Modified Skellam, Poisson and Gaussian distributions in semi-open systems at charge-like conservation law

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Poisson, Gaussian and Skellam distributions

Poisson distribution in statistics:

S.-D. Poisson, Poisson's Sorbonne lectures on probability and decision theory, Paris, 1837

$$P(N_i, n_i) = \frac{N_i^{n_i}}{n_i!}\exp(-N_i)$$

Gaussian distribution:

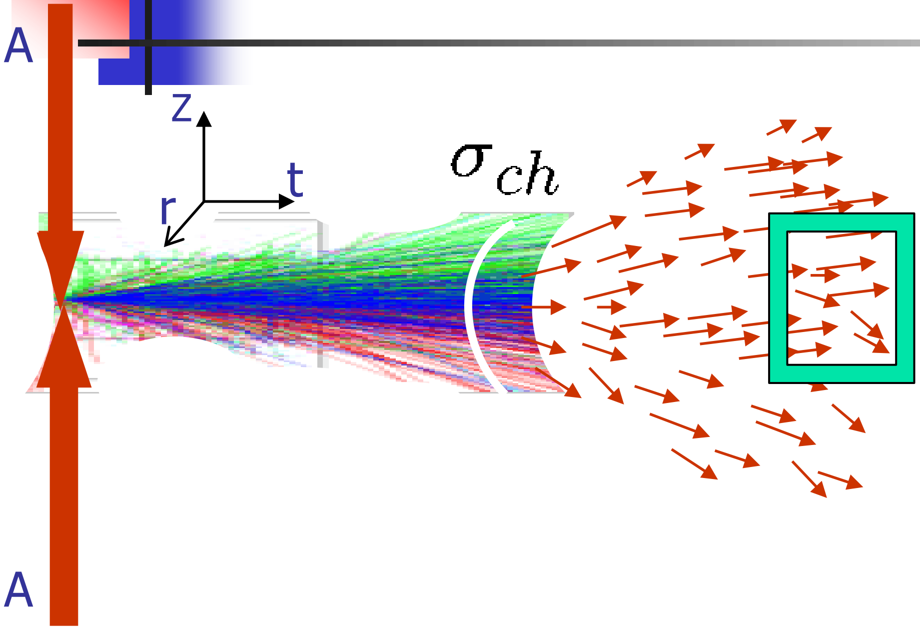
$$N \gg 1,\ \delta \ll 1, x = n = N(1+\delta)$$
$$x! \to \sqrt{2\pi x}\, e^{-x} x^x \quad \text{as} \quad x \to \infty$$
$$\Rightarrow P(N,n) \to \frac{1}{\sqrt{2\pi N}} e^{\frac{(x-N)^2}{2N}}$$

1840

Skellam, J. G. (1946). "The Frequency Distribution of the Difference Between Two Poisson Variates Belonging to Different Populations". Journal of the Royal Statistical Society. ***109*** *(3): 296*

$$p(k) = \sum_{\overline{n}}^{\infty} P(N, \overline{n}+k) P(\overline{N}, \overline{n}) \quad \Rightarrow \quad p(k; N, \overline{N}) = \exp(-N - \overline{N}) \left(\frac{N}{\overline{N}}\right)^{k/2} I_k(2\sqrt{N\overline{N}})$$

Statement of the problem



DETECTOR $\frac{N_D+\overline{N}_D}{N+\overline{N}} < 1$

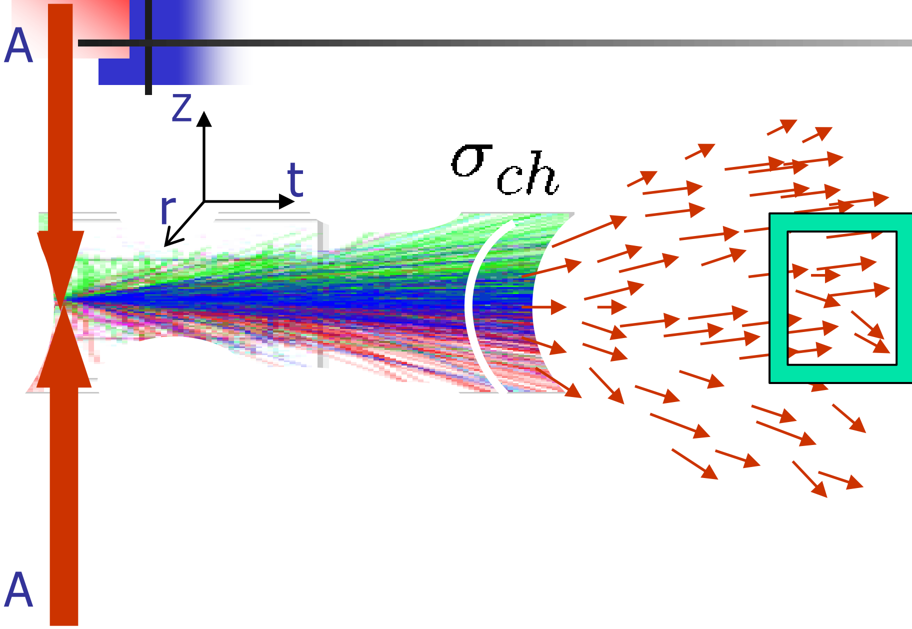
Net Baryon number

$$B = N - \overline{N}$$

of the particles that have undergone interactions in ultra-relativistic nucleus-nucleus collisions at RHIC and LHC is around 400.

