B = \int_{\partial K} (k_1 + k_2)/4\pi dA$ - mean Breadth;
$\chi = \int_{\partial K} k_1 k_2 / 4\pi dA$ - Euler characteristic;

Crash course in homology

Boundaries and holes

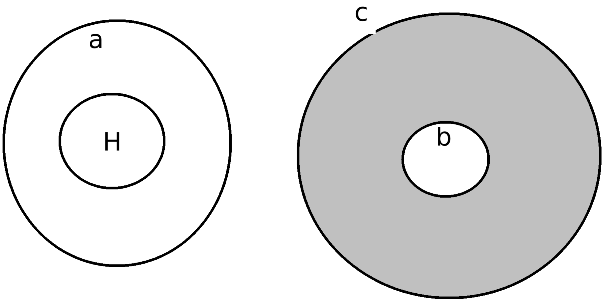
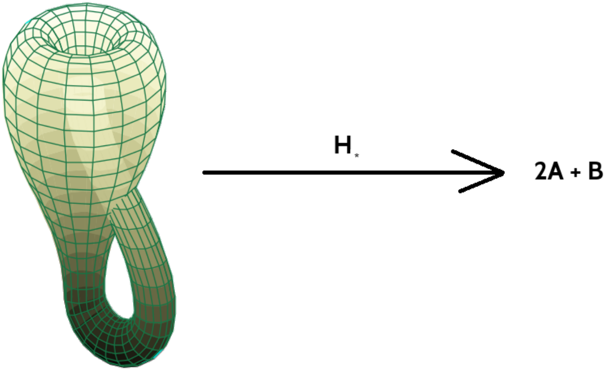


Figure: The curve a is not a boundary of any surface because of the hole H. $a \in Z_1$ is a $1-$cycle. On the other side the curves b and c are boundaries of the surface (shaded area) - more strictly, the sum of these curves with the signs depending on their orientation gives a boundary of 2dim surface, and therefore belongs to B_1.

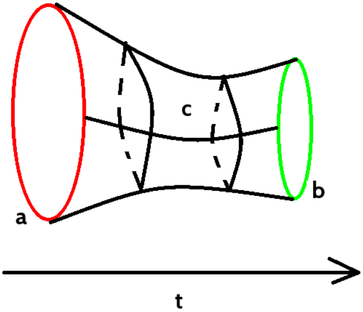
$$H_k = Z_k / B_k. \tag{8}$$

Homology



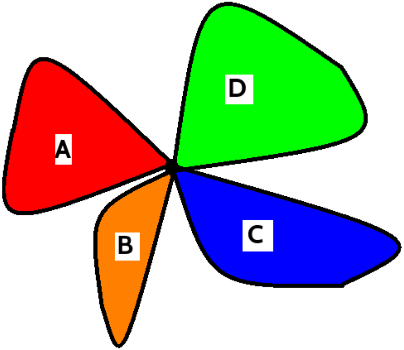
Homology H is a functor that maps topological space into algebraic structures - modules.

Homotopy



- Homotopy is a continous one-parameter map.
- Two homotopical cycles have the same homology, e.g., $a - b = \partial c$.

Wedge sum



$$A \vee B \vee C \vee D \tag{9}$$

The Eilenberg–Steenrod Axioms

- Exactness Axiom: For any pair (X, A) of topological spaces there is natural exact sequence:

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \ldots$$

- Homotopy Axiom: If $f \simeq g : (X, A) \to (Y, B)$ then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$.
- Excision Axiom: If $cl(U) \subset int(A)$ then $H_n(X - U, A - U) \cong H_n(X, A)$.
- Dimension Axiom: $H_0(\{*\}) = \mathbf{Z}$ and $H_{j>0}(\{*\}) = 0$
- Additivity Axiom: $H_k(\coprod_i X_i) = \bigoplus_i H_k(X_i)$

We will need extended Additivity Axiom:
If $X = \vee_{i=1}^n X_i$ then $H_k(X) = \bigoplus_{i=1}^n H_k(X_i)$ for $k > 0$; $H_0(X)$ is the number of connected components of X.

