Dynamics of relativistic spin-polarized fluids

Radoslaw Ryblewski

THE HENRYK NIEWODNICZAŃSKI
INSTITUTE OF NUCLEAR PHYSICS
POLISH ACADEMY OF SCIENCES

in collaboration with:
W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza

references:
arXiv:1705.00587, arXiv:1712.07676, forthcoming

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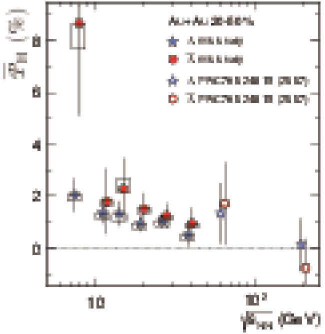
WPCF2018

Introduction & Motivation

- **measurements of the global spin polarization of Λ hyperons in relativistic heavy-ion collisions (HIC)**

see M. Lisa's talk



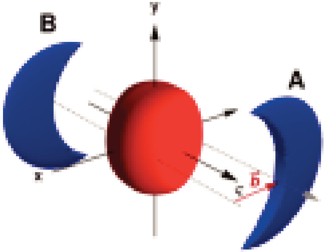
L. Adamczyk et al. (STAR), (2017), Nature 548 (2017) 62-65, arXiv:1701.06657

Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid

- **possible physical mechanisms responsible for polarization of matter?**

large global angular momenta in non-central HIC may generate global spin polarization of the matter and final hadrons (similarly to the Einstein-de Haas and Barnett effects)

A. Einstein and W. de Haas, Deutsche Physikalische Gesellschaft, Verhandlungen 17, 152 (1915); S. J. Barnett, Rev. Mod. Phys. 7, 129 (1935)



- **some models describe (some) data but the mechanism is not well understood yet**

F. Becattini, I. Karpenko, M. Lisa, I. Upsal, S. Voloshin, Phys. Rev. C 95, 054902 (2017)

Main physics objects

Lorentz symmetry leads to the basic conservation laws:

Conservation of energy and momentum

$$\partial_\mu \hat{T}^{\mu\nu}(x) = 0 \qquad \text{(4 equations)}$$

Conservation of total angular momentum

$$\partial_\mu \hat{J}^{\mu,\alpha\beta}(x) = 0 \qquad \hat{J}^{\mu,\alpha\beta}(x) = -\hat{J}^{\mu,\beta\alpha}(x) \qquad \text{(6 equations)}$$

Total angular momentum consists of orbital and spin parts:

$$\hat{J}^{\mu,\alpha\beta}(x) = \hat{L}^{\mu,\alpha\beta}(x) + \hat{S}^{\mu,\alpha\beta}(x)$$

$$\hat{L}^{\mu,\alpha\beta}(x) = x^\alpha \hat{T}^{\mu\beta}(x) - x^\beta \hat{T}^{\mu\alpha}$$

One has

$$\partial_\mu \hat{S}^{\mu,\alpha\beta}(x) = \hat{T}^{\beta\alpha}(x) - \hat{T}^{\alpha\beta}(x) \neq 0$$

For $\hat{T}^{\beta\alpha}(x) \neq \hat{T}^{\alpha\beta}(x)$ the spin tensor $\hat{S}^{\mu,\alpha\beta}(x)$ is not conserved
In our work $\hat{T}^{\beta\alpha}(x) = \hat{T}^{\alpha\beta}(x)$ (as expected for classical particles) and $\hat{S}^{\mu,\alpha\beta}(x)$ is conserved

Global thermodynamic equilibrium

Density operator for any quantum mechanical system

D. Zubarev, Nonequilibrium Statistical Thermodynamics (Springer, 1974)
F. Becattini, Phys. Rev. Lett. 108, 244502 (2012)

$$\hat{\rho}(t) = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)\beta_\nu(x) - \frac{1}{2}\hat{S}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x)\right)\right]$$

In global equilibrium $\hat{\rho}(t)$ should be independent of time hence $\beta_\nu = u_\nu/T$ satisfies the Killing equation

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0$$

and

$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right) = \mathrm{const} \quad \text{(thermal vorticity)}$$

Thermal vorticity $\varpi_{\mu\nu}$ is identified with the spin polarization tensor $\omega_{\mu\nu}$

$$\varpi_{\mu\nu} \longleftrightarrow \omega_{\mu\nu}$$

see A. Kumar's talk

Methodology

PRESENT PHENOMENOLOGICAL PRESCRIPTION USED TO DESCRIBE THE DATA:

1) Run any type of hydro, perfect or viscous, or transport, or whatsoever, ...
2) Find $\beta_\mu = u_\mu(x)/T(x)$
3) Calculate thermal vorticity $\varpi_{\mu\nu}$
4) Identify thermal vorticity $\varpi_{\mu\nu}$ with the spin polarization tensor $\omega_{\mu\nu}$
5) Make predictions about spin polarisation

THIS TALK:

In local equilibrium thermal vorticity and spin polarization tensor are independent

$\beta_\mu(x)$ and $\omega_{\mu\nu}(x)$ evolve independently — eventually, may become related if a system reaches global equilibrium

Local distribution functions for particles with spin-1/2

Phase-space distribution functions for spin-1/2 particles (+) and antiparticles (-) (2x2 spin density matrices)
F. Becattini et al., Annals Phys. 338 (2013) 32

$$f^{+}_{rs}(x,p) = \frac{1}{2m}\bar{u}_r(p)X^{+}u_s(p), \quad f^{-}_{rs}(x,p) = -\frac{1}{2m}\bar{v}_s(p)X^{-}v_r(p)$$

where

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu\right]M^{\pm}$$

$$M^{\pm} = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\hat{\Sigma}^{\mu\nu}\right]$$

with

$\beta^\mu = u^\mu/T$
$\xi = \mu/T$
m - mass of particles
T - temperature
μ - chemical potential
u^μ - four velocity ($u^2 = 1$)
$\omega_{\mu\nu}$ - spin-polarization tensor
$\hat{\Sigma}^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$ - spin operator