$N + \overline{N} =$ several thousands

Statement of the problem



DETECTOR $\frac{N_D+\overline{N}_D}{N+\overline{N}} < 1$

If

$$N_D + \overline{N}_D \ll N, \overline{N}$$

$$p(k; N_D, \overline{N}_D) = \exp(-N_D - \overline{N}_D)\left(\frac{N_D}{\overline{N}_D}\right)^{k/2} I_k(2\sqrt{N_D \overline{N}_D})$$

Net Baryon number

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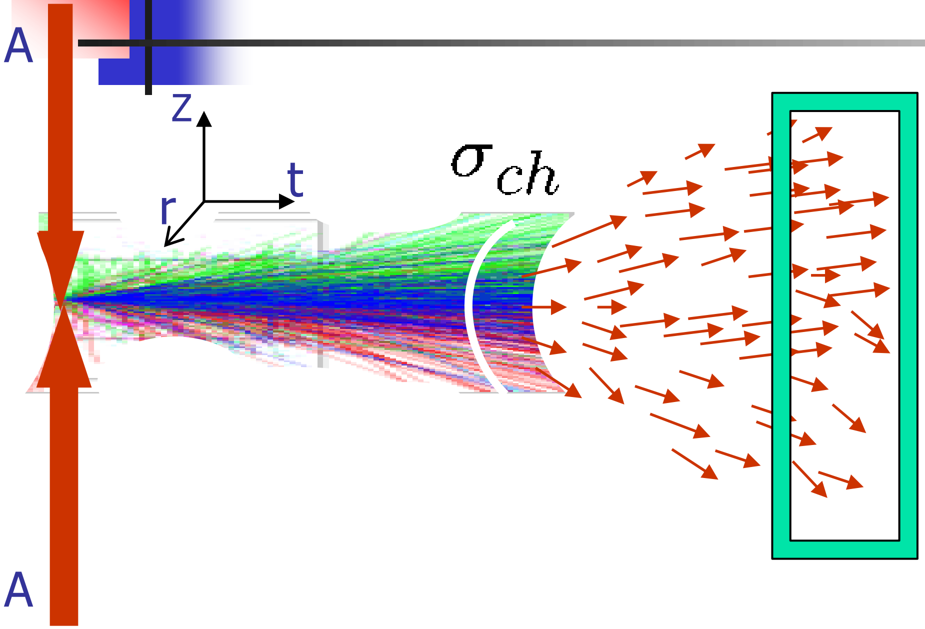


DETECTOR

$$\frac{N_D + \overline{N}_D}{N + \overline{N}} < 1$$

If

$$N_D + \overline{N}_D \ll N, \overline{N}$$

$$p(k; N_D, \overline{N}_D) = \exp(-N_D - \overline{N}_D) \left(\frac{N_D}{\overline{N}_D} \right)^{k/2} I_k(2\sqrt{N_D \overline{N}_D})$$

If

$$N_D + \overline{N}_D = N + \overline{N}$$

$$p(k; N_D, \overline{N}_D) = \delta^{Q}_{N_D - \overline{N}_D}$$

Net Baryon number

$$B = N - \overline{N}$$

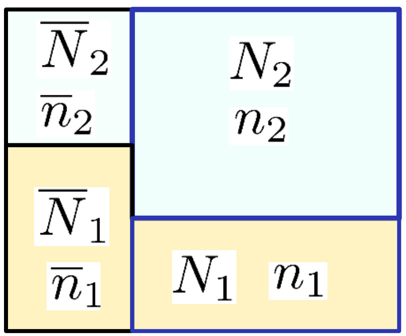
of the particles that have undergone interactions in ultra-relativistic nucleus-nucleus collisions at RHIC and LHC is around 400.

$N + \overline{N} =$ several thousands

Charge conservation law!

Possible models

$$p(k_i; N_i, \overline{N}_i) = \sum_{n_j, \overline{n}_l} \mathcal{F}(\{n_j, \bar{n}_l\}) \delta^{k_i}_{n_i - \overline{n}_i}$$



$$\mathcal{F} =$$

$$P(N_1, n_1) P(N_2, n_2) P(\bar{N}_1, \bar{n}_1) P(\bar{N}_2, \bar{n}_2) \delta^B_{n - \overline{n}}$$,

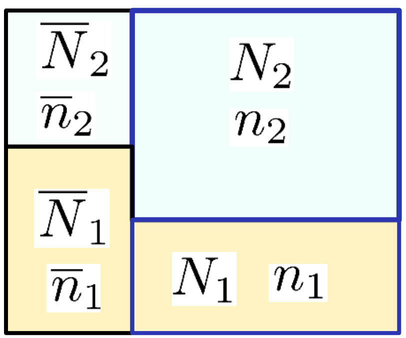
$$P(\sum N_i, \sum n_i) P(\bar{N}_1, \bar{n}_1) P(\bar{N}_2, \bar{n}_2) \delta^B_{n - \overline{n}}$$,

$$P(N_1, n_1) P(N_2, n_2) P(\sum \bar{N}_i, \sum \bar{n}_i) \delta^B_{n - \overline{n}}$$, etc.

M.I. Gorenstein, et al

Possible models

$$p(k_i; N_i, \overline{N}_i) = \sum_{n_j, \overline{n}_l} \mathcal{F}(\{n_j, \bar{n}_l\}) \delta^{k_i}_{n_i - \overline{n}_i}$$



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$$P(N_1, n_1)P(N_2, n_2)P(\bar{N}_1, \bar{n}_1)P(\bar{N}_2, \bar{n}_2)\delta^B_{n-\overline{n}} \quad ,$$

$$P(\sum N_i, \sum n_i)P(\bar{N}_1, \bar{n}_1)P(\bar{N}_2, \bar{n}_2)\delta^B_{n-\overline{n}} \quad ,$$

$$P(N_1, n_1)P(N_2, n_2)P(\sum \bar{N}_i, \sum \bar{n}_i)\delta^B_{n-\overline{n}}$$, etc.