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Betti numbers

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$b_k = rank(H_k/Tor(H_k))$

Euler characteristic

$\chi = \sum_{i=0}^{\infty}(-)^i b_i$

Examples

As an example take a **circle $\mathbf{S^1}$**, then

$$H_0(S^1) = \mathbb{Z}, \quad H_1(S^1) = \mathbb{Z}, \quad H_{k>1} = 0, \tag{10}$$

as S^1 has only one 1-dim hole and one connected component.
Similarly, for $\mathbf{S^n}$ one gets

$$H_0(S^n) = \mathbb{Z}, \quad H_n(S^n) = \mathbb{Z}, \quad H_{k\neq 0,n}(S^n) = 0. \tag{11}$$

We will also need the homology group of a **torus** $\mathbf{T^2} = \mathbf{S^1} \times \mathbf{S^1}$ which is

$$H_0(T^1) = \mathbb{Z}, \quad H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}, \quad H_2(T^2) = \mathbb{Z}, \quad H_{k>2}(T^2) = 0, \tag{12}$$

as it is connected, has two 1-dim holes, i.e., two closed curves that are 'boundaries' to these holes, and one two dimensional hole in the centre.

Examples

As an example take a **circle $\mathbf{S^1}$**, then

$$H_0(S^1) = \mathbb{Z}, \quad H_1(S^1) = \mathbb{Z}, \quad H_{k>1} = 0, \tag{10}$$

as S^1 has only one 1-dim hole and one connected component. Similarly, for $\mathbf{S^n}$ one gets

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Pasta phases

Lets go back to Pasta phases.

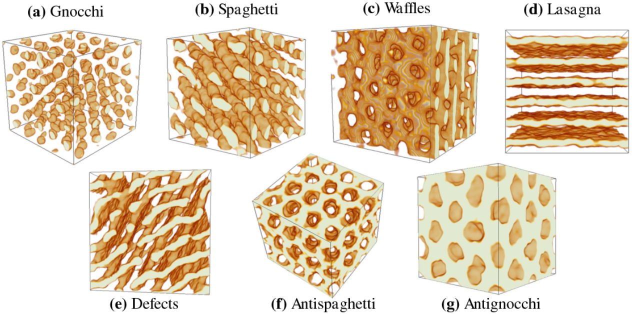

Figure: Pasta phases from [Caplan, Horowitz].

Gnocchi phase [Kycia, Kubis, Wójcik]

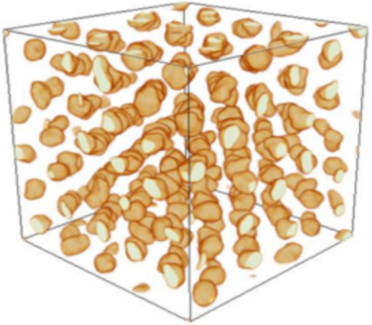

Figure: Gnocchi phase.

When we consider it as a ball B, which is contractible to the point, we have $b_0 = 1$ and other Betti numbers vanish. As a result $\chi(B) = 1$.

Spaghetti phase [Kycia, Kubis, Wójcik]

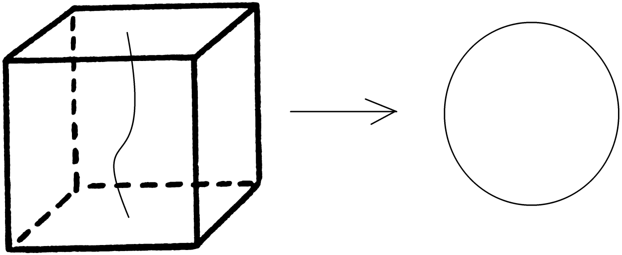

Figure: Spaghetti phase topology, which is a circle S^1.

