And now for something completely different

(a Monty Python quote)

How much motion in the expansion?
How does time really flow?
CREDObility of basic notions

CREDO Week 1.10-5.10 2018 @ IFJ PAN

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Delivered at CREDO Week 4 X 2018

CREDO enters the Unknown (sources of extreme energy radiation, the physics behind). It is hard to predict what would be the solution. Equally well, the known physics may be insufficient in this respect and our language not well suited to such problems.

In trying to find a solution, we are aware that even the basic concepts which we have been used so far to describe Nature with the known physics, have their limitations. That means that they are not always CREDIBLE. For some, we can simply deliniate the domain of their applicability. In this talk I will focus on two such tangible notions as an ilustration:

* the fundamental concept of *proper time* may be not realized by Nature, even at the classical level.

* some other, ‘less fundamental’ or secondary notions, eg. the concept of *relative velocity*, break already on cosmological scales, due to the presence of curvature.

Ideal clock and the clock hypothesis

Ideal clock I A fictious device of the relativity theory that always records its **proper time** in equal steps, independently of accelerations the clock is subject to.

What about extreme accelerations on the order of $\frac{mc^3}{\hbar}$ (10^{29} m/s^2 for the Dirac electron) ?

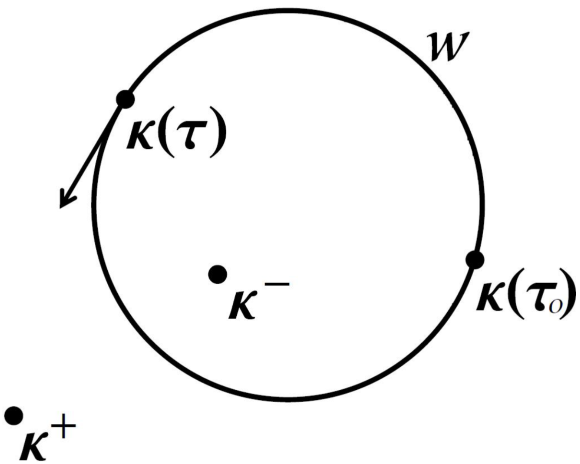
Ideal clock II A purely mathematical construct describing a perfect clocking mechanism with a finite number of degrees of freedom, built up according to the rules of classical relativistic mechanics (**resident of Plato's world of ideals**).

Ideal = mathematical + perfect; [perfect = the clocking mechanism /the intrinsic structure of the clock/ is insensitive to the influences of the external world]

Clock hypothesis asserts that ideal clock records its proper time even when accelerated.
This hypothesis should be tested.

Main idea: Ideal clock as a relativistic rotator *(A. Staruszkiewicz 'Fundamental Relativistic Rotator ' Acta Phys. Pol. B Proc. Suppl., vol. 1, pp. 109–112, 2008.:)*

Ideal clock as a geometric model of particle with spin
Interpretation on the plane of complex numbers



Fundamental correspondence:

1) Null directions (eg. EM wave vector k)
– points on complex plane

2) Spatial directions (eg. spin-pseudovector)
- circles on compex plane

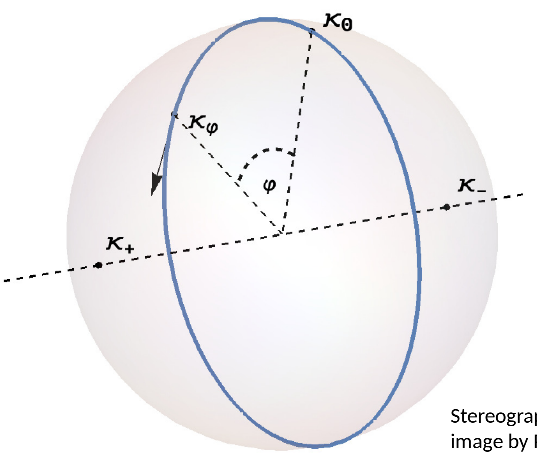
3) Lorentz transformations
– homographies on complex plane

null direction k (clock's pointer)
spatial direction W (clock's dial) fixed by the clock's spin direction

k.W=0 (point on a fixed circle)

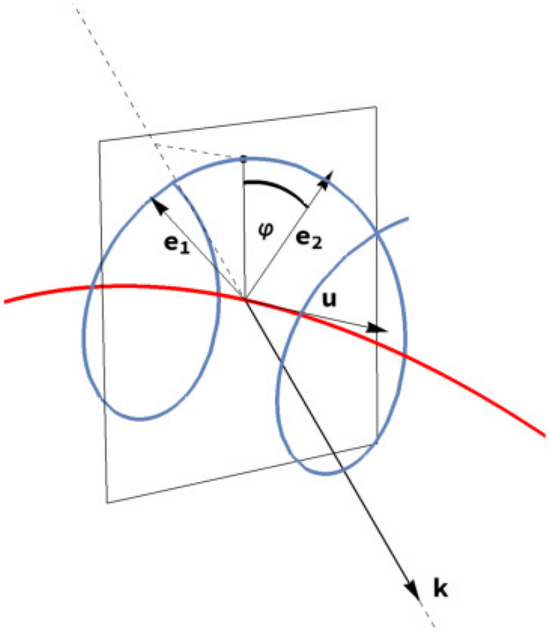
K+/- (fixed poles: determined by the momentum P and spin vector W)

The phase (phi) of the intrinsic circular motion is the clock's intrinsic time.



