Model-independent analysis of nearly Lévy source

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OUTLINE

Model-independent shape analysis:

- General introduction
- Edgeworth, Laguerre
- Lévy expansions
- Applications

Summary

MODEL - INDEPENDENT SHAPE ANALYIS I.

Model-independent method, proposed to analyze Bose-Einstein correlations IF experimental data satisfy

- The measured data *tend to a constant* for large values of the observable Q.
- There is a *non-trivial structure* at some definite value of Q, shift it to Q = 0.

Model-independent, but experimentally testable:

- t = Q R
- dimensionless scaling variable
- approximate form of the correlations *w(t)*
- *Identify w*(*t*) with a measure in an abstract Hilbert-space

$$\int dt w(t) h_n(t) h_m(t) = \delta_{n,m},$$

$$f(t) = \sum_{n=0}^{\infty} f_n \, h_n(t),$$

$$f_n = \int dt w(t) \, f(t) \, h_n(t).$$

e.g. $t = Q_I R_I$.

T. Csörgő and S: Hegyi, hep-ph/9912220, T. Csörgő, hep-ph/001233

MODEL - INDEPENDENT SHAPE ANALYIS II.

$$C_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{N_2(\mathbf{k}_1,\mathbf{k}_2)}{N_1(\mathbf{k}_1)\,N_1(\mathbf{k}_2)},$$

$$R_2(\mathbf{k}_1,\mathbf{k}_2) = C_2(\mathbf{k}_1,\mathbf{k}_2) - 1.$$

Let us assume, that the function $g(t) = R_2(t)/w(t)$ is also an element of the Hilbert space H. This is possible, if

$$\int dt\, w(t) g^2(t) = \int dt\, \left[R_2^2(t)/w(t)\right] < \infty, \tag{6}$$

Then the function *g* can be expanded as

$$g(t) = \sum_{n=0}^{\infty} g_n h_n(t),$$

$$g_n = \int dt\, R_2(t) h_n(t).$$

From the completeness of the Hilbert space, if *g*(*t*) is also in the Hilbert space:

$$R_2(t) = w(t) \sum_{n=0}^{\infty} g_n h_n(t).$$

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MODEL - INDEPENDENT SHAPE ANALYIS III.

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)},$$

$$C_2(t) = \mathcal{N}\left\{1 + \lambda_w\, w(t) \sum_{n=0}^{\infty} g_n h_n(t)\right\}.$$

Model-independent AND experimentally testable:

- method for any approximate shape $w(t)$
- the core-halo intercept parameter of the CF is $\lambda_* = \lambda_w \sum_{n=0}^{\infty} g_n h_n(0)$
- coefficients by numerical integration (fits to data) $g_n = \int dt\, R_2(t) h_n(t)$
- condition for applicability: experimentally $\int dt\, \left[R_2^2(t)/w(t)\right] < \infty$

  testable

GAUSSIAN w(t): EDGEWORTH EXPANSION

$$t = \sqrt{2} Q R_E,$$
$$w(t) = \exp(-t^2/2),$$
$$\int_{-\infty}^{\infty} dt \, \exp(-t^2/2) H_n(t) H_m(t) \propto \delta_{n,m},$$

$$H_n(t) = \exp(t^2/2) \left(-\frac{d}{dt}\right)^n \exp(-t^2/2).$$

$$H_1(t) = t,$$
$$H_2(t) = t^2 - 1,$$
$$H_3(t) = t^3 - 3t,$$
$$H_4(t) = t^4 - 6t^2 + 3, \; ...$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_E \exp(-Q^2 R_E^2) \times \left[1 + \frac{\kappa_3}{3!} H_3(\sqrt{2} Q R_E) + \frac{\kappa_4}{4!} H_4(\sqrt{2} Q R_E) + ... \right] \right\}.$$

3d generalization straightforward

- Applied by NA22, L3, STAR, PHENIX, ALICE, CMS (LHCb)

EXPONENTIAL w(t): LAGUERRE EXPANSIONS

Model-independent but experimentally tested:

- $w(t)$ exponential
- t: dimensionless
- Laguerre polynomials

$$t = QR_L,$$
$$w(t) = \exp(-t)$$
$$\int_0^\infty dt\, \exp(-t) L_n(t) L_m(t) \propto \delta_{n,m},$$

$$L_n(t) = \exp(t) \frac{d^n}{dt^n} (-t)^n \exp(-t).$$

$$L_0(t) = 1,$$
$$L_1(t) = t - 1,$$

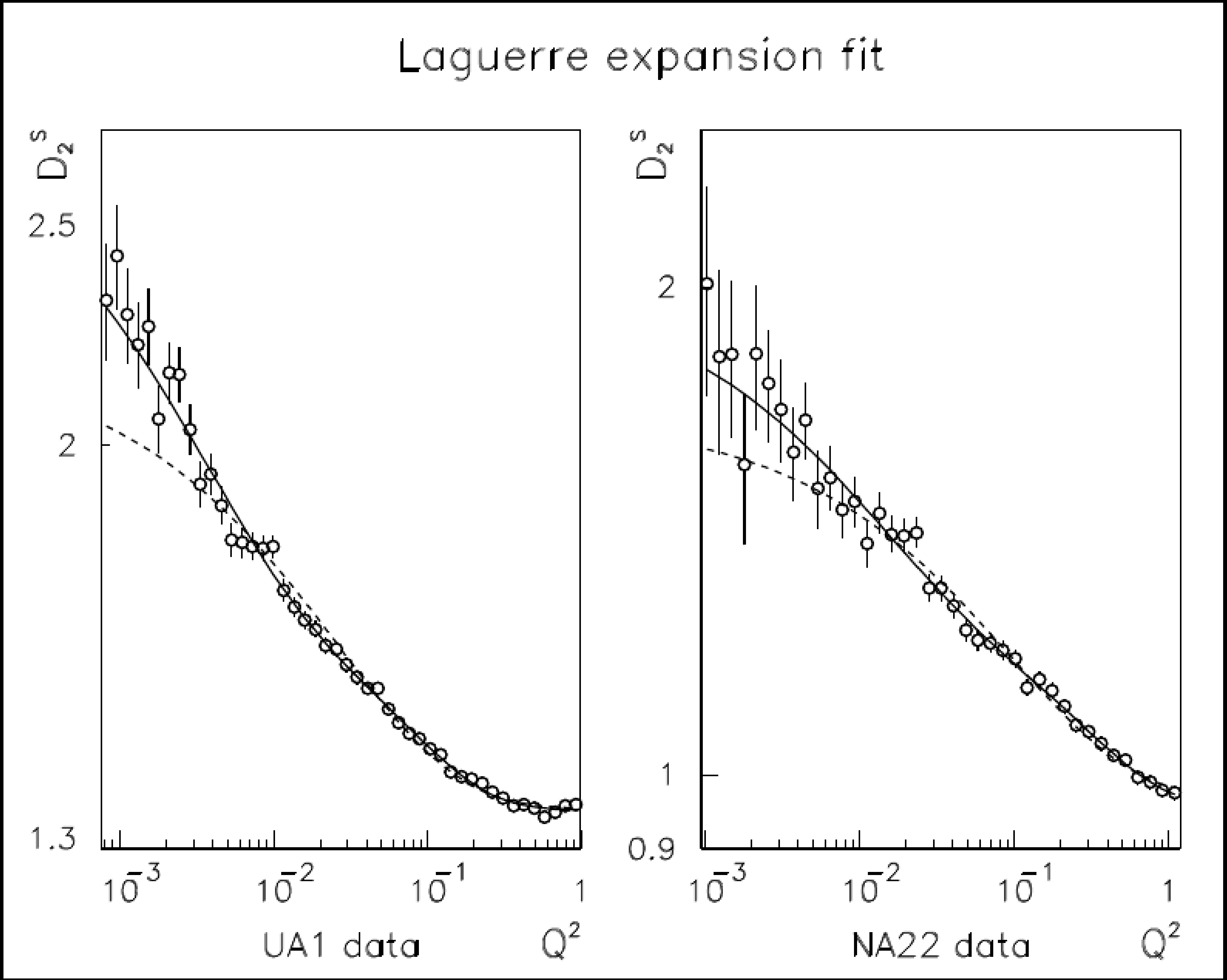
$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_L \exp(-QR_L) \left[1 + c_1 L_1(QR_L) + \frac{c_2}{2!} L_2(QR_L) + \ldots \right] \right\}$$