Energy-momentum tensor

The **energy-momentum tensor** is
S. de Groot, W. van Leeuwen, and C. van Weert, *Relativistic Kinetic Theory: Principles and Applications* (1980)

$$T^{\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu p^\nu \left[\mathrm{tr}_4(X^+) + \mathrm{tr}_4(X^-)\right] = (\varepsilon + P) u^\mu u^\nu - P g^{\mu\nu}$$

where the **energy density** and **pressure** are

$$\varepsilon = 4\,\cosh(\zeta)\cosh(\xi)\,\varepsilon_{(0)}(T)$$

$$P = 4\,\cosh(\zeta)\cosh(\xi)\,P_{(0)}(T)$$

and
$\varepsilon_{(0)}(T) = \langle (u \cdot p)^2 \rangle_0$ - the energy density of spin-0, neutral Boltzmann particles
$P_{(0)}(T) = -\frac{1}{3}\langle p \cdot p - (u \cdot p)^2 \rangle_0$ - the pressure of spin-0, neutral Boltzmann particles
$\langle \cdots \rangle_0 \equiv \int \frac{d^3p}{(2\pi)^3 E_p} (\cdots)\, e^{-\beta \cdot p}$ - thermal average
$E_p = \sqrt{m^2 + p^2}$

Variable ζ may be thus treated as a (spin) chemical potential Ω divided by temperature
Expressing $\zeta = \frac{1}{2}\sqrt{\frac{1}{2}\omega^{\mu\nu}\omega_{\mu\nu}}$ with $\zeta = \Omega/T$ means that all thermodynamic variables are functions of μ, Ω and T

Charge current

The **charge current** is

S. de Groot, W. van Leeuwen, and C. van Weert, *Relativistic Kinetic Theory: Principles and Applications* (1980)

$$N^\mu = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu \left[\mathrm{tr}_4(X^+) - \mathrm{tr}_4(X^-) \right] = n u^\mu$$

where the **charge density** is

$$n = 4 \cosh(\zeta) \sinh(\xi) \, n_{(0)}(T)$$

$n_{(0)}(T) = \langle (u \cdot p) \rangle_0$ - the charge density of spin-0, neutral Boltzmann particles

Entropy current

The **entropy current** is

$$S^\mu = -\int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu \left(\mathrm{tr}_4 \left[X^+ (\ln X^+ - 1) \right] + \mathrm{tr}_4 \left[X^- (\ln X^- - 1) \right] \right) = s u^\mu$$

with

$$s = u_\mu S^\mu = \frac{\varepsilon + P - \mu\, n - \Omega w}{T}$$

where Ω is defined through $\zeta = \Omega / T$ and

$$w = 4 \sinh(\zeta) \cosh(\zeta) n_{(0)}.$$

Relations above suggest that Ω should be treated as a thermodynamic variable of the grand canonical potential, in addition to T and μ

Taking the pressure $P = P(T, \mu, \Omega)$ one has

$$s = \left.\frac{\partial P}{\partial T}\right|_{\mu,\Omega}, \quad n = \left.\frac{\partial P}{\partial \mu}\right|_{T,\Omega}, \quad w = \left.\frac{\partial P}{\partial \Omega}\right|_{T,\mu}$$

Relativistic hydrodynamics for particles with spin $^1/_2$

W. Florkowski et al., Phys.Rev. C97 (2018) no.4, 041901, arXiv:1705.00587 [nucl-th]
W. Florkowski et al., arXiv:1712.07676 [nucl-th]

Basic hydrodynamic equations follow from energy and momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

Its transverse projection (with respect to u^μ) gives the Euler equation

$$(\varepsilon + P)\frac{du^\mu}{d\tau} = (g^{\mu\alpha} - u^\mu u^\alpha)\partial_\alpha P$$

while its longitudinal projection (with respect to u^μ) is

$$\partial_\mu[(\varepsilon + P)u^\mu] - u^\mu\partial_\mu P = \frac{dP}{d\tau} \quad \Rightarrow \quad T\partial_\mu(su^\mu) + \mu\partial_\mu(nu^\mu) + \Omega\partial_\mu(wu^\mu) = 0$$

which, assuming the charge conservation $\partial_\mu(nu^\mu) = 0$ and entropy conservation $\partial_\mu(su^\mu) = 0$, results in spin density conservation

$$\partial_\mu(wu^\mu) = 0$$

CHANGE OF THE POLARIZATION MAGNITUDE DESCRIBED BY THE BACKGROUND EQUATIONS

Spin dynamics

Since we have $T^{\mu\nu} = T^{\nu\mu}$, the **spin tensor** $S^{\lambda,\mu\nu}$ has to be conserved separately

$$\partial_\lambda S^{\lambda,\mu\nu} = 0.$$

for $S^{\lambda,\mu\nu}$ we use the form
F. Becattini and L. Tinti, Annals Phys. 325 (2010) 1566–1594

$$S^{\lambda,\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p} \, p^\lambda \, \mathrm{tr}_4 \left[(X^+ - X^-) \hat{\Sigma}^{\mu\nu} \right] = \frac{w u^\lambda}{4\zeta} \omega^{\mu\nu}$$

using the spin density conservation law and introducing the **rescaled spin-polarization tensor**

$$\bar{\omega}^{\mu\nu} = \omega^{\mu\nu}/(2\zeta) \qquad (\bar{\omega}_{\mu\nu}\bar{\omega}^{\mu\nu} = 2)$$

one gets

$$u^\lambda \partial_\lambda \bar{\omega}^{\mu\nu} = \frac{d\bar{\omega}^{\mu\nu}}{d\tau} = 0$$