Stereographic re-projection:
image by Paweł Rzońca,
Msc thesis, AGH 2018.

Ideal clock as a geometric model of particle with spin
A toy model: CHRONOMETRIC CURVES



Kinematical constraints from ideal clock theory

null worldline

$$\frac{\dot{x}\cdot\dot{x}}{\dot{x}\cdot k}=0$$

clocking frequency

$$-\ell^2\frac{\dot{k}\cdot\dot{k}}{(k\cdot\dot{x})^2}=1$$

null directions conjugated w.r.t. CM frame 'u'

$$\frac{\dot{x}}{u\cdot\dot{x}}=2u-\frac{k}{u\cdot k}$$

+ a definition of phase w.r.t. a Fermi-Walker transported reference position on the clock dial

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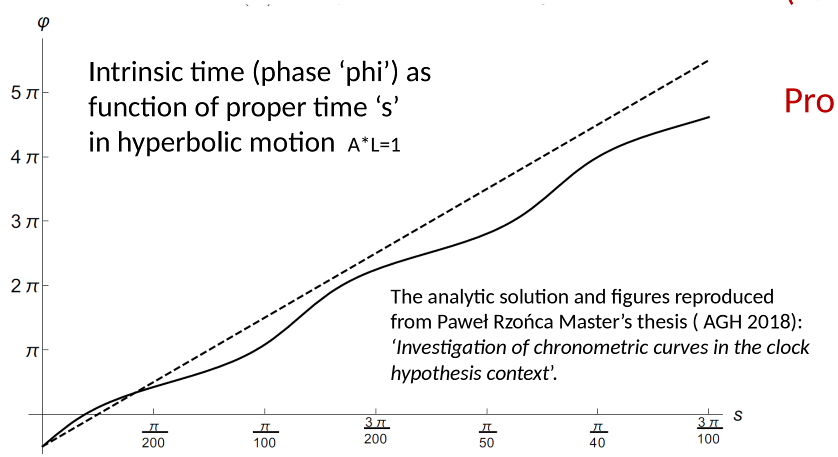


Application to hyperbolic motion (relativistic uniformly accelerated frame)

$$(u^\mu)=(\cosh\alpha s,\ \sinh\alpha s,\ 0,\ 0),$$

$$(A^\mu)=(\alpha\sinh\alpha s,\ \alpha\cosh\alpha s,\ 0,\ 0).$$

Problem exactly integrable

$$\alpha=\sqrt{-A^\mu A_\mu}$$

$$\varphi=\pi+2\mathrm{arctg}\left(\sqrt{1-\frac{\alpha^2\ell^2}{4}}\mathrm{tg}\left(\pm\sqrt{1-\frac{\alpha^2\ell^2}{4}}(s+s_0)/\ell\right)\mp\frac{\alpha\ell}{2}\right)$$

$$\varphi=\pm\frac{2}{\ell}s-\frac{\pi}{2}+\frac{\alpha\ell}{2}\sin(2s/\ell)+O(\alpha^2)$$

Summary

* It is uncertain whether it is possible to devise a relativistic clocking mechanism that would always measure its proper time.

* the *chronometric curve* model suggests, that even the simplest such mechanism will depend on accelerations (will not be perfect).
However, it is not yet known if the prototype ideal clock (Fundamental Relativistic Rotator with helical null worldline) violates the clock hypothesis in non-free motion.

(1927) Lemaitre-Hubble Expansion Law (1929)

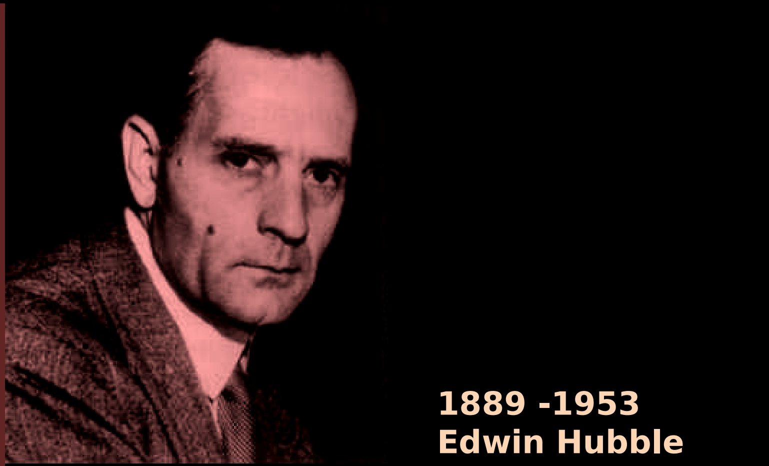
UN UNIVERS HOMOGÈNE DE MASSE CONSTANTE ET DE RAYON CROISSANT RENDANT COMPTE DE LA VITESSE RADIALE DES NÉBULEUSES EXTRAGALACTIQUES.