First successful tests

- NA22, UA1 data
- convergence criteria satisfied

$$\int_0^\infty dt\, R_2^2(t) \exp(+t) < \infty,$$

EXAMPLE, LAGUERRE EXPANSIONS



T. Csörgő and S: Hegyi, hep-ph/9912220, T. Csörgő, hep-ph/001233

STRETCHED w(t): LÉVY EXPANSIONS

$$w(t|\alpha) = \exp(-t^\alpha) = \exp(-Q^\alpha R^\alpha)$$

Model-independent but:

- Lévy: stretched exponential
- generalizes exponentials and Gaussians
- ubiquitous in nature
- How far from a Lévy?
- Need new set of polynomials orthonormal to a Lévy weight

$$L_0(t\,|\,\alpha) = \mu_{0,\alpha},$$

$$L_1(t\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & t \end{pmatrix},$$

$$L_2(t\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & t & t^2 \end{pmatrix},$$

$$L_3(t\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} & \mu_{4,\alpha} \\ \mu_{2,\alpha} & \mu_{3,\alpha} & \mu_{4,\alpha} & \mu_{5,\alpha} \\ 1 & t & t^2 & t^3 \end{pmatrix},$$

$$\mu_{r,\alpha} = \int_0^\infty dx\ x^r f(x\,|\,\alpha) = \frac{1}{\alpha}\Gamma\left(\frac{r+1}{\alpha}\right)$$

STRETCHED w(t) = exp(-t$^{\alpha}$): LÉVY EXPANSIONS

In case of α = 1,
in 1 dimension Laguerre expansion is recovered

$$
\begin{aligned}
L_0(t\,|\,\alpha=1) &= 1, \\
L_1(t\,|\,\alpha=1) &= t-1, \\
L_2(t\,|\,\alpha=1) &= t^2-4t+2, \\
L_3(t\,|\,\alpha=1) &= 4t^3-36t^2+72t-24...
\end{aligned}
$$

These reduce to the Laguerre expansions and Laguerre polynomials.

STRETCHED w(t)=exp(-t$^\alpha$): LEVY EXPANSIONS

In case of α = 2 ,
a new formulae for one-sided Gaussians:

$$
\begin{aligned}
L_0(t\,|\,\alpha=2) &= \frac{\sqrt{\pi}}{2}, \\
L_1(t\,|\,\alpha=2) &= t\frac{\sqrt{\pi}}{2} - \frac{1}{2}, \\
L_2(t\,|\,\alpha=2) &= t^2\left[\frac{\pi}{8} - \frac{1}{4}\right] - t\frac{\sqrt{\pi}}{8} - \frac{\pi}{16} + \frac{1}{4}, \\
L_3(t\,|\,\alpha=2) &= t^3\left[\frac{\pi^{3/2}}{32} - \frac{3\sqrt{\pi}}{32}\right] + t^2\left[\frac{1}{8} - \frac{3\pi}{64}\right] + t\left[\frac{5\sqrt{\pi}}{32} - \frac{3\pi^{3/2}}{64}\right] + \frac{5\pi}{128} - \frac{1}{8}
\end{aligned}
$$

Provides a new expansion around a Gaussian shape that is defined for the non-negative values of *t* only.

Edgeworth expansion different, its around two-sided Gaussian, includes non-negative values of *t* also.

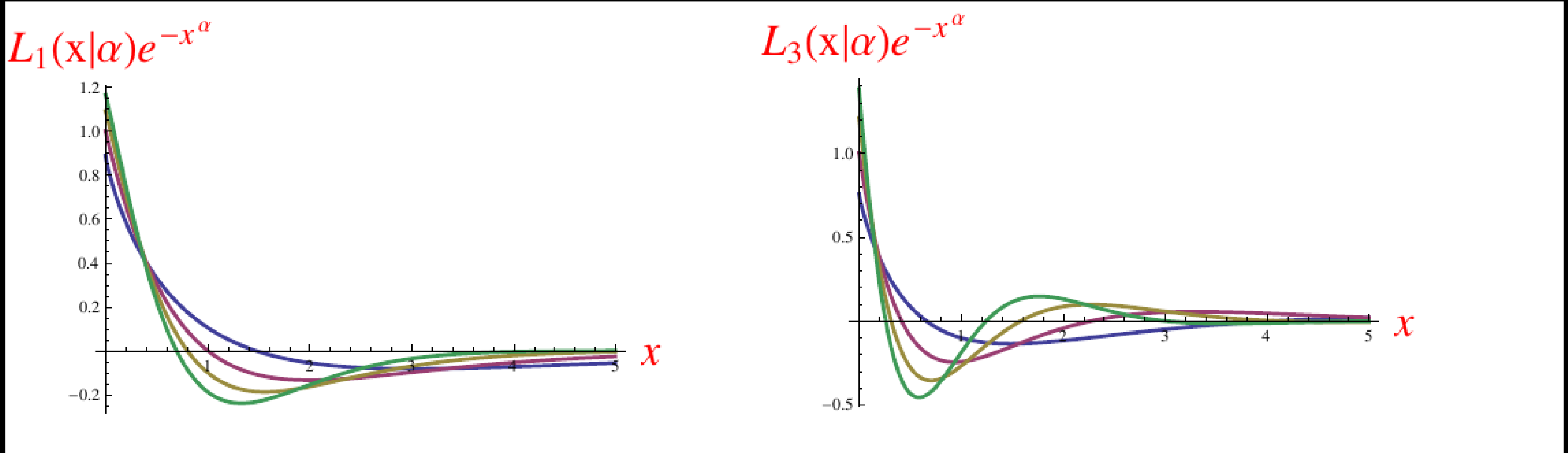
arXiv:1604.05513 [physics.data-an]