TRANSPORT OF THE SPIN POLARIZATION DIRECTION ALONG THE FLUID STREAM LINES

Stationary vortex - global equilibrium with rigid rotation

Consider a stationary cylindrically symmetric hydrodynamic flow with the components

$$u^\mu = \gamma(1, v_x, v_y, v_z) = \gamma(1, -\tilde{\Omega} y, \tilde{\Omega} x, 0)$$

$\gamma = 1/\sqrt{1-\tilde{\Omega}^2 r^2}$ - Lorentz factor ($\tilde{\Omega}$ is a constant)

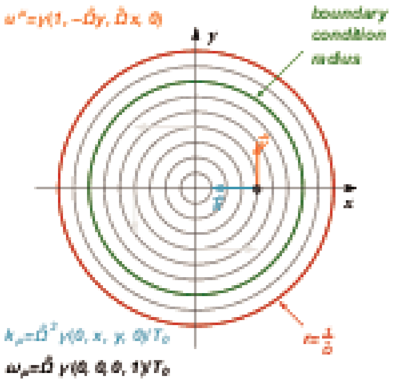
$r = \sqrt{x^2+y^2}$ - distance from the vortex (z) axis

Due to limiting light speed one has to have $0 \le r < R = 1/\tilde{\Omega}$.

Hydrodynamic background equations are satisfied if

$$T = T_0\gamma, \quad \mu = \mu_0\gamma, \quad \Omega = \Omega_0\gamma$$

with T_0, μ_0 and Ω_0 being arbitrary constants.



Stationary vortex - global equilibrium with rigid rotation

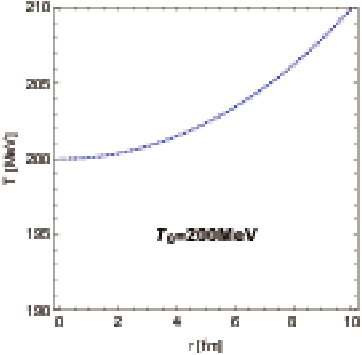


Figure: Temperature

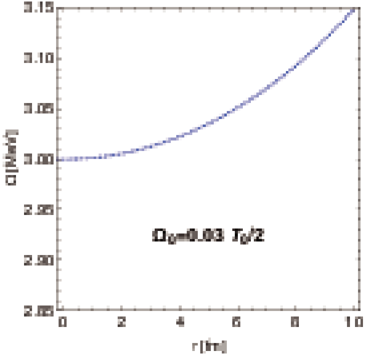


Figure: Spin chemical potential

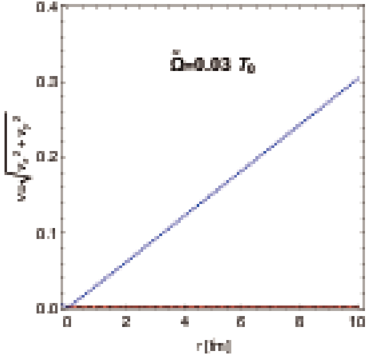


Figure: Flow magnitude $v = \sqrt{v_x^2 + v_y^2}$

Initial parameters are: $T_0 = 200$ MeV, $\mu_0 = 200$ MeV, $\Omega_0 = 0.03 T_0/2$ [Becattini et al, Phys. Rev. C77 (2008) 024906], $m = 1$ GeV.

Stationary vortex - global equilibrium with rigid rotation

The possible solutions for the spin tensor components are:

- **Unpolarized fluid** with $\omega_{\mu\nu} = 0 \Rightarrow \Omega_0 = 0$
- **Polarized fluid** with $\Omega_0 \neq 0$
  In this case the spin-polarization tensor has the structure

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\Omega}/T_0 & 0 \\ 0 & -\tilde{\Omega}/T_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the parameter T_0 has been introduced to keep $\omega_{\mu\nu}$ dimensionless.

As a result $\zeta = \frac{1}{2}\sqrt{\frac{1}{2}\omega^{\mu\nu}\omega_{\mu\nu}} = \tilde{\Omega}/(2T_0)$, which, due to $\zeta = \Omega/T$, implies $\tilde{\Omega} = 2\,\Omega_0$

In this case the **spin polarization tensor agrees with the thermal vorticity**, namely

$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right) \longleftrightarrow \omega_{\mu\nu}$$

In this case only $\varpi_{xy} = -\varpi_{yx} = 2\zeta = \frac{\Omega}{T_0}$ is different from zero

Vortex in the static medium

What will happen if the external boundary is reduced?

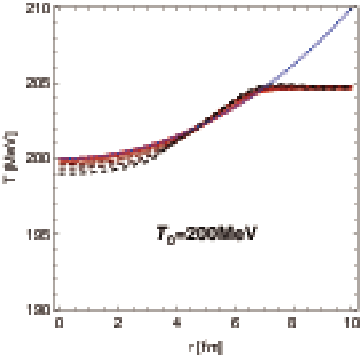


Figure: Temperature

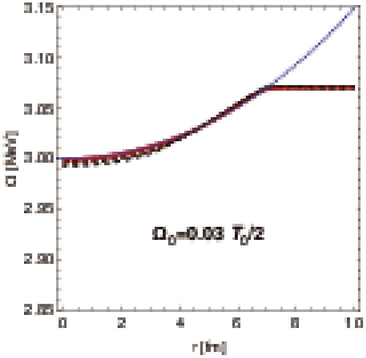


Figure: Spin chemical potential

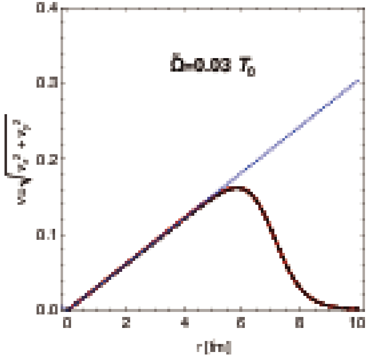


Figure: Flow magnitude $v = \sqrt{v_x^2 + v_y^2}$

Profiles for the system at times: $t = 0.1, 2, 4, 6, 8, 10$ fm (lines ranging from red to black).
Initial parameters are: $T_0 = 200$ MeV, $\mu_0 = 200$ MeV, $\Omega_0 = 0.03 T_0/2$ [Becattini et al, Phys. Rev. C77 (2008) 024906], $m = 1$ GeV.