Using identification we obtain a filled torus $T^2_f = S^1 \times D^2$, where D^2 is the two dimensional disc. We can contract T^2_f to S^1 by shrinking the disc D to the point, and the Betti numbers are $b_0 = 1$, $b_1 = 1$; other numbers vanishes. Therefore $\chi(T^2_f) = \chi(S^1) = 0$.

Waffles phase and its creation [Kycia, Kubis, Wójcik]

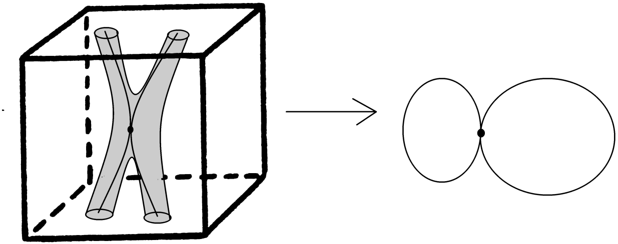

Figure: Waffle phase topology - merging of two strings deformed to filled tori $T^2_f \bigvee T^2_f \sim S^1 \bigvee S^1$.

This structure can be modelled by merging $S^1 \sim T^2_f$ that occurred in the Spaghetti phase as such merging generates $\bigvee_{i=1}^n S^1$ which has $b_0 = 1$, $b_1 = n$ and zero otherwise, therefore $\chi = 1 - n$ in this case.

Waffles phase and its creation [Kycia, Kubis, Wójcik]

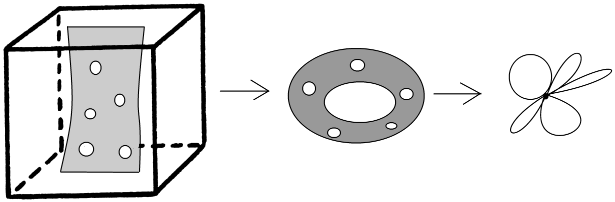

Figure: Waffle phase topology - a cylinder(closed 'ribbon') with holes, that can be deformed to $\bigvee_{i=1}^{n} S_i^2$.

At some point it can also be treated as a cylinder with holes which can be deformed to $\bigvee_{i=1}^{n} S_i^2$, where n is the number of holes. It is the best to consider first the cylinder with two holes and expand these holes until they meet each other and then deform them slightly. The Betti numbers then are $b_0 = 1$, $b_1 = n$ and other numbers vanishes, which produces $\chi = 1 - n$.

Waffles phase and its creation [Kycia, Kubis, Wójcik]

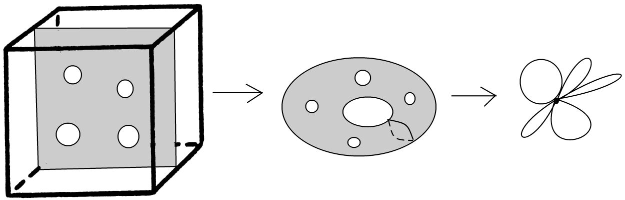

Figure: Waffle phase topology - a torus with holes that can be deformed to $\bigvee_{i=1}^{n+1} S_i^2$.

When the multi hole surface is wide enough to reach all four sides of the simulation cell then we can wrap the surface into the surface of the torus with holes. This torus with holes can be deformed into $\bigvee_{i=1}^{n+1} S_i^2$ for n holes. Therefore, again $b_0 = 1$, $b_1 = n + 1$ and other Betti numbers vanishes, that gives $\chi = -n$ again.

Lasagna phase [Kycia, Kubis, Wójcik]

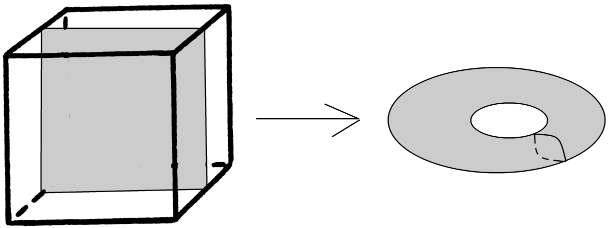

Figure: Lasagna phase topology - a torus.