EXAMPLE, LÉVY EXPANSIONS

Model-independent but:

- Levy generalizes exponentials and Gaussians
- ubiquitous in nature
- How far from a Lévy?
- Not necessarily positive definit !

$$
\begin{aligned}
L_0(t\,|\,\alpha) &= \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right), \\
L_1(t\,|\,\alpha) &= \frac{1}{\alpha}\left\{\Gamma\left(\frac{1}{\alpha}\right)t - \Gamma\left(\frac{2}{\alpha}\right)\right\}, \\
L_2(t\,|\,\alpha) &= \frac{1}{\alpha^2}\left\{\left[\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{3}{\alpha}\right) - \Gamma^2\left(\frac{2}{\alpha}\right)\right]t^2 - \right. \\
&\quad - \left[\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{4}{\alpha}\right) - \Gamma\left(\frac{3}{\alpha}\right)\Gamma\left(\frac{2}{\alpha}\right)\right]t + \\
&\quad \left. + \left[\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\frac{4}{\alpha}\right) - \Gamma^2\left(\frac{3}{\alpha}\right)\right]\right\}.
\end{aligned}
$$

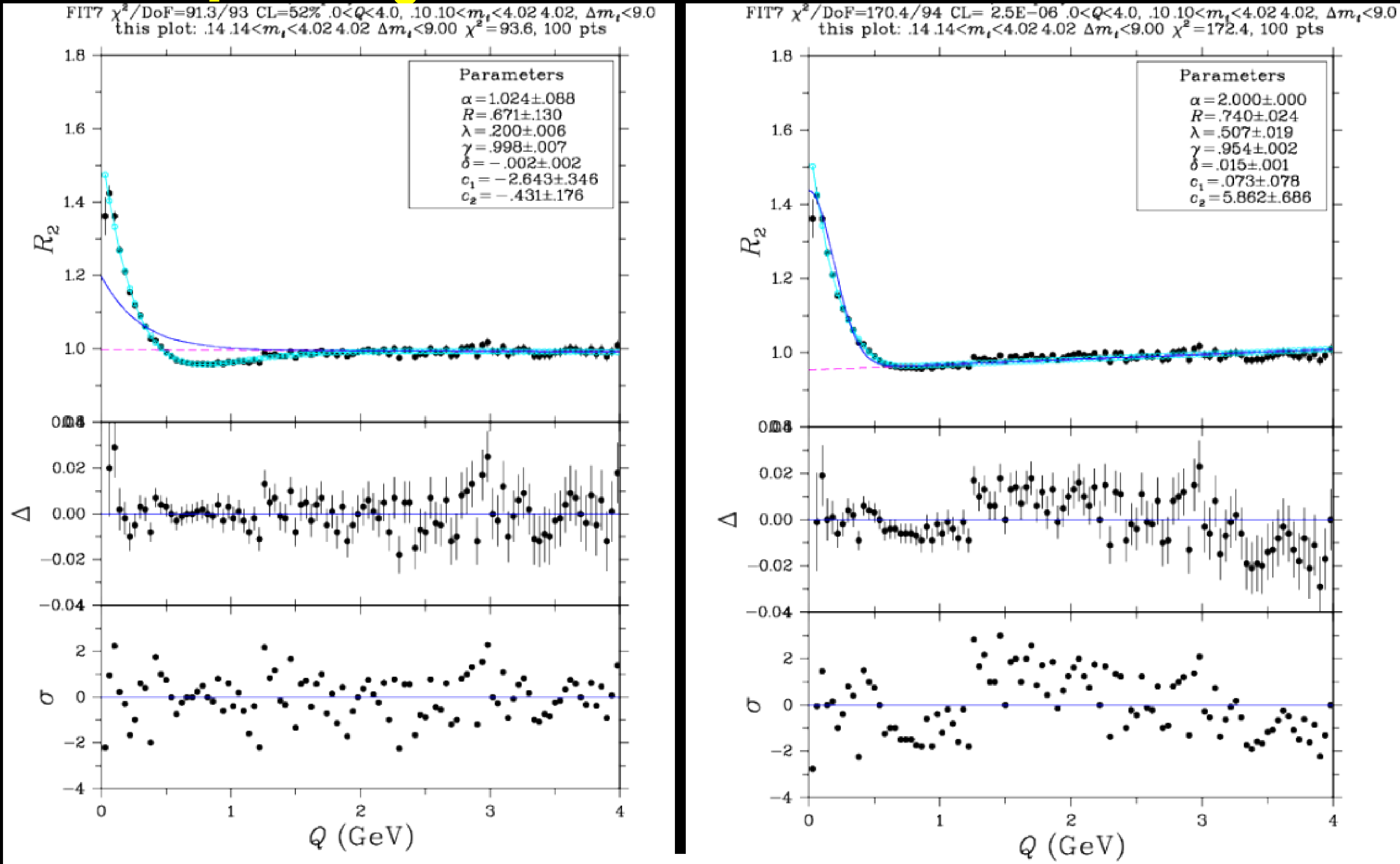


Lévy polynomials of first and third order times the weight function e^{-x^α} for $\alpha = 0.8, 1.0, 1.2, 1.4$.

1st-order Lévy polynomial	$\gamma\left[1+\lambda e^{-R^\alpha Q^\alpha}\left[1+c_1 L_1(Q\|\alpha,R)\right]\right]$
3rd-order Lévy polynomial	$\gamma\left[1+\lambda e^{-R^\alpha Q^\alpha}\left[1+c_1 L_1(Q\|\alpha,R)+c_3 L_3(Q\|\alpha,R)\right]\right]$

M. de Kock, H. C. Eggers, T. Cs: arXiv:1206.1680v1 [nucl-th]

Example, L_3 BE correlation function



$$C_2(t) = N\left\{1+\lambda\exp(-t^{\alpha})\left[1+\sum_{n=1}^{\infty}c_nL_n(t|\alpha)\right]\right\} \quad t = QR$$

