Thermodynamic versus kinetic approach to polarization-vorticity coupling and hydrodynamics with spin

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Rotation and Polarization

Barnett Effect

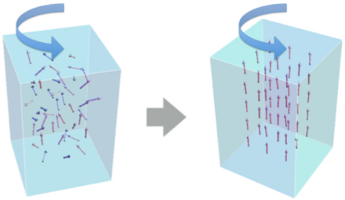

Figure: Mechanical rotation of an unmagnetized metallic object induces magnetization, an effective magnetic field emerges.

$$B_\Omega = \Omega/\gamma$$

Einstein-de Haas Effect

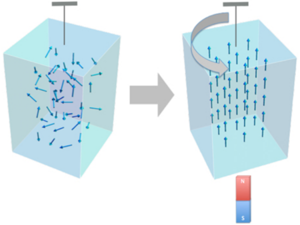

Figure: Application of magnetic field on an unmagnetized metallic object induces magnetization, body start rotating (mechanical angular momentum emerges)

Case of Heavy ion collision experiment

• Global angular momentum $J \approx 10^4\hbar$ (RHIC Au-Au 200 GeV, b=2.5 fm)[arXiv:0711.1253v3 [nucl-th] 18 Feb 2008].

• One can think of a Fluid with the highest vorticity.

• Emerging particle are expected to be globally polarized with their spins on average pointing along the system angular momentum.

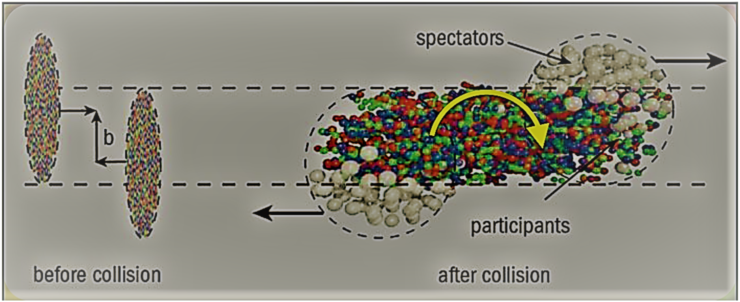

Figure: Geometry of a non-central heavy ion collision

Global Λ-polarization in RHIC experiment

Evidence of the fluid with highest vorticity:

The average polarization $\bar{P}_H$ (where $H = \Lambda$ or $\bar{\Lambda}$) from $20 - 50\%$ central Au+Au collisions [Nature 548 (2017) 62-65, arXiv:1701.06657 (nucl-ex)]

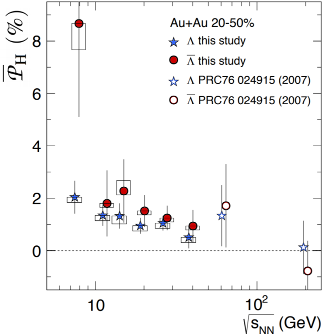

Figure: The average polarization versus collision energy

• Present phenomenological prescription used to describe the data make use of the fact that thermal vorticity is equal to spin polarization tensor.

• In this work using the equilibrium distributions functions for particles of spin 1/2 as an input to the Wigner function and its semi-classical expansion we will show how a kinetic approach can lead us to the fact that the thermal vorticity and spin polarization tensor are constant. However, no such conclusion can be drawn whether they are equal or not.

• I will also discuss the procedure to construct the hydrodynamic framework that can deal with the spin physics.

Equilibrium Wigner Functions

We start with the equilibrium Wigner functions [de-Groot 1980]

$$\mathcal{W}^{+}_{\rm eq}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k-p)u^{r}(p)\bar{u}^{s}(p)f^{+}_{rs}(x,p),$$

$$\mathcal{W}^{-}_{\rm eq}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k+p)v^{s}(p)\bar{v}^{r}(p)f^{-}_{rs}(x,p).$$

We take $f^{+}_{rs}(x,p)$ and $f^{-}_{rs}(x,p)$ [F. Becattini et al. Annals Phys. 338 (2013) 32]

$$f^{+}_{rs}(x,p) = \frac{1}{2m}\bar{u}_r(p)X^{+}u_s(p), \qquad f^{-}_{rs}(x,p) = -\frac{1}{2m}\bar{v}_s(p)X^{-}v_r(p).$$

m is the (anti)particle mass, while $u_r(p)$ and $v_r(p)$ are Dirac bispinors.

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu\right]M^{\pm}, \qquad M^{\pm} = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right].$$

- $\beta^\mu(x) = u^\mu(x)/T(x)$ and $\xi(x) = \mu(x)/T(x)$, with $\mu(x)$ being the chemical potential.
- The quantity $\omega_{\mu\nu}(x)$ is the spin polarization tensor, while $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu,\gamma^\nu]$.