Binomial-based approach

V. Koch, et al
A. Rustamov, PBM, et al

$$\mathcal{F}(\{n_j, \bar{n}_l\}) =$$

$$C \sum_{n, \overline{n}} \delta^B_{n-\overline{n}} \delta^n_{n_1+n_2} \delta^{\overline{n}}_{\overline{n}_1+\overline{n}_2} P(D, n) P(\overline{D}, \overline{n})$$

$$\times\ q^{n_1}(1-q)^{n_2}\overline{q}^{\overline{n}_1}(1-\overline{q})^{\overline{n}_2} \frac{n!}{n_1!\ n_2!} \frac{\overline{n}!}{\overline{n}_1!\ \overline{n}_2!}$$

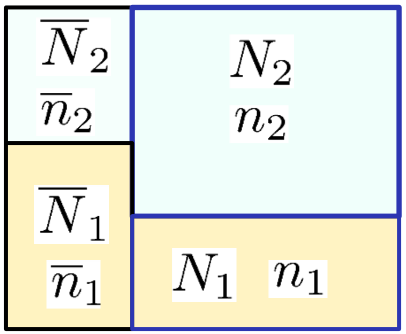
→

$$N_i = f_i(D_1, D_2, q, \overline{q})$$

$$\overline{N}_i = \overline{f}_i(D, \overline{D}, q, \overline{q})$$

Property → $q = \frac{N_1}{N_1 + N_2}$, $\overline{q} = \frac{\overline{N}_1}{\overline{N}_1 + \overline{N}_2}$

Simple alternative approach



$$B = n_1 + n_2 - \overline{n}_1 - \overline{n}_2 = k_1 + k_2 = N_1 + N_2 - \overline{N}_1 - \overline{N}_2$$

For each of the two subsystems $i = 1, 2$

$$p(k_i; N_i, \overline{N}_i) \to \widetilde{p}(k_i; N_i, \overline{N}_i) = e^{-(M_i + \overline{M}_i)} \left(\frac{M_i}{\overline{M}_i} \right)^{(k_i - k_i^0)/2} I_{k_i - k_i^0} (2\sqrt{M_i \overline{M}_i}\ )$$

Where one should find $M_i, \overline{M}_i,\ k_i^0$ so that to satisfy the following obvious conditions:

(C1) $\sum_{k_i=-\infty}^{\infty} \widetilde{p}(k_i; N_i, \overline{N}_i) = 1$ ⟹ k_i^0 is integer.

$$k_i \to q_i = k_i - k_i^0$$

Conditions

(C2) Mean value $m_i = \sum_{k_i=-\infty}^{\infty} k_i \, \widetilde{p}(k_i; N_i, \overline{N}_i) = N_i - \overline{N}_1$

⇩

$k_i^0 = N_i + \overline{G}_i - \overline{N}_i - G_i$ Replacement of notation: $M_i, \overline{M}_i \to G_i, \overline{G}_i$.

(C3)

Because of the symmetry of the conservation law constraint B) with respect to the mutual permutations $N_1 \leftrightarrow N_2$, $\overline{N}_1 \leftrightarrow \overline{N}_2$ and $k_1 \leftrightarrow k_2$, the analytical expressions for modified Skellam distributions for the two subsystems transform to each other $\widetilde{p}(k_1; N_1, \overline{N}_1) \leftrightarrow \widetilde{p}(k_2; N_2, \overline{N}_2)$ under these permutations.

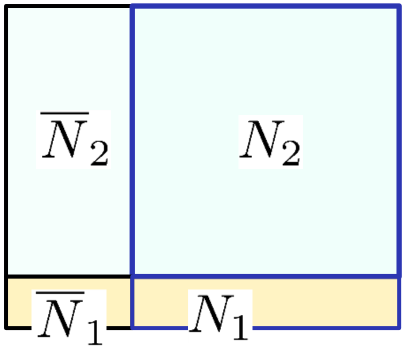
$B =$
$k_1 + k_2 = N_1 + N_2 - \overline{N}_1 - \overline{N}_2$

⇩

$$G_1 = \overline{G}_2, \; G_2 = \overline{G}_1$$

Conditions in limiting cases

(C4)



If one of the subsystems (say "1") is much smaller then the other one, $N_1 + \overline{N}_1 \ll N_2, \overline{N}_2$, then fluctuations of the two components, baryon and anti-baryon, in this smaller subsystem are uncorrelated Poisson ones (the second subsystem just plays the role of an "thermal bath").

$$\widetilde{p}(k_1; N_1, \overline{N}_1) \to p(k_1; N_1, \overline{N}_1), \; N_1 + \overline{N}_1 \ll N_2, \overline{N}_2$$

(C5)