HB *et al.* observed oscillations

Mon. Not. R. astr. Soc. (1974) **167**, 121–136.

THE ANGULAR DIAMETERS OF 32 STARS

R. Hanbury Brown, J. Davis and L. R. Allen

The normalized correlation also depends upon whether a star is single or multiple. It was shown in Paper II that, if a star is binary and the angular separation of the two components is completely resolved by the interferometer at the shortest baseline, then the normalized zero-baseline correlation $\overline{c_N(0)'}$ averaged over a range of position angles is reduced relative to a single star $\overline{c_N(0)}$ by the factor,

$$\overline{c_N(0)'}/\overline{c_N(0)} = (I_1{}^2+I_2{}^2)/(I_1+I_2)^2 \tag{9}$$

where I_1, I_2 are the brightness of the two components. It is simple to extend this analysis to a multiple star with n components and to show that, if the angular separation between all the components is resolved, the zero-baseline correlation is reduced relative to a single star by the factor,

$$\overline{c_N(0)'}/\overline{c_N(0)} = \sum_n I^2 \Big/ \Big(\sum_n I\Big)^2. \tag{10}$$

It follows that if a star yields a correlation which is significantly less than that expected from a single star, then it must be multiple.

LÉVY EXPANSIONS for POSITIVE DEFINIT FORMS

experimental conditions:
(i) The correlation function tends to a constant for large values of the relative momentum Q.
(ii) The correlation function deviates from its asymptotic, large Q value in a certain domain of its argument.

(iii) The two-particle correlation function is related to a Fourier transformed space-time distribution of the source.