When the holes vanishes the parallel planes are obtained, that under identification of opposite simulation cell walls results in tori. It is well known fact [3] that for torus the Betti numbers are $b_0 = 1$, $b_1 = 2$ and $b_2 = 1$ which gives well known fact $\chi(T^2) = 2 - 2g = 0$, where the genus $g = 1$.

Defect phase - general [Kycia, Kubis, Wójcik]

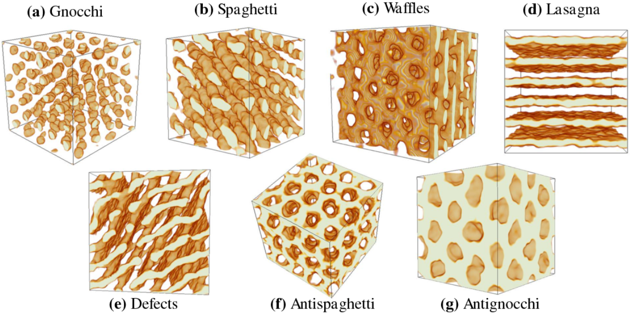

Figure: Pasta phases from [Caplan, Horowitz].

We assume that the defect appears when some connections between planes appears, which is equivalent to the merging of the tori, which results in increasing the genius g and therefore the Euler characteristics $\chi = 2 - 2g$

Defect phase - spiral phase [Kycia, Kubis, Wójcik]

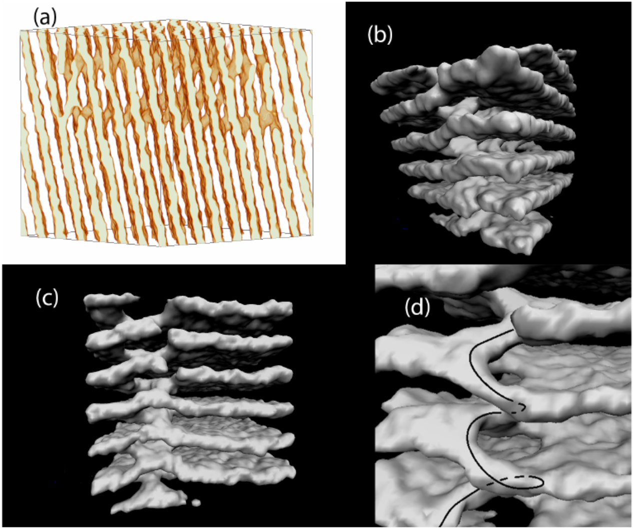

Figure: Spiral defect figure taken from [2]. See especially d).

Defect phase - spiral phase [Kycia, Kubis, Wójcik]

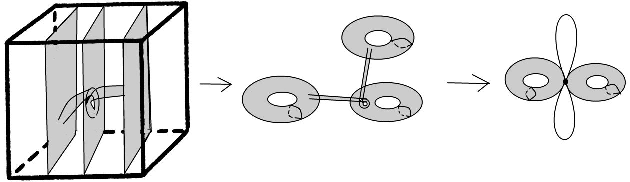

Figure: Two boundary planes connected with one middle plane $k = 1$ by the spiral ribbon.

Call the surface M, then deforming homeomorphically k tori with holes to the $\bigvee_{i=1}^{2} S^1_{k;i}$ and shrinking spiral to the interval and then to the point we get that $M \sim T^2 \bigvee [\bigvee_{l=1}^{k} (S^1_l \bigvee S^1_l)] \bigvee T^2$ and adding the Betti numbers of all factors we obtain $b_0 = 1$, $b_1 = 4 + 2k$ and $b_2 = 2$, therefore, $\chi(M) = -(1 + 2k)$.