When one of the subsystems (say "1") vanishes, $N_1, \overline{N}_1 \to 0$, the subsystem "2" occupies, in fact, the total system and then, according to the net baryon charge conservation law,

$$\widetilde{p}(k_2; N_2, \overline{N}_2) \to \delta^{k_2}_{N_2 - \overline{N}_2},$$

$$N_2 \to N, \; \overline{N}_2 \to \overline{N}.$$

(C4) + (C5) ⟹

$$G_1 = \overline{G}_2 = \frac{N_1 \overline{N}_2}{N_1 + \overline{N}_2}, \; G_2 = \overline{G}_1 = \frac{N_2 \overline{N}_1}{N_2 + \overline{N}_1}$$

$$k_i^0 = N_i + \overline{G}_i - \overline{N}_i - G_i$$

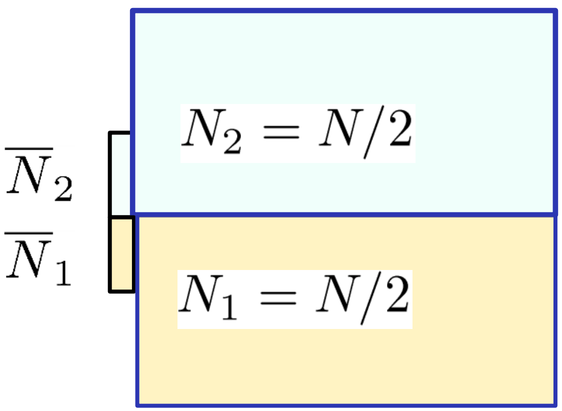
⟹

$$M_i = G_i + Q(N_i, \overline{N}_i)$$

$$\overline{M}_i = \overline{G}_i + Q(N_i, \overline{N}_i)$$

Conditions in limiting cases

(C6)



$$Q = \frac{\left|k_1^0 k_2^0\right|}{2(N + \overline{N})}$$

One more restricting condition appears if the total system is only one-component, e.g. when $\overline{N}_i \to 0$ for both $i = 1, 2$. Then the fluctuations in k_i inside selected subsystem arise only because of the fluctuations of the baryons between the subsystems "1" and "2". It is obviously, that when $N_1 = N_2$, the fluctuations in any single subsystem will be twice suppressed because of the conservation law: fluctuation in the single subsystem enforce the double fluctuation in the total system. The width of fluctuations distributed according (8) in equal subsystems $\frac{N}{2}$ is defined, similar to Skellam distribution, by dispersion $\sigma = \sqrt{M_1 + M_2}$ and for independent Poisson subsystems when $M_i \to N/2$ is $\sigma_{\text{ind}} = \sqrt{N}$. So, when the conservation law constraint is imposed, $M_i = N/8$, then $\sigma = \sigma_{\text{ind}}/2$.

On the other hand, if $N_i \to N$, it must be: $\tilde{p} \to \delta^{B}_{k_i - n_i}$.

The modified Skellam distribution

$$\widetilde{p}(k_i; N_i, \overline{N}_i) = e^{-(M_i+\overline{M}_i)} \left(\frac{M_i}{\overline{M}_i}\right)^{(k_i-k_i^0)/2} I_{k_i-k_i^0}(2\sqrt{M_i\overline{M}_i}\ )$$

where

$$k_i^0 = N_i + \overline{G}_i - \overline{N}_i - G_i \qquad G_1 = \overline{G}_2 = \frac{N_1\overline{N}_2}{N_1+\overline{N}_2},\ G_2 = \overline{G}_1 = \frac{N_2\overline{N}_1}{N_2+\overline{N}_1}$$

$$M_i = G_i + \frac{|k_1^0 k_2^0|}{2(N+\overline{N})},\ \overline{M}_i = \overline{G}_i + \frac{|k_1^0 k_2^0|}{2(N+\overline{N})}$$

The modified Skellam distribution

$$\widetilde{p}(k_i; N_i, \overline{N}_i) = e^{-(M_i+\overline{M}_i)} \left(\frac{M_i}{\overline{M}_i}\right)^{(k_i-k_i^0)/2} I_{k_i-k_i^0}(2\sqrt{M_i\overline{M}_i}\ )$$

where

$$k_i^0 = N_i + \overline{G}_i - \overline{N}_i - G_i \qquad G_1 = \overline{G}_2 = \frac{N_1\overline{N}_2}{N_1+\overline{N}_2},\ G_2 = \overline{G}_1 = \frac{N_2\overline{N}_1}{N_2+\overline{N}_1}$$

$$M_i = G_i + \frac{|k_1^0 k_2^0|}{2(N+\overline{N})},\ \overline{M}_i = \overline{G}_i + \frac{|k_1^0 k_2^0|}{2(N+\overline{N})}$$

Mean value

$$m_i = \sum_{k_i=-\infty}^{\infty} k_i\ \widetilde{p}(k_i; N_i, \overline{N}_i) = N_i - \overline{N}_i$$

Variance

$$\sigma_i^2 = \sum_{k=-\infty}^{\infty} (k-m_i)^2 \widetilde{p}(k; N_i, \overline{N}_i) = (M_i + \overline{M}_i)$$