Model-independent but:

- Assumes that Coulomb can be corrected
- No assumptions about analyticity yet
- For simplicity, consider 1d case first
- For simplicity, consider factorizable *x k*
- Normalizations :
  - density
  - multiplicity
  - single-particle spectra

$$C_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{N_2(\mathbf{k}_1,\mathbf{k}_2)}{N_1(\mathbf{k}_1)\,N_1(\mathbf{k}_2)}$$

$$S(x,k) = f(x)\,g(k)$$

$$\int \mathrm{d}x\, f(x) = 1, \qquad \int \mathrm{d}k\, g(k) = \langle n \rangle,$$

$$N_1(k) = \int \mathrm{d}x\, S(x,k) = g(k).$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MINIMAL MODEL ASSUMPTION: LÉVY

Model-independent but:

- not assumes analyticity
- C_2 measures a modulus squared Fourier-transform vs relative momentum
- Correlations non-Gaussian
- Radius not a variance
- $0 < \alpha \leq 2$

$$C_2(k_1, k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int \mathrm{d}x \, \exp(\mathrm{i} q_{12} x) \, f(x),$$

$$C(q;\alpha) = 1 + \lambda \exp\left(-|qR|^{\alpha}\right).$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

UNIVARIATE LEVY EXAMPLES

Include some well known cases:

- $\alpha = 2$
  - Gaussian source, Gaussian C_2

$$f(x) = \frac{1}{(2\pi R^2)^{1/2}} \exp\left[-\frac{(x-x_0)^2}{2R^2}\right]$$

$$C(q) = 1 + \exp\left(-q^2 R^2\right)$$

- $\alpha = 1$
  - Lorentzian source, exponential C_2

$$f(x) = \frac{1}{\pi} \frac{R}{R^2 + (x-x_0)^2},$$

$$C(q) = 1 + \exp\left(-|q\, R|\right).$$

- asymmetric Lévy:
  - asymmetric support
  - Streched exponential

$$f(x) = \sqrt{\frac{R}{8\pi}} \frac{1}{(x-x_0)^{3/2}} \exp\left(-\frac{R}{8(x-x_0)}\right)$$