Antispaghetti phase [Kycia, Kubis, Wójcik]

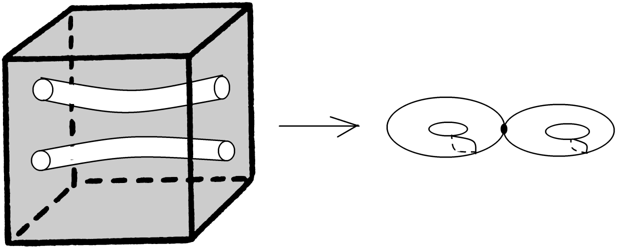

Figure: Antispaghetti phase topology - a T^3 with the holes which are T^2, that can be deformed to $\bigvee_{i=1}^{n} T_i^2$.

The antispaghetti phase can be considered as $T^3 \setminus \cup_{i=1}^{n} T_i^2$, where n is the number of tori-shape holes. The phase can be deformed into $\bigvee_{i=1}^{n} T_i^2$ and that results in $b_0 = 1$, $b_1 = 2n$ and $b_2 = n$, that gives the Euler characteristic $\chi = 1 - n$. This phase occurs when n is large and decreases to 0.

Antignocchi phase [Kycia, Kubis, Wójcik]

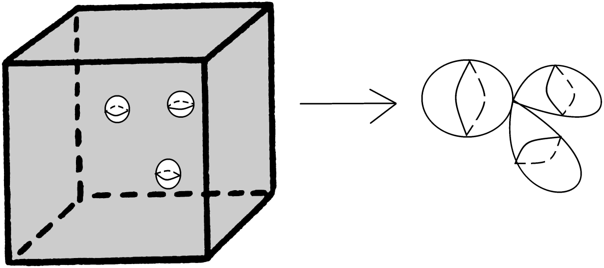

Figure: Antignocchi phase topology - a T^3 with the holes which are two dimensiona empty spheres S^2 that can be deformed to $\bigvee_{i=1}^{n} S_i^2$.

Finally, the tori holes become an empty balls. Taking its boundary we have that $T^3 \setminus \cup_{i=1}^{n} S_i^2$, where n is the number of holes, that can be deformed to $\bigvee_{i=1}^{n} S_i^2$ and this provides $b_0 = 1$, $b_2 = n$ and other Betti numbers vanishes, which gives $\chi = 1 + n$ - a linear growth.

Conclusions

Why topology is useful:

- It helps to derive some characteristics of phenomena even when analytic description is not available.
- The methods are general enough to be applied to classify other materials with holes - work in progress.
- Topology is present in physics - Noble Prize 2016: David J. Thouless, F. Duncan M. Haldane, and J. Michael Kosterlitz.
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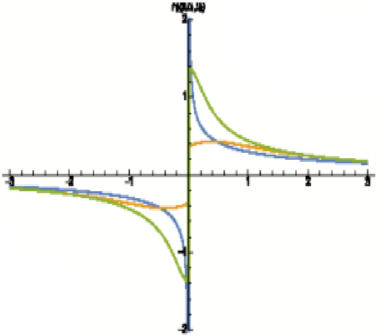
Distributions & Special functions



Particle physics

- Removing singular behaviour of matrix elements by regularization and renormalization - ill-defined multiplication of distributions.
- Propagators as a source of singularities.
- Complex plane methods for dealing with poles and branch cuts.