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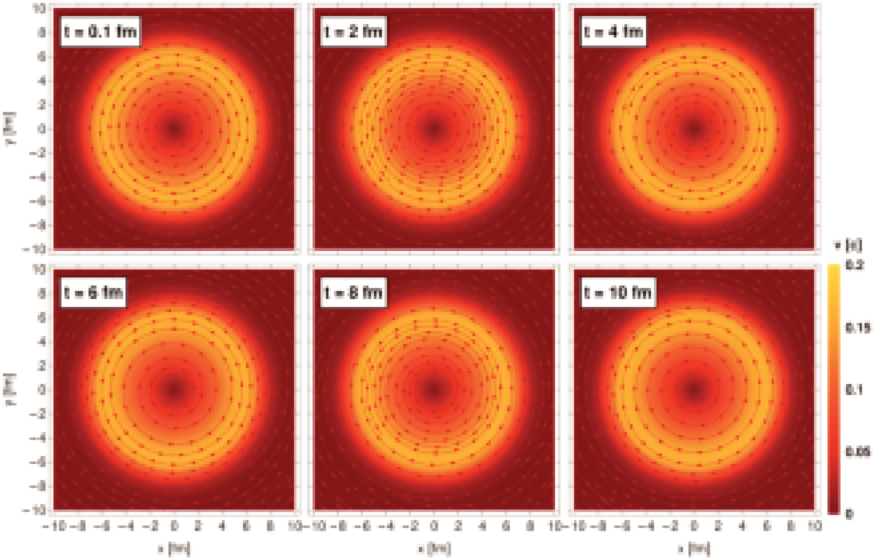

Vortex in the static medium

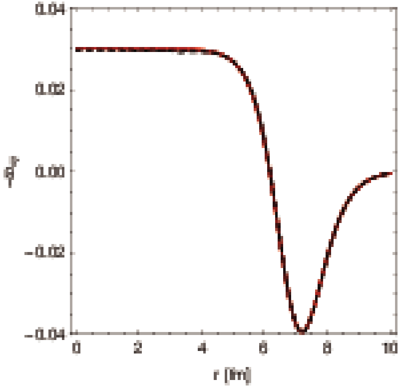

Figure: Thermal vorticity $\varpi_{xy} = (\partial_y \beta_x - \partial_x \beta_y)/2$

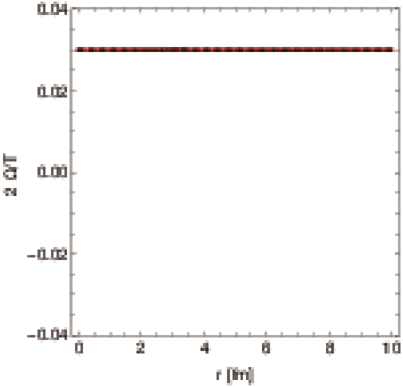

Figure: $2\Omega/T$

Profiles for the system at times: $t = 0.1, 2, 4, 6, 8, 10$ fm (lines ranging from red to black).

Initial parameters are: $T_0 = 200$ MeV, $\mu_0 = 200$ MeV, $\Omega_0 = 0.03 T_0/2$ [Becattini et al, Phys. Rev. C77 (2008) 024906], $m = 1$ GeV.

Vortex in vacuum

What will happen if the external boundary is removed? Expansion into external vacuum.

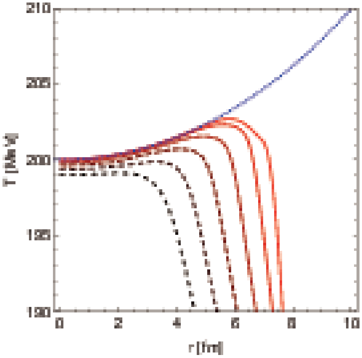

Figure: Temperature profile

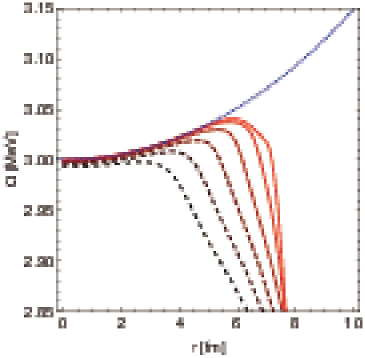

Figure: Velocity profile

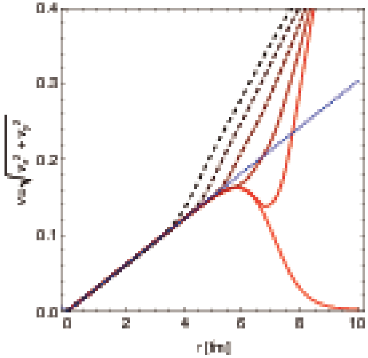

Figure: $2\Omega/T$

Profiles for the system at times: $t = 0.1, 2, 4, 6, 8, 10$ fm (lines ranging from red to black).