$$x_0 < x < \infty,$$

$$C(q) = 1 + \exp\left(-\sqrt{|q\, R|}\right).$$

T. Cs, hep-ph/0001233, T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

Multivariate, nearly Lévy correlations

If the BE correlation function is $C_2(k_1, k_2) = 1 + |\tilde{f}(q_{12})|^2,$

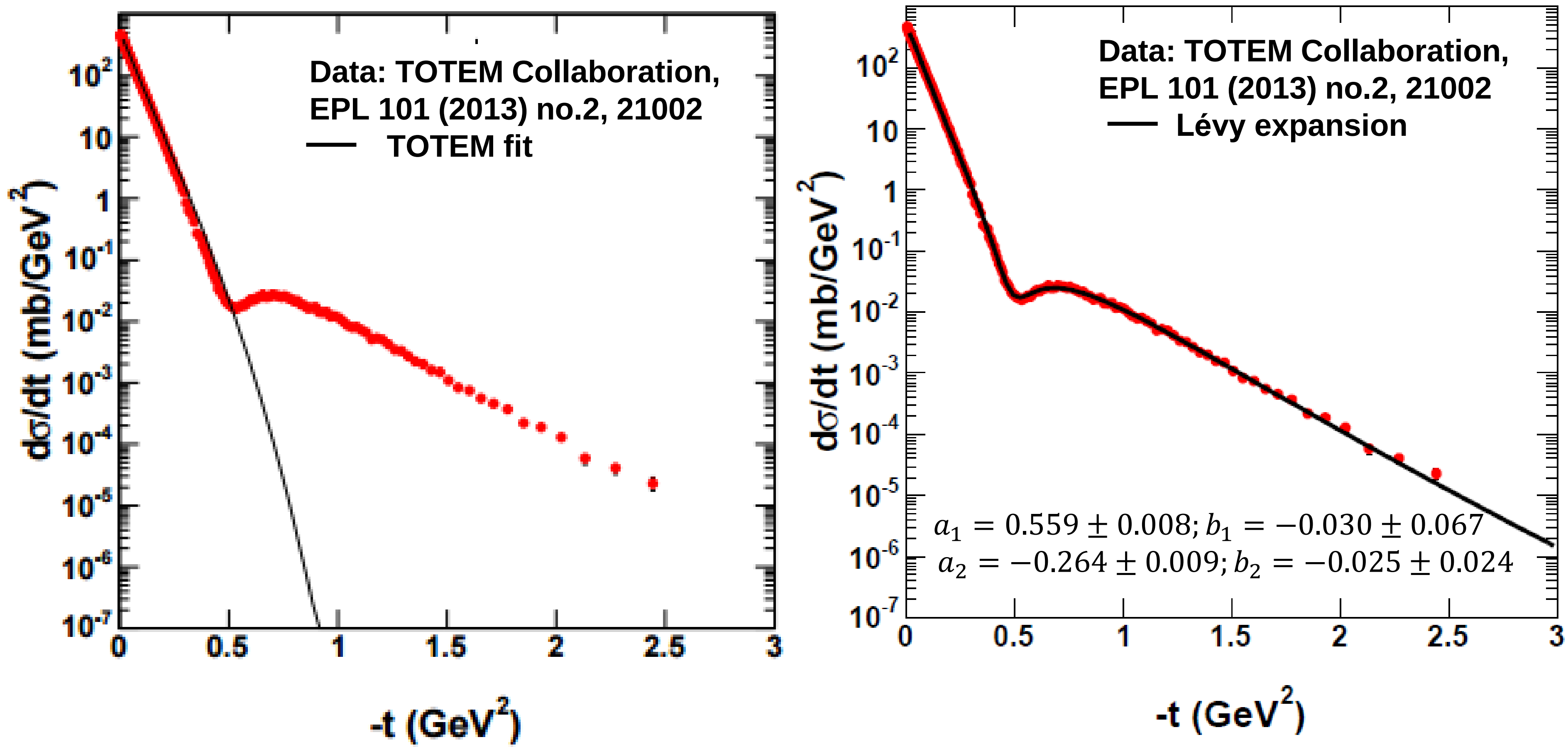
then

$$t = \left(\sum_{i,j=\mathrm{side,out,long}} R_{i,j}^2 q_i q_j \right)^{1/2},$$

$$C_2(t) = N \left\{ 1 + \lambda \exp(-t^\alpha) \left| 1 + \sum_{n=1}^{\infty} (a_n + i b_n) L_n(t|\alpha) \right|^2 \right\}$$

where $\{c_n = a_n + i b_n\}_{n=1}^{\infty}$ are now complex valued expansion coefficients,

Lévy expansion fit to pp elastic scattering at sqrt(s) = 7 TeV



Towards imaging

$$C(q) = 1 + \exp\left(-(qR)^{\alpha}\right) = 1 + |\tilde{f}(q)|$$

$$\tilde{f}(q) = \exp\left(-\frac{1}{2}(qR)^{\alpha}\right) \rightarrow \tilde{f}(q) = \exp\left(-\frac{1}{2}(qR)^{\alpha}\right)\left(\sum_j c_j L_j(q|\alpha)\right)$$

$$c_j = a_j + ib_j$$

$$\sum_j c_j L_j(q|\alpha) = \sum_j d_j \frac{q^j}{j!}$$

$$d_j = \left(\frac{\mathrm{d}}{\mathrm{d}q}\right)^j \left(\sum_j c_j L_j(q|\alpha)\right)|_{q=0}$$

$$f(r) = \frac{1}{2\pi}\int \mathrm{d}q e^{-iqr} \tilde{f}(q)$$

$$f(r) = \frac{1}{2\pi}\int \mathrm{d}q e^{-iqr} \exp\left(-\frac{1}{2}(qR)^{\alpha}\right)\left(\sum_j d_j \frac{q^j}{j!}\right)$$

$$f(r) = \frac{1}{2\pi}\sum_j \frac{d_j}{j!}\int \mathrm{d}q e^{-iqr} q^j \exp\left(-\frac{1}{2}(qR)^{\alpha}\right)$$

$$f(r) = \frac{1}{2\pi}\sum_j \frac{d_j}{j!}(-i)^j \left(\frac{d}{dr}\right)^j \mathrm{LS}\left(\frac{r}{R}, \alpha\right)$$

SUMMARY AND CONCLUSIONS

Several model-independent methods:

- Based on matching an abstract measure in *H* to the approximate shape of data
- Gaussian: Edgeworth expansions
- Exponential: Laguerre expansions
- Lévy ($0 < \alpha \leq 2$): Lévy expansions
- Lévy expansions for positive definit functions

Thank you for your attention!