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Real-life accelerator beams [Kycia, Jadach]

A resonance seen through the Gaussian beam dispersion is

$$V_2(E;\mu,\Gamma,\sigma) = \int_{-\infty}^{\infty} \frac{\mu\Gamma}{\pi} \frac{1}{(E'^2-\mu^2)^2+(\mu\Gamma)^2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(E'-E)^2}{2\sigma^2}} dE'. \quad (13)$$

Introducing the variable

$$t = \frac{E-E'}{\sqrt{2}\sigma}, \quad (14)$$

we obtain

$$V_2(E;\mu,\Gamma,\sigma) = \frac{\mu\Gamma}{\pi\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{(E-\sqrt{2}\sigma t-\mu)^2(E-\sqrt{2}\sigma t-\mu)^2+(\mu\Gamma)^2} \sigma\sqrt{2}dt. \quad (15)$$

Relativistic Voigt profile [Kycia, Jadach]

Defining new variables

$$u_1 := \frac{E-\mu}{\sqrt{2}\sigma}, \quad u_2 := \frac{E+\mu}{\sqrt{2}\sigma}, \quad a := \frac{\Gamma\mu}{2\sigma^2}, \tag{16}$$

the distribution can be rewritten in the form

$$V_2(E;\mu,\Gamma,\sigma) = \frac{1}{\sigma^2 2\sqrt{\pi}} \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{(u_1-t)^2(u_2-t)^2+a^2} dt. \tag{17}$$

New function which can be called relativistic line broadening function and is defined as

$$H_2(a,u_1,u_2) := \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{(u_1-t)^2(u_2-t)^2+a^2} dt. \tag{18}$$

and (17) can be rewritten in the following form

$$V_2(E;\mu,\Gamma,\sigma) = \frac{H_2(a,u_1,u_2)}{\sigma^2 2\sqrt{\pi}}. \tag{19}$$

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Proposition 1

We have

$$\begin{aligned} H_2(a, u_1, u_2) &= H_2(a, u_2, u_1) \\ H_2(a, -u_1, u_2) &= H_2(a, u_1, -u_2) \\ H_2(-a, u_1, u_2) &= -H_2(a, u_1, u_2), \end{aligned} \tag{21}$$

from which results that

$$H_2(a, -u_1, -u_2) = H_2(a, u_1, u_2). \tag{22}$$

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Proposition 2

For $u_1 \neq u_2$

$$lim_{a \to 0^{\pm}} H_2(a, u_1, u_2) = \frac{\pm}{|u_1 - u_2|} (e^{-u_1^2} + e^{-u_2^2}). \qquad (24)$$

For $u_1 = u_2$ and nonzero a the H_2 function is finite and when $a \to 0$ it becomes unbounded with assymptotics

$$H_2(a, u_1 = u, u_2 = u) = \frac{e^{-u^2}}{\sqrt{2a}} + \frac{e^{-u^2}}{\sqrt{2}} (2u^2 - 1)\sqrt{a} + O(a). \qquad (25)$$

When $|u_1|$ and $|u_2|$ tends to infinity for $a \neq 0$ then as the integrand in (23) tends to zero, so H_2 also vanishes. It is also true when $a \to 0^{\pm}$ and $|u_1 - u_2| \to \infty$.

Properties [Kycia, Jadach]

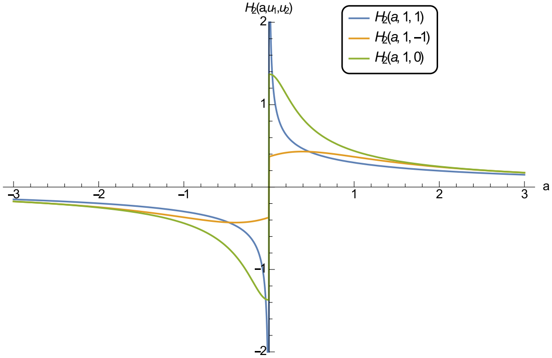

Figure: H_2 for a few selected values [Kycia, Jadach].

Conclusions

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- More properties in [Kycia, Jadach].