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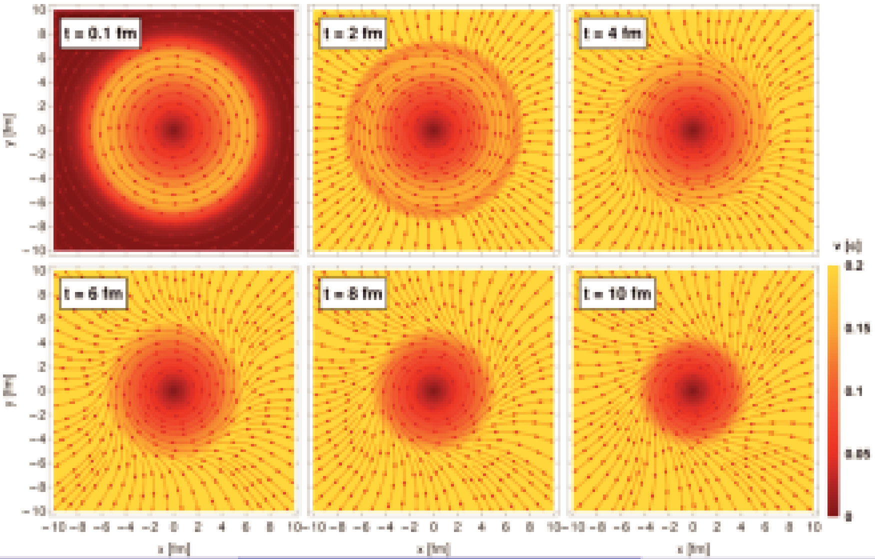
Vortex in vacuum



Vortex in vacuum

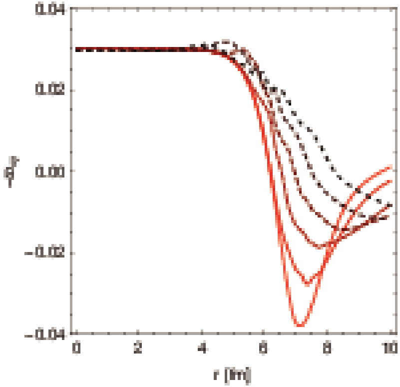


Figure: Thermal vorticity $\varpi_{xy} = (\partial_y \beta_x - \partial_x \beta_y)/2$

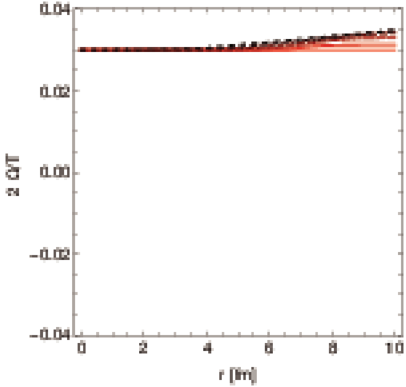


Figure: $2\Omega/T$

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Quasi-realistic 3D model for low-energy collisions

Initial gaussian temperature profile

$$T_i = T_0\, g(x,y,z) \qquad g(x,y,z) = \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right)$$

$\sigma_z = 1$ (beam direction, one can possibly use the Landau model)

$\sigma_y = 2.6$ and $\sigma_x = 2$ (from GLISSANDO version of the Glauber Model, Au+Au, 20-30%)

W. Broniowski, M. Rybczynski, P. Bozek, Comput.Phys.Commun. 180 (2009) 69-83
M. Rybczynski, G. Stefanek, W. Broniowski, P. Bozek, Comput.Phys.Commun. 185 (2014) 1759-1772

Initial spin chemical potential profile

$\Omega_i = 0.03\, T_i/2$, hence we still have $2\Omega_i/T_i = 0.03$, as in the previous cases

Initial baryon chemical potential profile

$$\mu_i = \mu_0\, g(x,y,z)$$

Initial flow profile

$$\vec{\Omega} \to \frac{1}{r} \tanh\frac{r}{r_0}, \quad \Rightarrow \quad v_x = -\frac{y}{r}\tanh\frac{r}{r_0}, \quad v_y = \frac{x}{r}\tanh\frac{r}{r_0}$$

The parameter r_0 controls the magnitude of the initial angular velocity.

In the limit $r_0 \to \infty$ the initial angular velocity vanishes.

Quasi-realistic 3D model for low-energy collisions

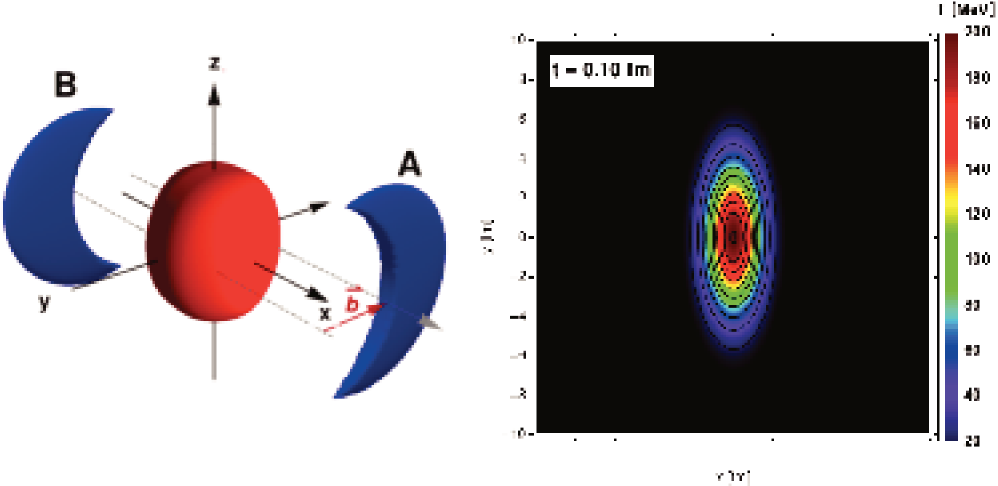
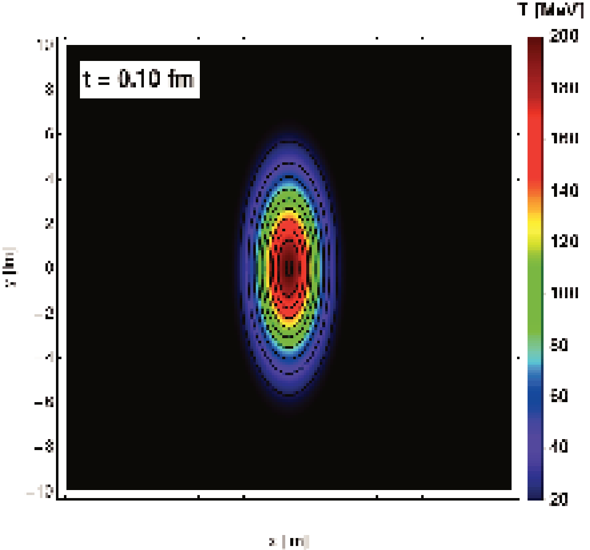