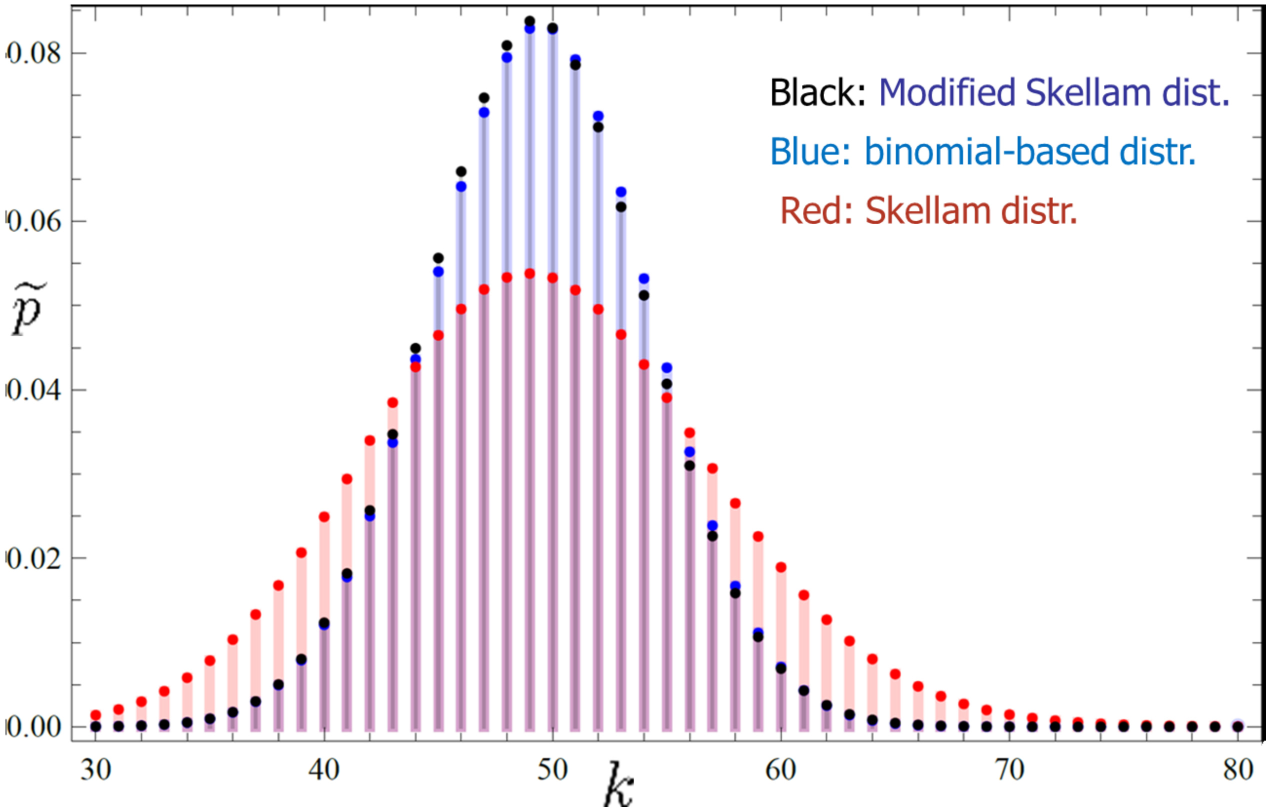
Skewness

$$S_i = \sum_{k=-\infty}^{\infty} \left(\frac{k-m_i}{\sigma}\right)^3 \widetilde{p}(k; N_i, \overline{N}_i) = \frac{M_i - \overline{M}_i}{(M_i+\overline{M}_i)^{3/2}}$$

Kurtosis

$$K_i = \sum_{k=-\infty}^{\infty} \left(\frac{k-m_i}{\sigma}\right)^4 \widetilde{p}(k; N_i, \overline{N}_i) - 3 = \frac{1}{M_i + \overline{M}_i}$$

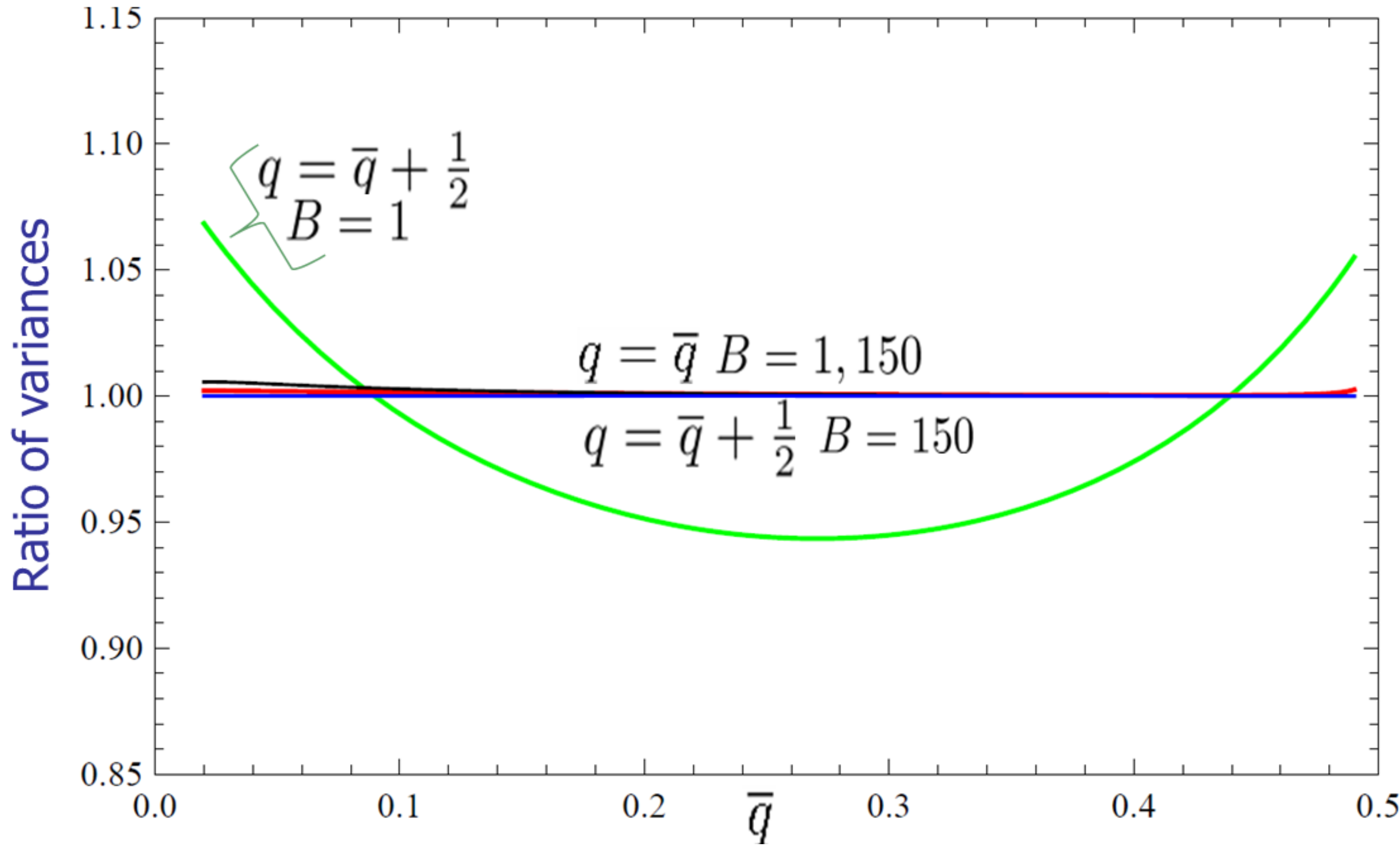
The Mod-Skellam, Binomial-based and Skellam distributions