Annals of the Scientific Society of Brussels 47A: 41 (1927)

Monthly Notices of the Royal Astronomical Society
(1931) 91: 483–490

A HOMOGENEOUS UNIVERSE OF CONSTANT MASS AND GROWING RADIUS ACCOUNTING FOR THE RADIAL VELOCITY OF EXTRAGALACTIC NEBULAE

1894 – 1966
Georges Lemaitre



1889 -1953
Edwin Hubble

$$\int_{\tau_{em}}^{\tau_{obs}} \frac{d\tau}{a(\tau)} = \Delta\psi = \int_{\tau_{em}+\delta\tau_{em}}^{\tau_{obs}+\delta\tau_{obs}} \frac{d\tau}{a(\tau)} \quad \Rightarrow \quad \frac{\delta\tau_{obs}}{a(\tau_{obs})} \underset{\lim}{=} \frac{\delta\tau_{em}}{a(\tau_{em})}$$

$$z = \frac{\omega_{em}}{\omega_{obs}} - 1 = \lim_{\delta\tau_{em}\to 0} \frac{\delta\tau_{obs}}{\delta\tau_{em}} - 1 = \frac{a(\tau_{obs})}{a(\tau_{em})} - 1 \approx \underbrace{\frac{\dot{a}(\tau_{em})}{a(\tau_{em})}}_{H} \Delta\tau = \frac{v}{c}$$

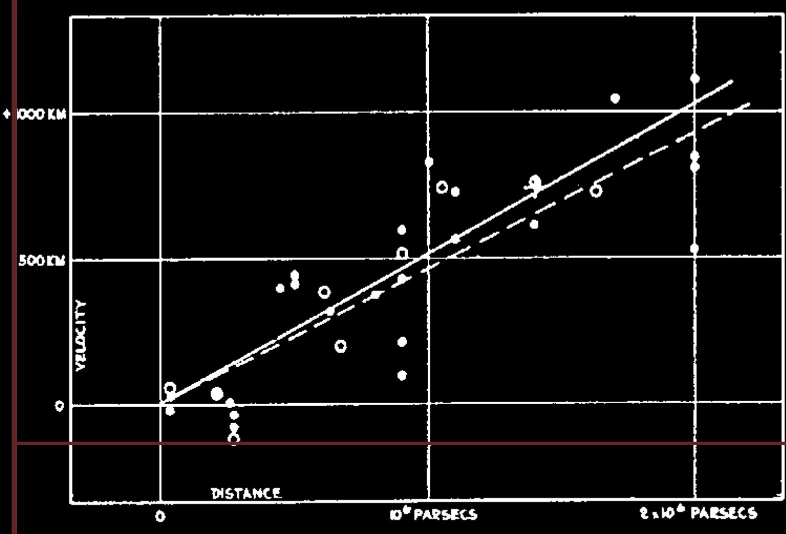
$$ds^2 = -d\tau^2 + a^2(\tau) \begin{cases} d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta d\phi^2)\,, & \psi \in (0,\infty) \\ d\psi^2 + \psi^2\,(d\theta^2 + \sin^2\theta d\phi^2)\,, & \psi \in (0,\infty) \\ d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta d\phi^2)\,, & \psi \in (0,\pi) \end{cases}$$

??? $$z = \frac{\omega_{em}}{\omega_{obs}} - 1 =: \sqrt{\frac{1+v/c}{1-v/c}} - 1$$???

A RELATION BETWEEN DISTANCE AND RADIAL VELOCITY AMONG EXTRA-GALACTIC NEBULAE

Proceedings of the National Academy of Sciences
Vol. 15 : 1929 : Number 3

PARSEC ≈ 206 000 AU ≈ 3.26 light years



homogenous and isotropic spaces of constant time

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2 \left[\frac{\mathrm{d}\varrho^2}{1 - \kappa\,\varrho^2} + \varrho^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \right) \right]$$

Three homogenous
and isotropic spaces
of constant cosmic time:

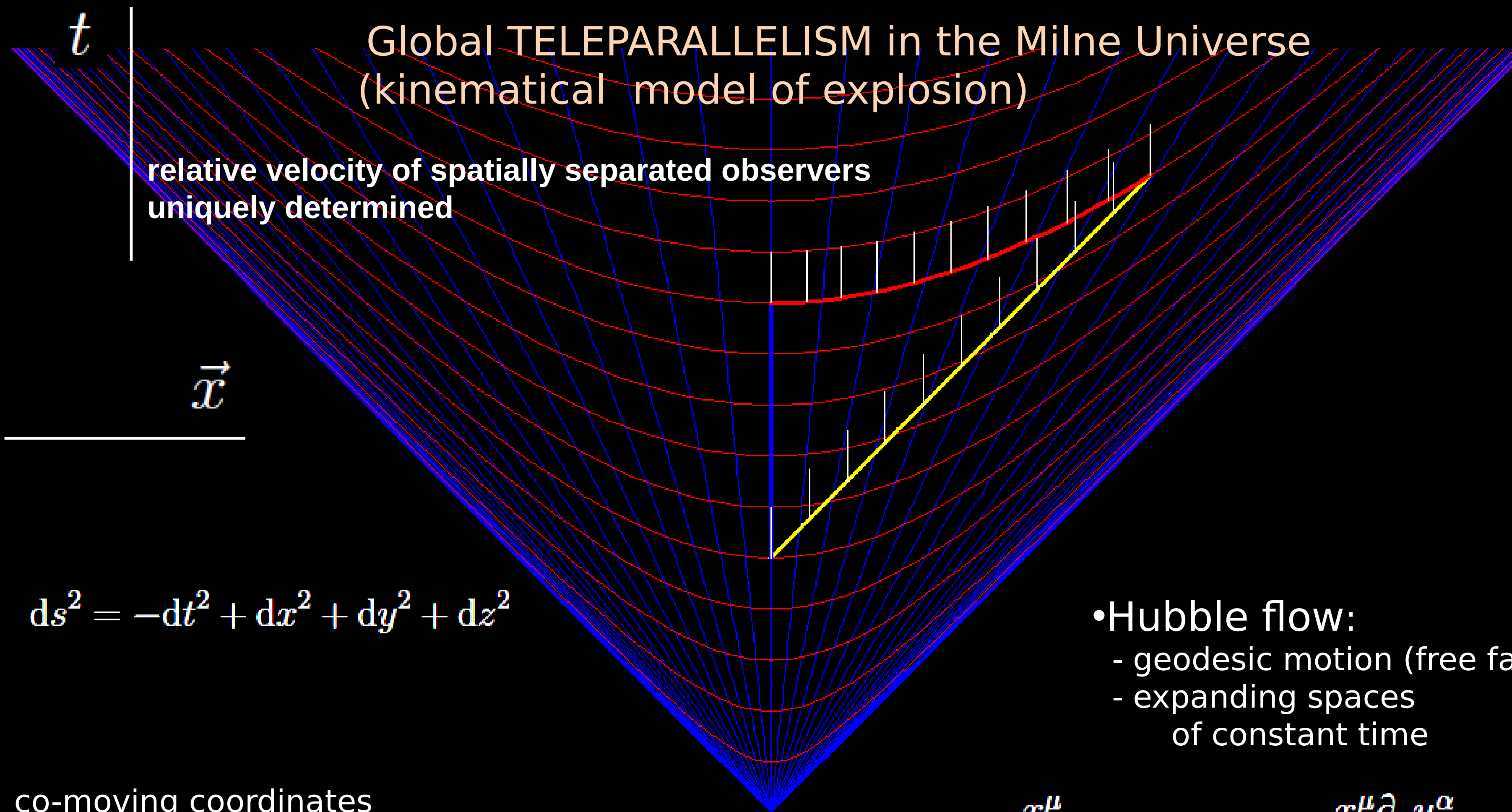
$$\begin{cases} \kappa = -1 & \varrho = \sinh\psi \\ \kappa = 0 & \varrho = \psi \\ \kappa = +1 & \varrho = \sin\psi \end{cases}$$

Scalar curvature $R = 6\,\dfrac{\kappa + \dot{a}^2 + a\ddot{a}}{a^2},$ $\quad \Theta^{\mu}{}_{\mu} = 3\,\dfrac{\dot{a}}{a}$ Dilation scalar

MILNE UNIVERSE: $\kappa = -1 \qquad a(\tau) = \tau \qquad \Theta^{\mu}{}_{\mu} = \dfrac{3}{\tau}$

FLAT UNIVERSE: $\kappa = 0 \qquad a(\tau) = \tau^{2/3} \qquad \Theta^{\mu}{}_{\mu} = \dfrac{2}{\tau}$

Global TELEPARALLELISM in the Milne Universe
(kinematical model of explosion)



$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

•Hubble flow:
- geodesic motion (free fall)
- expanding spaces of constant time

<u>co-moving coordinates</u>

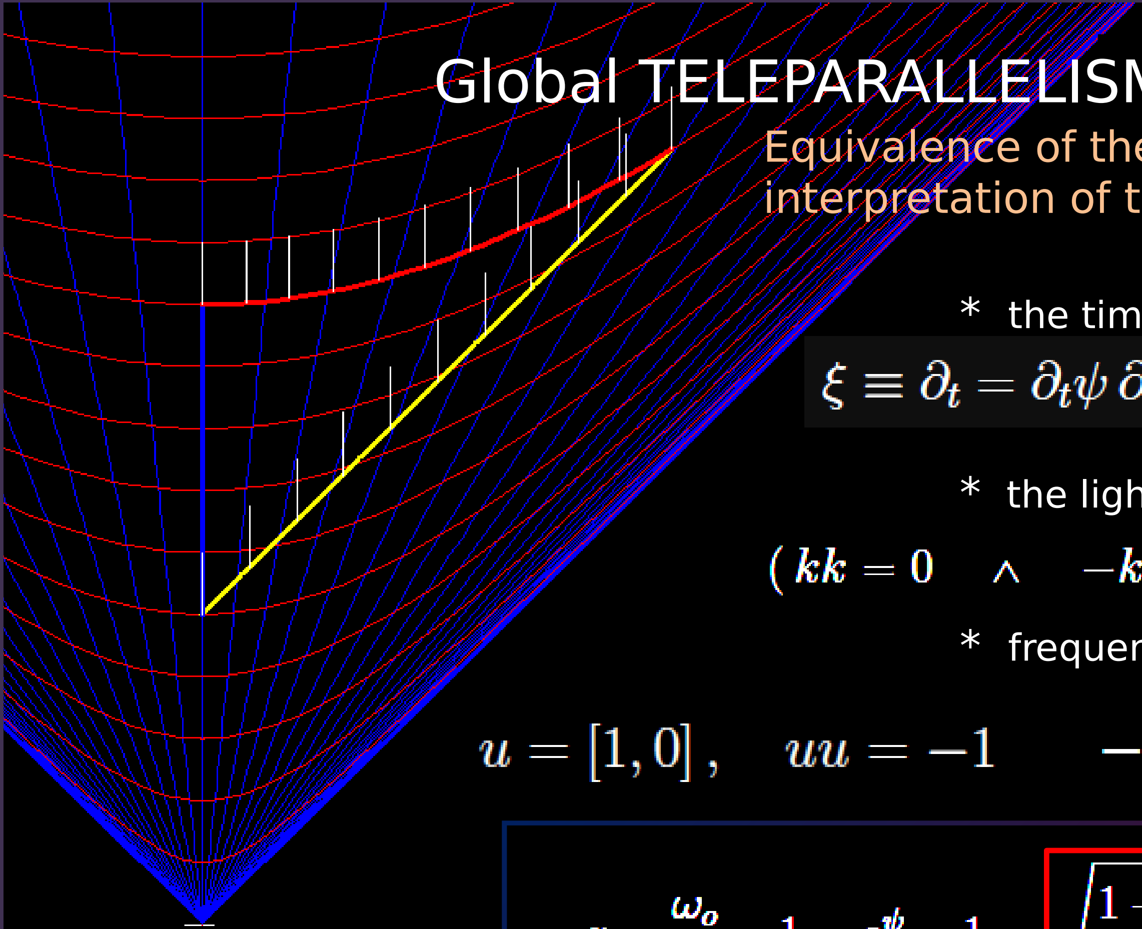
$$t = (\tau \cosh \psi)$$
$$\vec{x} = (\tau \sinh \psi) \cdot \vec{n}(\theta, \phi)$$