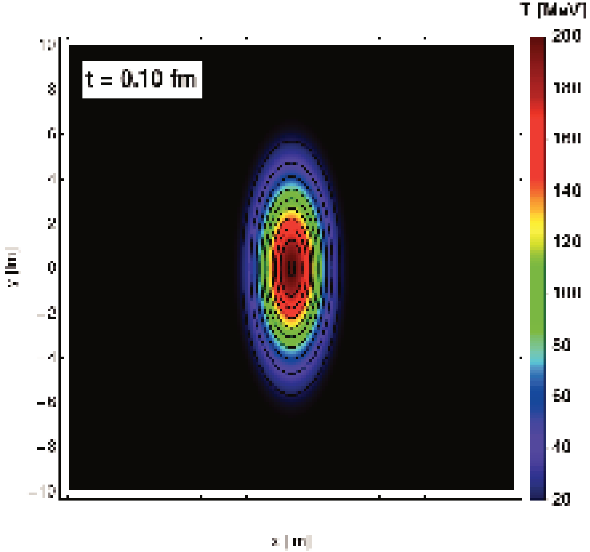
Figure: Initial conditions for the quasi-realistic model

Quasi-realistic 3D model for low-energy collisions ($r_0 = 10$)



- For animation open the presentation file with Adobe Reader.

Quasi-realistic 3D model for low-energy collisions ($r_0 = 1$)



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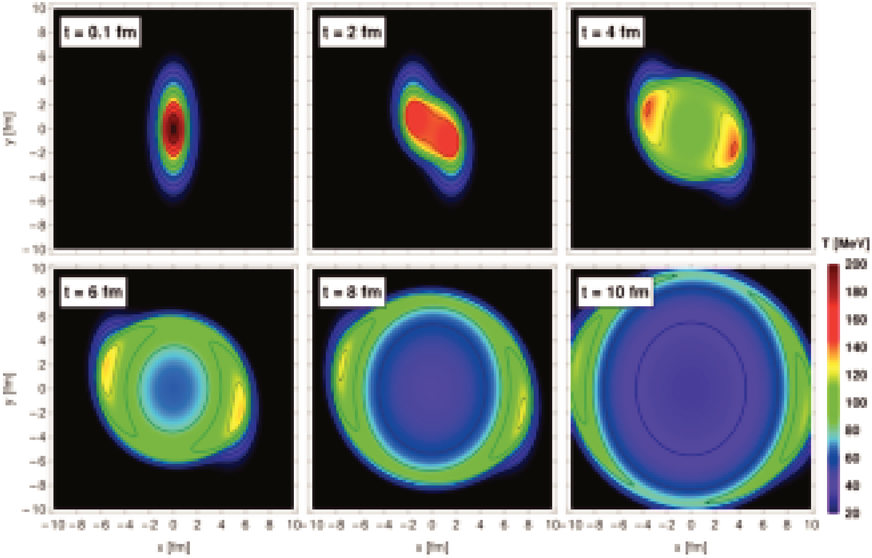
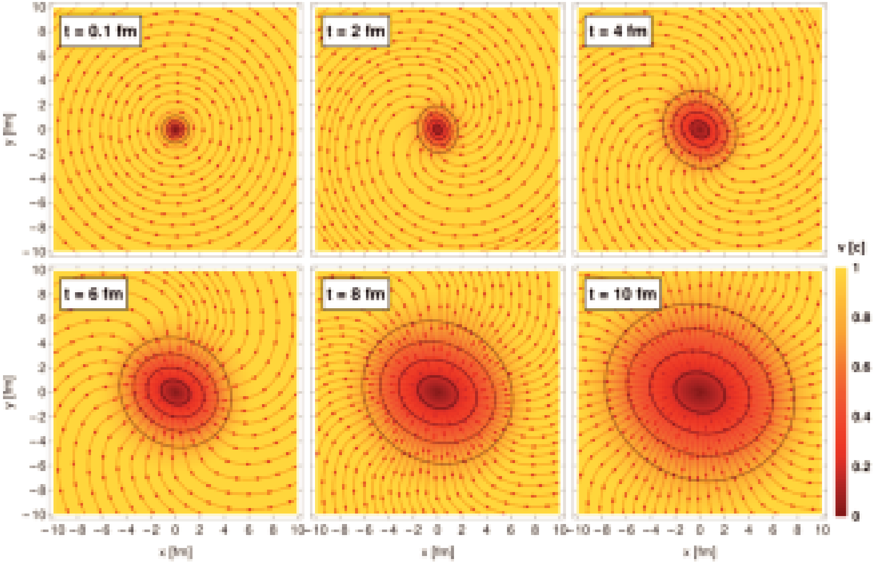
Quasi-realistic 3D model for low-energy collisions