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ODE

- R. Conte (editor) *The Painlevé property. One century later* CRM Series in Mathematical Physics, Springer 1999
- A. Goriely, *Integrability and Nonintegrability of Dynamical Systems* , World Scientific (2001)
- C. Hunter *Series solutions for polytropes and the isothermal sphere*, Mon. Not. R. Astron. Soc. **328** 839-847 (2001)
- P. Mach, *All solutions of the $n = 5$ Lane-Emden equation*, J. Math. Phys. 53, 062503 (2012)
- D.L. Yu-Qing Lou, J. Esimbek, *Lane-Emden equation with inertial force and general polytropic dynamic model for molecular cloud cores.*, Mon Not R Astron Soc 473 (2):2441–2464 (2018)

ODE

- R. Kycia, G. Filipuk, *On the singularities of the Emden-Fowler type equations*, Proceedings of ISAAC Conference (2014)
- R. Kycia, G. Filipuk, *On the generalized Emden–Fowler and isothermal spheres equations*, Applied Mathematics and Computation, Volume 265, 2015, Pages 1003–1010
- R. Kycia *Scheme for numerical investigation of movable singularities of the complex valued solutions of ordinary differential equations* V.P. Gerdt et al. (Eds.): CASC Workshop 2014, LNCS 8660, pp. 288–303, 2014
- R.A. Kycia, *Perturbed Lane–Emden Equations as a Boundary Value Problem with Singular Endpoints*, J Dyn Control Syst 26, 333–347 (2020)

- P. Bizoń, D. Maison, A. Wasserman, *Self-similar solutions of semilinear wave equations with a focusing nonlinearity*, Nonlinearity **20** 2061-2074 (2007); arXiv: math.AP/0702156v1
- P. Breitenlohner, P. Forgács, D. Maison, *Static spherically symetric solutions of the Einstein-Yang-Mills equations*, Commun. Math. Phys. **163** 141-172 (1994)
- J. Eggers, M.A. Fontelos, *The role of self-similarity in singularities of partial differential equations*, Nonlinearity **22** R1-R44 (2009)
- R. Kycia, *On self-similar solutions of semilinear wave equations in higher space dimensions*, Appl. Math Comput. **217** 9451-9466 (2011)
- Y. Masafumi, *Movable singularity of semi linear Heun equation and application to blowup phenomenon*, NODEA, 26 1 (2019)

Topology

- A.S. Schneider, C.J. Horowitz, J. Hughto, D.K. Berry, *Nuclear Pasta Formation*, Phys. Rev. C 88, 065807 (2013);DOI: 10.1103/PhysRevC.88.065807; arXiv:1307.1678 [nucl-th]
- C.J. Horowitz, D. K. Berry, C.M. Briggs, M.E. Caplan, A. Cumming, A.S. Schneider, *Disordered nuclear pasta, magnetic field decay, and crust cooling in neutron stars*, Phys. Rev. Lett. 114, 031102 (2015); DOI: 10.1103/PhysRevLett.114.031102; arXiv:1410.2197 [astro-ph.HE]
- R.A. Kycia, S. Kubis, W. Wójcik, *Topological analysis of nuclear pasta phases*, Phys. Rev. C 96, 025803 (2017)
- M.E. Caplan, C. J. Horowitz, *Astromaterial Science and Nuclear Pasta*, arXiv:1606.03646 [astro-ph.HE]
- A. Hatcher, *Algebraic topology*, Cambridge University Press, 1st edition 2001;

Particle physics

- R.A. Kycia, S. Jadach, *Relativistic Voigt profile for unstable particles in high energy physics*, J. Math. Anal. Appl. 463, 2 1040–1051 (2018)
- S. Jadach, R.A. Kycia, *Lineshape of the Higgs boson in future lepton colliders*, Physics Letters B, 755, 58–63 (2016)

Differential forms

R.A. Kycia, *The Poincare Lemma, Antiexact Forms, and Fermionic Quantum Harmonic Oscillator*, Results Math 75, 122 (2020)

R.A. Kycia, *The Poincare lemma for codifferential, anticoexact forms, and applications to physics*, arXiv:2009.08542 [math.DG]

Thank You for Your Attention