A comparision of the modified Skellam distribution with the Skellam-like one based on the Poissonian-binomial distribution, and also with the original Skellam distribution.

Variance ratios for Binomial-based over Mod-Skellam distr.

$$\frac{\sigma^2_{bin}}{\sigma^2_{Mod-Skellam}}$$

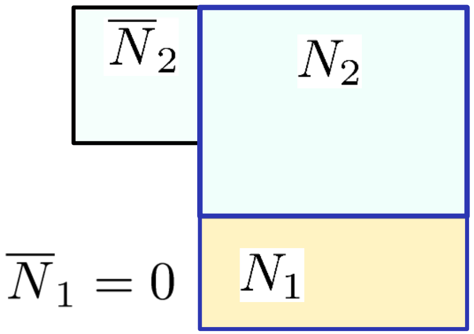


$q = \frac{N_1}{N_1+N_2}$, $\overline{q} = \frac{\overline{N}_1}{\overline{N}_1+\overline{N}_2}$

If $q = \overline{q}$ ⟹ $\frac{\sigma^2}{\sigma^2_{Skellam}} = 1 - q$ A. Rustamov, PBM, J. Stachel (2017)

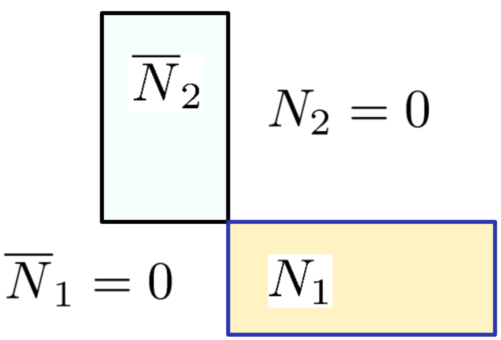
Agreement within < 0.5%

Modification of Poisson Distribution



$$k_1 = n_1 - \overline{n}_1 = n_1$$

$$\widetilde{P}_{1+2} = \widetilde{p}(n_1; N_i, \overline{N}_2)$$ at $$\begin{cases} M_1 = \overline{M}_2 = \frac{N_1 \overline{N}_2}{N_1 + \overline{N}_2} + \frac{|k_1^0 k_2^0|}{2(N + \overline{N}_2)} \\ M_2 = \overline{M}_1 = Q = \frac{|k_1^0 k_2^0|}{2(N + \overline{N}_2)} \end{cases}$$



$N_2 = 0$

$$\overline{N}_2 = \overline{N} \qquad M_1 \equiv M = \frac{N\overline{N}}{N + \overline{N}}$$