$$u^\mu = \frac{x^\mu}{\sqrt{-xx}}, \quad u^\mu \partial_\mu u^\alpha = \frac{x^\mu \partial_\mu u^\alpha}{\sqrt{-xx}} = 0,$$

$$\Theta = \frac{1}{3}\partial_\mu u^\mu = \frac{1}{\tau}$$

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^2 \left[\mathrm{d}\psi^2 + \sinh^2\psi \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right]$$

$$u = [1, 0]\,, \quad uu = -1$$



Global TELEPARALLELISM in the Milne Universe

Equivalence of the kinematical and cosmological interpretation of the red-shift formula

* the time-like Killing vector

$$\xi \equiv \partial_t = \partial_t\psi\,\partial_\psi + \partial_t\tau\,\partial_\tau \hat{=} \left[\cosh\psi, -\tau^{-1}\sinh\psi\right]$$

* the light-ray vector

$$(\,kk = 0 \quad \wedge \quad -k\xi = \omega_o\,) \quad \Rightarrow \quad k = \omega_o\,\mathrm{e}^{-\psi}\left[1, \tau^{-1}\right]$$

* frequency perceived in the fluid rest frame

$$u = [1, 0]\,, \quad uu = -1 \qquad -ku = \omega \quad \Rightarrow \quad \boxed{\omega = \omega_o\,\mathrm{e}^{-\psi}}$$

$$z = \frac{\omega_o}{\omega} - 1 = \mathrm{e}^{\psi} - 1 = \boxed{\sqrt{\frac{1+\mathrm{tgh}\,\psi}{1-\mathrm{tgh}\,\psi}} - 1} \quad \wr \quad \boxed{v = \mathrm{tgh}\,\psi \quad !!!}$$

$$\begin{aligned} \tau\cosh\psi &= \tau_o + \lambda \\ \tau\sinh\psi &= \lambda \end{aligned} \quad \Rightarrow \quad \mathrm{e}^{\psi(\tau)} = \frac{\tau}{\tau_o} = \frac{a(\tau)}{a(\tau_o)} \quad \wr \quad z = \boxed{\frac{a(\tau)}{a(\tau_o)} - 1}$$

$$d(\tau) = \tau\psi(\tau) = \tau\ln\frac{\tau}{\tau_o} \quad \Rightarrow \quad d'(\tau) = 1 + \ln\frac{\tau}{\tau_o} > 1$$

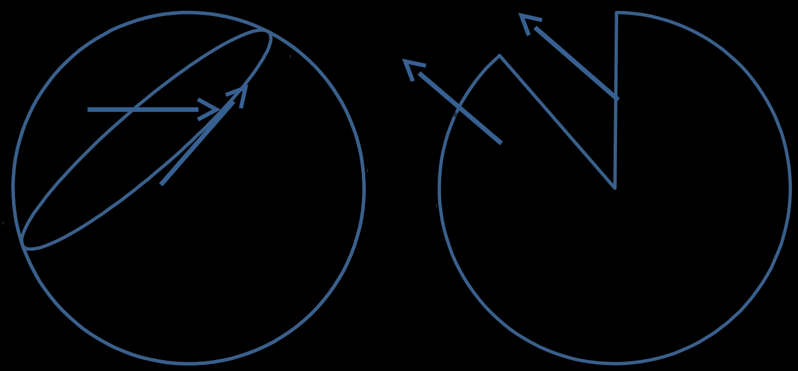
Heuresis of the covariant law of parallel transport

$$0 = \dot{w}^a = w^a_{,\nu}\,\dot{x}^\nu, \qquad X : x^{\mu'} \to x^\mu = X^\mu(x^{1'}, \ldots, x^{n'})$$