Quasi-realistic 3D model for low-energy collisions

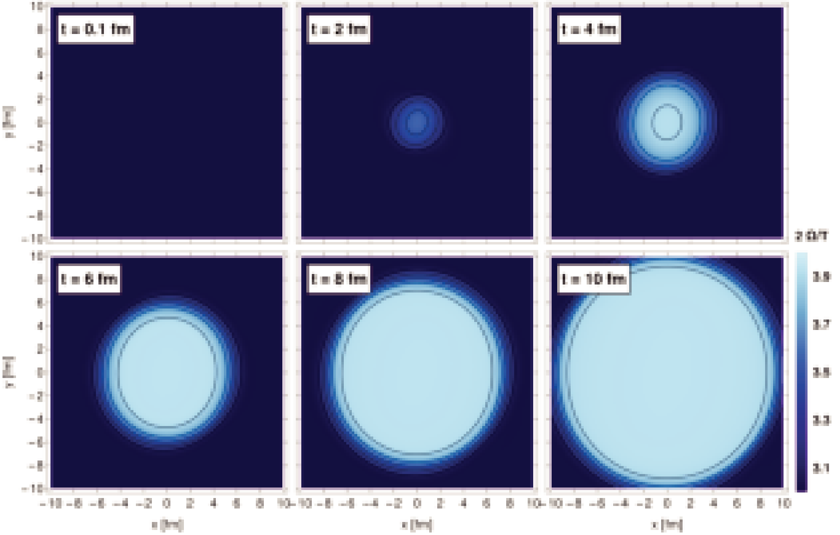


Quasi-realistic 3D model for low-energy collisions

$$\varpi_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{\varpi}_{xy} & \bar{\varpi}_{zx} \\ 0 & \bar{\varpi}_{xy} & 0 & -\bar{\varpi}_{yz} \\ -0 & -\bar{\varpi}_{zx} & \bar{\varpi}_{yz} & 0 \end{bmatrix}$$

$$2\Omega/T = 2\zeta = \sqrt{\varpi^{\mu\nu}\varpi_{\mu\nu}/8}$$

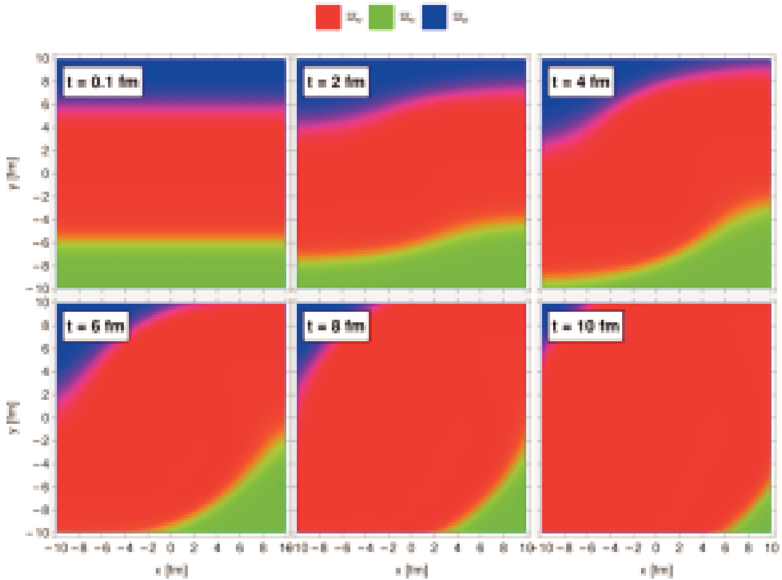
$$\tilde{\varpi}^{\mu\nu} = 0$$



Quasi-realistic 3D model for low-energy collisions

$$\bar{\omega}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{\omega}_{xy} & \bar{\omega}_{xz} \\ 0 & \bar{\omega}_{xy} & 0 & -\bar{\omega}_{yz} \\ -0 & -\bar{\omega}_{xz} & \bar{\omega}_{yz} & 0 \end{bmatrix}$$

$$\frac{dz^{\mu}}{d\tau} = 0$$



Conclusions and Summary

- Hydrodynamic framework which includes evolution of spin density in a consistent fashion
- Minimal extension of well-established perfect-fluid hydrodynamics
- Polarization evolution in heavy-ion collisions
- Advantage to study dynamics of systems in local equilibrium

Outlook: spin-orbit interactions, asymmetric $T_{\mu\nu}$, dissipation, ...

Thank you for your attention!

and ...

Welcome to

Quark Matter 2021

in Kraków, October 3-9, 2021

Backup slides

1. Pseudo-gauge transformation

defined with the help of arbitrary tensors $\Phi^{\lambda,\mu\nu}$ and $Z^{\alpha\lambda,\mu\nu}$

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\Phi^{\lambda,\mu\nu} + \Phi^{\mu,\nu\lambda} + \Phi^{\nu,\mu\lambda}\right) = T^{\mu\nu} + \frac{1}{2}\partial_\lambda G^{\lambda,\mu\nu}$$

$$S'^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\alpha Z^{\alpha\lambda,\mu\nu}$$

does not change global charges, new tensors, $T'^{\mu\nu}$ and $J'^{\lambda,\mu\nu}$, are also conserved

Bellinfante prescription: $\Phi^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} \quad \rightarrow \quad S'^{\lambda,\mu\nu} = 0, \quad T'^{\mu\nu} = T'^{\nu\mu}$

2. Spin polarization – standard QM treatment

Expansion in terms of Pauli matrices

$$f^{\pm}(x,p) = e^{\pm\xi - p\cdot\beta}\left[\cosh(\zeta) \mp \frac{\sinh(\zeta)}{2\zeta}\,\boldsymbol{P}\cdot\boldsymbol{\sigma}\right]$$

average polarization vector

$$\mathcal{P} = \frac{1}{2}\frac{\mathrm{tr}_2[(f^+ + f^-)\boldsymbol{\sigma}]}{\mathrm{tr}_2[f^+ + f^-]} = -\frac{1}{2}\tanh(\zeta)\frac{\boldsymbol{P}}{2\zeta}$$

$$\mathcal{P} = -\frac{1}{2}\tanh\left[\frac{1}{2}\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}\right]\frac{\boldsymbol{b}_*}{\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}}$$

where the spin polarization is expressed by the matrix

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}$$

$*$ denotes the PARTICLE REST FRAME

3. Pauli-Lubański four-vector (phase-space density $\Pi_\mu(x,p)$)

$J^{\lambda,\nu\alpha}(x,p)$ is the phase-space density of the angular momentum of particles