$$N_1 = N \qquad M_2 = 0 \ , \ k_1^0 \equiv k_0 = N - M$$

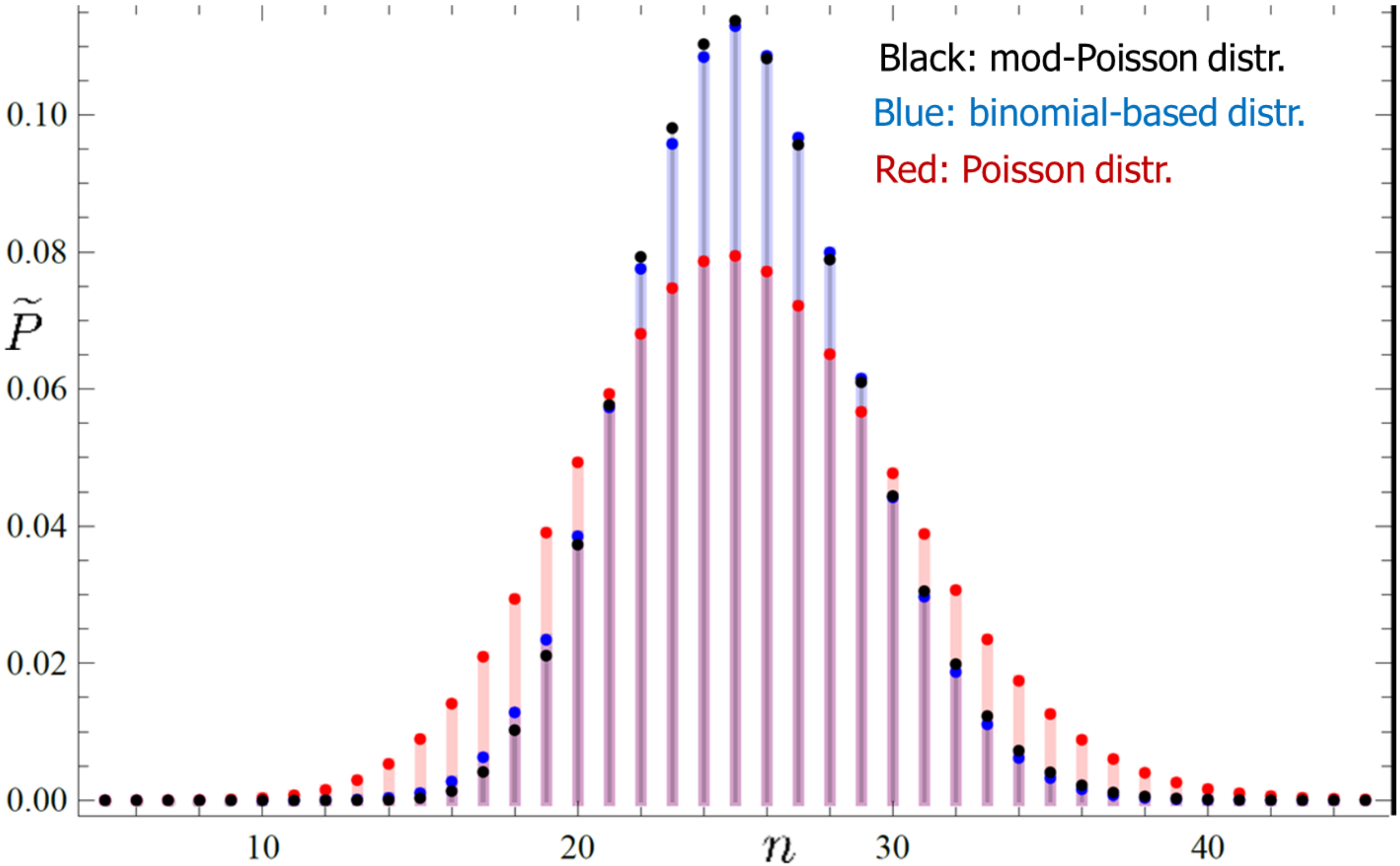
$$Q \le M_1/8$$

$\downarrow$

0

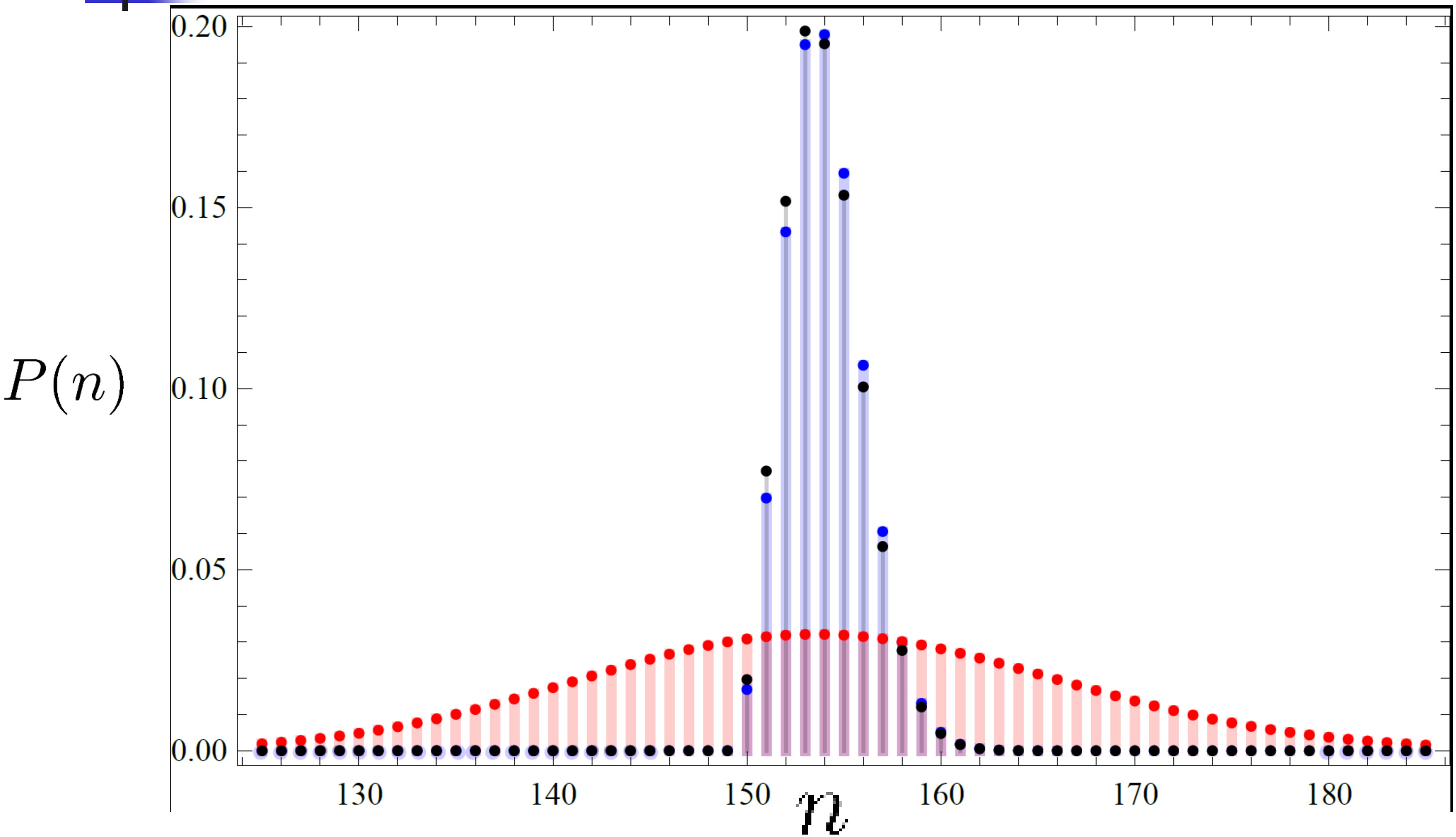
$$\widetilde{P}(n; N) = \frac{e^{-M} M^{(n-k_0)}}{(n-k_0)!}$$

Binomial-based, Mod-Poisson and Poisson Distr.