$$\dot{w}^a = X^a_{,a'}\left[\dot{w}^{a'} + \gamma^{a'}_{b'\nu'}\,w^{b'}\,\dot{x}^{\nu'}\right], \quad \gamma^{a'}_{b'\nu'} = (X^{-1})^{a'}_{,b}X^b_{,b'\nu'}$$

$$\boxed{\dot{w}^a + \Gamma^a_{b\nu}\,w^b\,\dot{x}^\nu = 0}$$

Parallel transport along a closed loop. Connection with the curvature

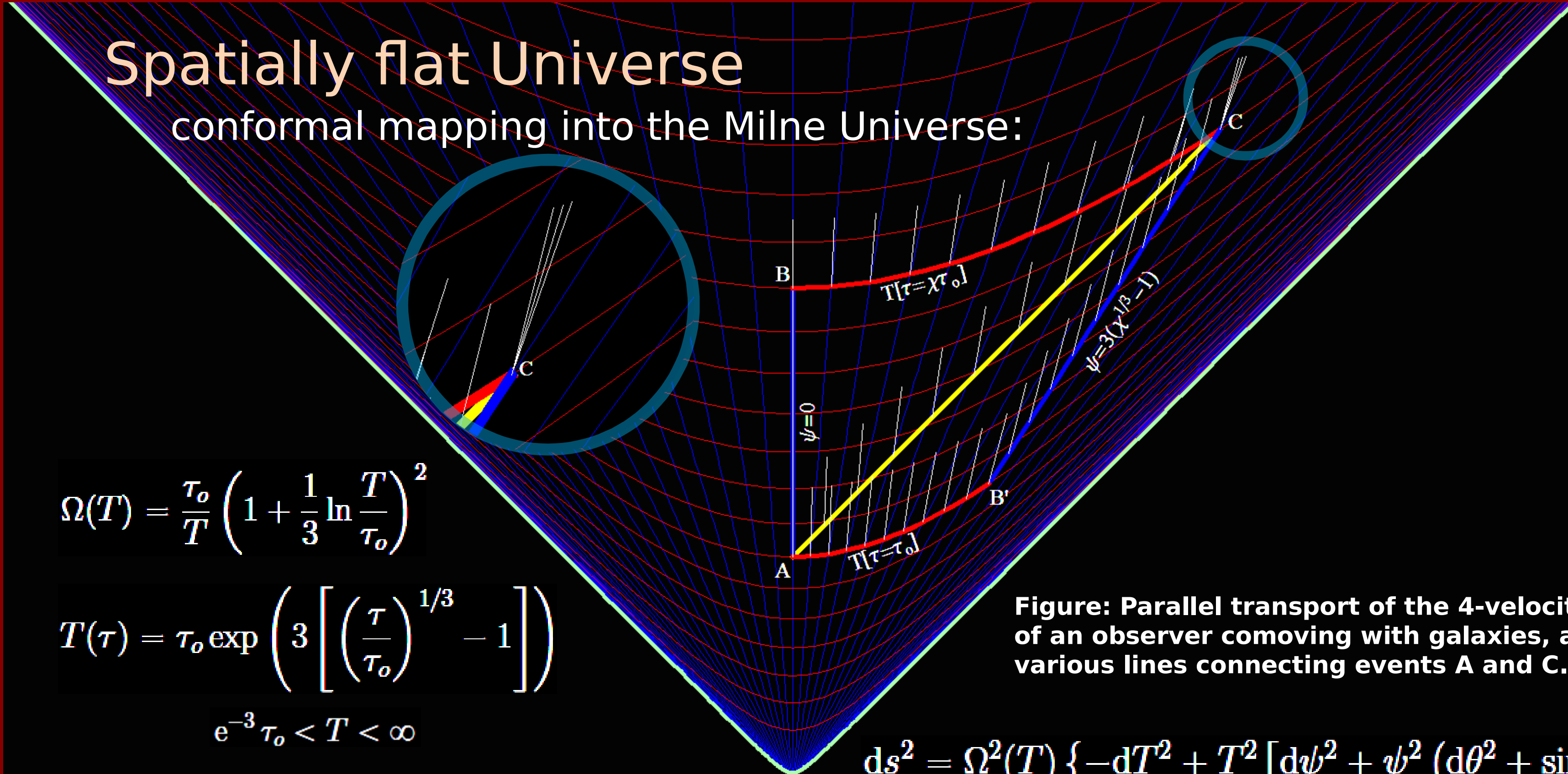


$$\Delta w^a = -\oint_{\partial S} \Gamma^a_{b\nu}\,w^b \mathrm{d}x^\nu = \iint_S \left(\Gamma^a_{b\nu}\,w^b\right)_{,\mu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \equiv \frac{1}{2}\iint_S R^a{}_{b\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,w^b$$

$$\boxed{R^a{}_{b\mu\nu} := \Gamma^a_{b\mu,\nu} - \Gamma^a_{b\nu,\mu} + \Gamma^c_{b\mu}\Gamma^a_{c\nu} - \Gamma^c_{b\nu}\Gamma^a_{c\mu}}$$