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x) E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\frac{p^\beta}{m}$$

$$E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p} = \frac{\kappa}{2} p^\lambda \left(x^\nu p^\alpha - x^\alpha p^\nu\right) \mathrm{tr}_4(X^+ + X^-) + \frac{\kappa}{2} p^\lambda \mathrm{tr}_4\left[\left(X^+ - X^-\right)\Sigma^{\nu\alpha}\right]$$

particle density in the volume $\Delta\Sigma$

$$E_p \frac{d\Delta N}{d^3p} = \frac{\kappa}{2}\Delta\Sigma \cdot p\, \mathrm{tr}_4\left(X^+ + X^-\right)$$

$$\pi_\mu(x,p) = \frac{\Delta\Pi_\mu(x,p)}{\Delta N(x,p)}$$

By applying the Lorentz transformation we find that the PL four-vector calculated in the PRF agrees with the spin polarization (!)

$$\pi_*^0 = 0, \quad \boldsymbol{\pi}_* = \boldsymbol{P} = -\frac{1}{2}\tanh(\zeta)\hat{\boldsymbol{P}}$$

4. Two-component system

In the absence of a net spin polarization, i.e., for $\zeta = 0$, we find the standard expression for the net charge density $n = 4\sinh(\xi)\, n_{(0)}$.

On the other hand, one may consider two linear combinations of the form $\partial_\mu \left[(n \pm w)u^\mu\right] = 0$. Then, we find $n \pm w = 4\sinh[(\mu \pm \Omega)/T]\, n_{(0)}$, which indicates that thermodynamic quantities corresponding to charge and spin of the particles couple.

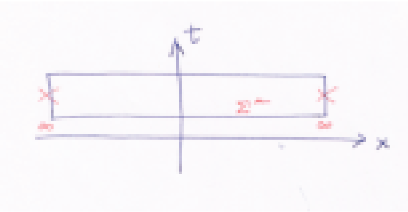
Ω can be interpreted as a chemical potential related with spin — from a thermodynamic point of view, a system of particles with spin 1/2 can be seen as a two component mixture of scalar particles with chemical potentials $\mu \pm \Omega$.

5. Global thermodynamic equilibrium (Zubarev, Becattini)

Density operator for any quantum mechanical system

$$\hat{\rho}(t) = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x) b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x)\right)\right]$$

$d^3\Sigma_\mu$ is an element of a space-like, 3-dimensional hypersurface Σ_μ,
we can take, for example, $d^3\Sigma_\mu = (dV, 0, 0, 0)$
in global equilibrium $\hat{\rho}(t)$ should be independent of time



$$\partial_\mu\left(\hat{T}^{\mu\nu}(x) b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x)\right) = \hat{T}^{\mu\nu}(x)\left(\partial_\mu b_\nu(x)\right) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\left(\partial_\mu \omega_{\alpha\beta}(x)\right) = 0$$

for asymmetric energy-momentum tensor:

$$b_\nu = \text{const.}, \quad \omega_{\alpha\beta} = \text{const.}$$

5. Global thermodynamic equilibrium (Zubarev, Becattini)

splitting angular momentum into its orbital and spin part

$$
\begin{aligned}
\hat{\rho}_{EQ} &= \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)b_\nu - \frac{1}{2}\left(\hat{L}^{\mu,\alpha\beta}(x) + \hat{S}^{\mu,\alpha\beta}(x)\right)\varpi_{\alpha\beta}\right)\right] \\
&= \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)b_\nu - \frac{1}{2}\left(x^\alpha \hat{T}^{\mu\beta}(x) - x^\beta \hat{T}^{\mu\alpha} + \hat{S}^{\mu,\alpha\beta}(x)\right)\varpi_{\alpha\beta}\right)\right] \\
&= \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)\left(b_\nu + \varpi_{\nu\alpha}x^\alpha\right) - \frac{1}{2}\hat{S}^{\mu,\alpha\beta}(x)\varpi_{\alpha\beta}\right)\right]
\end{aligned}
$$

Introducing the notation

$$\beta_\nu = b_\nu + \varpi_{\nu\alpha}x^\alpha$$

we may write

$$\hat{\rho}_{EQ} = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)\beta_\nu - \frac{1}{2}\hat{S}^{\mu,\alpha\beta}(x)\varpi_{\alpha\beta}\right)\right]$$

We note that β_ν is the Killing vector, satisfies the equations

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0, \quad -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right) = \varpi_{\mu\nu} = \text{const} \quad \text{(thermal vorticity)}$$

6. Spin-polarization tensor

as the **spin-polarization tensor** is antisymmetric it is parametrized in terms of k_μ and ω_μ

$$\omega_{\mu\nu} = k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\alpha\beta} u^\alpha \omega^\beta$$

assuming $k \cdot u = \omega \cdot u = 0$, one has

$$k_\mu = \omega_{\mu\nu} u^\nu, \quad \omega_\mu = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\nu\alpha} u^\beta$$ electric- and magnetic-like components!

convenient to introduce the **dual spin-polarization tensor**

$$\tilde{\omega}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \omega_{\alpha\beta} = \omega^\mu u^\nu - \omega^\nu u^\mu + \epsilon^{\mu\nu\alpha\beta} k_\alpha u_\beta$$

one finds $\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} = k \cdot k - \omega \cdot \omega$ and $\frac{1}{2}\tilde{\omega}_{\mu\nu}\omega^{\mu\nu} = 2k \cdot \omega$

using the constraint $k \cdot \omega = 0$ we find the compact form

$$M^{\pm} = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta} \omega_{\mu\nu} \Sigma^{\mu\nu}$$

where $\zeta = \frac{1}{2}\sqrt{k \cdot k - \omega \cdot \omega} = \frac{1}{2\sqrt{2}}\sqrt{\omega_{\mu\nu}\omega^{\mu\nu}}$ with $k \cdot k - \omega \cdot \omega \geq 0$