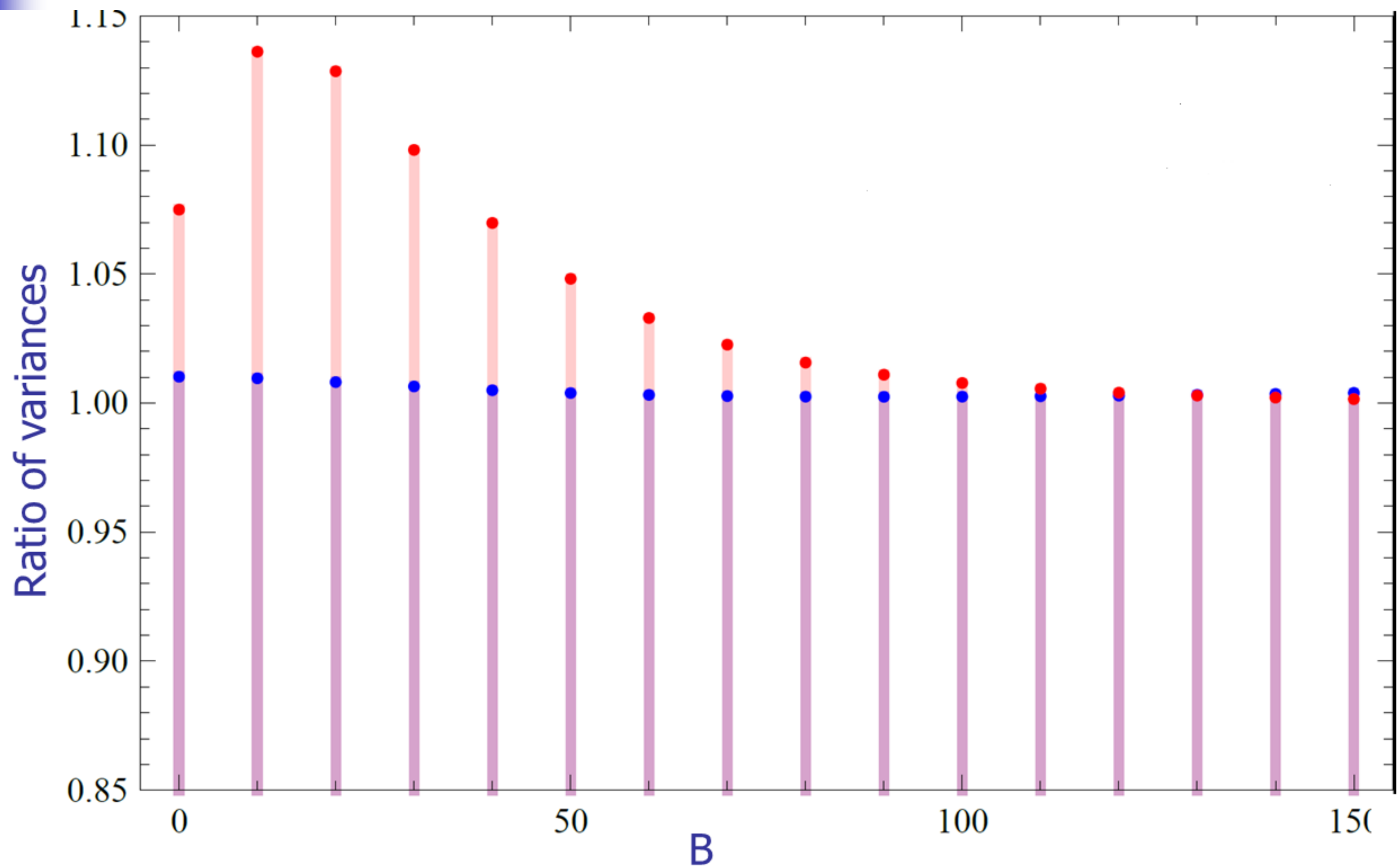
$$B = 1$$

Binomial-based, Mod-Poisson and Poisson Distr.



$B = 150$

Variance ratios for Binomial-based over Mod-Poisson distr.



Blue $\widetilde{P}(N, n)$ Red $\widetilde{P}_{1+2} = \widetilde{p}(n; N_i, \overline{N}_2)$

Summary

- Modified Skellam $$\tilde{p}(k_i; N_i, \overline{N}_i) = e^{-(M_i+\overline{M}_i)} \left(\frac{M_i}{\overline{M}_i}\right)^{(k_i-k_i^0)/2} I_{k_i-k_i^0}(2\sqrt{M_i\overline{M}_i}\,)$$

  where $M_i, \overline{M}_i, k_i^0$ are the simple rational expressions of $N_j, \overline{N}_l$

- Modified Poisson $$\widetilde{P}(n;N) = \frac{e^{-M} M^{(n-k_0)}}{(n-k_0)!}$$

  where $M = \frac{N\overline{N}}{N+\overline{N}}$, $k_0 = N - M$

- Modified Gaussian $$\widetilde{P}(x, N) = \frac{1}{\sqrt{2\pi M}} e^{\frac{(x-N)^2}{2M}}$$

Yu.M. Sinyukov, arXiv:1805.03884,
11 May 2018