Spatially flat Universe

conformal mapping into the Milne Universe:



$$\Omega(T) = \frac{\tau_o}{T}\left(1 + \frac{1}{3}\ln\frac{T}{\tau_o}\right)^2$$

$$T(\tau) = \tau_o \exp\left(3\left[\left(\frac{\tau}{\tau_o}\right)^{1/3} - 1\right]\right)$$

$$\mathrm{e}^{-3}\,\tau_o < T < \infty$$

Figure: Parallel transport of the 4-velocity vector of an observer comoving with galaxies, along various lines connecting events A and C.

$$\mathrm{d}s^2 = \Omega^2(T)\left\{-\mathrm{d}T^2 + T^2\left[\mathrm{d}\psi^2 + \psi^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]\right\}$$

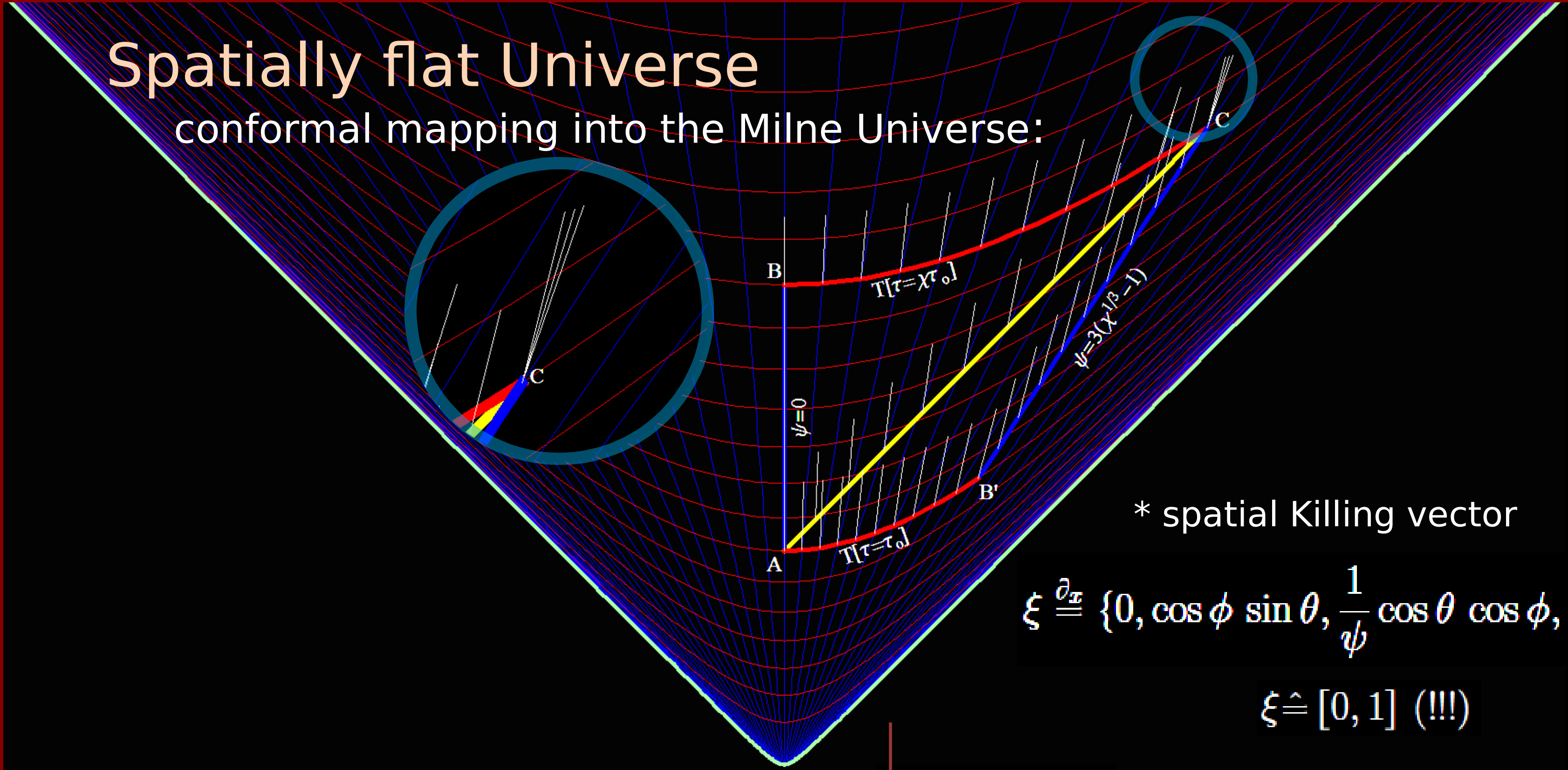
$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \left(\frac{\tau}{\tau_o}\right)^{4/3}\tau_o^2\left[\mathrm{d}\psi^2 + \psi^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right], \qquad \psi \in (0,\infty)$$

	$\tau_{min} < \tau(x) < \tau_{max}$	$\psi_{min} < \psi(x) < \psi_{max}$	$\lambda_{min} < \lambda(x) < \lambda_{max}$	$\lambda\|_{\chi\to 1+\epsilon}$
$A \to B$	$\tau_o < x < \chi\tau_o$	0	0	
$B \to C$	$\chi\tau_o$	$0 < \frac{3}{2}\chi^{1/3}x < 3\left(\chi^{1/3} - 1\right)$	$0 < x < 2\left(1 - \chi^{-1/3}\right)$	$\frac{2}{3}\epsilon - \frac{4}{9}\epsilon^2$
$A \to C$	$\tau_o < \tau_o \exp\left(\frac{3}{2}x\right) < \chi\tau_o$	$0 < 3\left(\exp\left(\frac{1}{2}x\right) - 1\right) < 3\left(\chi^{1/3} - 1\right)$	$0 < x < \frac{2}{3}\ln\chi$	$\frac{2}{3}\epsilon - \frac{3}{9}\epsilon^2$
$A \to B'$	τ_o	$0 < \frac{3}{2}x < 3\left(\chi^{1/3} - 1\right)$	$0 < x < 2\left(\chi^{1/3} - 1\right)$	
$B' \to C$	$\tau_o < x < \chi\tau_o$	$3\left(\chi^{1/3} - 1\right)$	$2\left(\chi^{1/3} - 1\right)$	$\frac{2}{3}\epsilon - \frac{2}{9}\epsilon^2$

Table: hyperbolic angle (lambda) [relative velocity measure – rapidity] between local and transported velocity vector.

Spatially flat Universe

conformal mapping into the Milne Universe:



* spatial Killing vector

$$\xi \overset{\partial_x}{=} \{0, \cos\phi\,\sin\theta, \frac{1}{\psi}\cos\theta\,\cos\phi, -\frac{1}{\psi}\frac{\sin\phi}{\sin\theta}\}$$

$$\xi \hat{=} [0, 1] \ \ (!!!)$$

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \left(\frac{\tau}{\tau_o}\right)^{4/3} \tau_o^2 \left[\mathrm{d}\psi^2 + \psi^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right]$$

* a light-ray vector

$$(\, kk = 0 \quad \wedge \quad -k\xi = const. \quad \wedge \quad -uk|_{\tau_o} = \omega_o)$$

$$\Rightarrow \quad k = \omega_o \left(\frac{\tau_o}{\tau}\right)^{2/3} \left[1, \left(\frac{\tau_o}{\tau}\right)^{2/3} \frac{1}{\tau_o}\right]$$

$$\frac{\omega_o}{\omega} = \left(\frac{\tau}{\tau_o}\right)^{3/2}\Bigg|_{\tau=\chi\tau_o} = \mathrm{e}^{\lambda}, \quad \lambda = \frac{2}{3}\ln\chi$$

$$-ku = \omega$$

* Cosmological red-shift

CONCLUSIONS

* For distances small enough (neighbouring galaxies, low redshifts z<<1), the relative motion is indistinguishable from space expansion (the neighbouring galaxies truly receed one from another).

* For larger distances, the kinematical interpretation of the redshift breaks, also the notion of relative velocity between spatially separated objects no longer makes sense.

Parallel transport along a null geodesics in a spatially flat Universe

- **spatially flat universe:** $-\mathrm{d}\tau^2 + \tau^{4/3}\left(\mathrm{d}\psi^2 + \psi^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right)$
- **a null geodesics** in the plane $\theta = \pi/2,\ \phi = 0:\quad x(s) = \{A^3(s), B(s)\}$

$$\dot{x}\dot{x} = 0 \quad \rightarrow \quad A(s)^4\left(-9\,A'(s)^2 + B'(s)^2\right) = 0$$

- **geodesic equations** $\begin{cases} A(s)\left(6\,A'(s)^2 + (2/3)\,B'(s)^2 + 3\,A(s)\,A''(s)\right) = 0, \\ 4\frac{A'(s)}{A(s)}B'(s) + B''(s) = 0 \end{cases}$

$$x(0) := (T_o, 0) \quad \rightarrow \quad x(s) = \{(1+s)^{\frac{3}{5}}\,T_o\,,\ -3\,T_o{}^{\frac{1}{3}} + 3\,(1+s)^{\frac{1}{5}}\,T_o{}^{\frac{1}{3}}\}$$

- **parallel transport** of $v(s) = \{\alpha(s), \beta(s)\}$: $v(0) = \{1, 0\}$, $\tau \in (T_o, \chi T_o)$:

$$\frac{2\,T_o{}^{\frac{2}{3}}\,\beta(s)}{5\,(1+s)^{\frac{3}{5}}} + \alpha'(s) = 0, \qquad \frac{2}{5}\left(\frac{\alpha(s)}{(1+s)^{\frac{7}{5}}\,T_o{}^{\frac{2}{3}}} + \frac{\beta(s)}{1+s}\right) + \beta'(s) = 0$$

$$\rightarrow \quad v(s) = \left\{\frac{1 + (1+s)^{\frac{4}{5}}}{2\,(1+s)^{\frac{2}{5}}}, \frac{1 - (1+s)^{\frac{4}{5}}}{2\,(1+s)^{\frac{4}{5}}\,T_o{}^{\frac{2}{3}}}\right\}$$

- **new parameterization** $uv := \cosh\lambda \quad \rightarrow \quad s(\lambda) = -1 + e^{\frac{5\,\lambda}{2}}$

$$x(\lambda) = \left(e^{\frac{3\,\lambda}{2}}\,T_o\,,\ 3\,\left(-1 + e^{\frac{\lambda}{2}}\right)\,T_o{}^{\frac{1}{3}}\right), \qquad v(\lambda) = \{\cosh(\lambda)\,,\ -\frac{\sinh(\lambda)}{e^{\lambda}\,T_o{}^{\frac{2}{3}}}\}$$