Equilibrium Wigner functions

Spin polarization tensor $\omega_{\mu\nu}$ satisfies the two conditions [W. Florkowski et al. 2017]

$$\omega_{\mu\nu}\omega^{\mu\nu} \geq 0, \quad \omega_{\mu\nu}\tilde{\omega}^{\mu\nu} = 0, \quad \text{where } \tilde{\omega}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}$$

$$M^{\pm} = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\,\omega_{\mu\nu}\Sigma^{\mu\nu}.$$

ζ is defined by the expression

$$\zeta = \frac{\Omega}{T} = \frac{1}{2}\sqrt{\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}}.$$

Ω plays a role of the spin chemical potential.
The equilibrium Wigner functions

$$\mathcal{W}^{+}_{\rm eq}(x,k) = \frac{e^{\xi}}{4m}\int dP\, e^{-\beta\cdot p}\;\delta^{(4)}(k-p)\left[2m(m+\not p)\cosh(\zeta) + \frac{\sinh(\zeta)}{2\zeta}\,\omega_{\mu\nu}\,(\not p+m)\Sigma^{\mu\nu}(\not p+m)\right],$$

$$\mathcal{W}^{-}_{\rm eq}(x,k) = \frac{e^{-\xi}}{4m}\int dP\, e^{-\beta\cdot p}\;\delta^{(4)}(k+p)\left[2m(m-\not p)\cosh(\zeta) - \frac{\sinh(\zeta)}{2\zeta}\,\omega_{\mu\nu}\,(\not p-m)\Sigma^{\mu\nu}(\not p-m)\right].$$

The total Wigner function

$$\mathcal{W}_{\rm eq}(x,k) = \mathcal{W}^{+}_{\rm eq}(x,k) + \mathcal{W}^{-}_{\rm eq}(x,k).$$

Spinor decomposition of the equilibrium Wigner function

The Wigner functions $\mathcal{W}^{\pm}_{\rm eq}(x,k)$, being four-by-four matrices satisfying the relations $\mathcal{W}^{\pm}_{\rm eq}(x,k) = \gamma_0 \mathcal{W}^{\pm}_{\rm eq}(x,k)^{\dagger}\gamma_0$, can always be expanded in terms of the 16 independent generators of the Clifford algebra

$$\begin{aligned}\mathcal{W}^{\pm}_{\rm eq}(x,k) &= \frac{1}{4}\left[\mathcal{F}^{\pm}_{\rm eq}(x,k) + i\gamma_5 \mathcal{P}^{\pm}_{\rm eq}(x,k) + \gamma^{\mu}\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k)\right.\\ &\qquad \left. + \gamma_5\gamma^{\mu}\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k)\right].\end{aligned}$$

The coefficient functions in the equilibrium Wigner function expansion can be obtained by the following traces:

$$\begin{aligned}
\mathcal{F}^{\pm}_{\rm eq}(x,k) &= {\rm tr}\left[\mathcal{W}^{\pm}_{\rm eq}({\rm x},{\rm k})\right] = 2{\rm m}\cosh(\zeta)\int {\rm dP}\ {\rm e}^{-\beta\cdot{\rm p}\pm\xi}\ \delta^{(4)}({\rm k}\mp{\rm p}),\\
\mathcal{P}^{\pm}_{\rm eq}(x,k) &= -i\,{\rm tr}\left[\gamma^5\mathcal{W}^{\pm}_{\rm eq}({\rm x},{\rm k})\right] = 0,\\
\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) &= {\rm tr}\left[\gamma_{\mu}\mathcal{W}^{\pm}_{\rm eq}({\rm x},{\rm k})\right] = \pm\,2\cosh(\zeta)\int {\rm dP}\,{\rm e}^{-\beta\cdot{\rm p}\pm\xi}\ \delta^{(4)}({\rm k}\mp{\rm p})\,{\rm p}_{\mu},\\
\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) &= {\rm tr}\left[\gamma_{\mu}\gamma^5\mathcal{W}^{\pm}_{\rm eq}({\rm x},{\rm k})\right] = -\frac{\sinh(\zeta)}{\zeta}\int {\rm dP}\,{\rm e}^{-\beta\cdot{\rm p}\pm\xi}\ \delta^{(4)}({\rm k}\mp{\rm p})\,\tilde{\omega}_{\mu\nu}\,{\rm p}^{\nu},\\
\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k) &= 2\,{\rm tr}\left[\circ^{\mu\nu}\mathcal{W}^{\pm}_{\rm eq}({\rm x},{\rm k})\right] = \pm\frac{\sinh(\zeta)}{{\rm m}\zeta}\int {\rm dP}\,{\rm e}^{-\beta\cdot{\rm p}\pm\xi}\ \delta^{(4)}({\rm k}\mp{\rm p})\left[\left({\rm p}_{\mu}\omega_{\nu\alpha} - {\rm p}_{\nu}\omega_{\mu\alpha}\right){\rm p}^{\alpha} + {\rm m}^2\omega_{\mu\nu}\right].
\end{aligned}$$

Leading-order terms of the coefficient functions in the spin-polarization tensor,

$$
\begin{aligned}
F_{\rm eq}(x,k) &= 2m \int dP\; e^{-\beta\cdot p \pm \xi}\;\delta^{(4)}(k \mp p),\\
P_{\rm eq}(x,k) &= 0,\\
V^{\pm}_{{\rm eq},\mu}(x,k) &= \pm 2 \int dP\, e^{-\beta\cdot p \pm \xi}\;\delta^{(4)}(k \mp p)\, p_\mu\,,\\
A^{\pm}_{{\rm eq},\mu}(x,k) &= -\int dP\, e^{-\beta\cdot p \pm \xi}\;\delta^{(4)}(k \mp p)\,\tilde{\omega}_{\mu\nu}\, p^\nu\,,\\
S^{\pm}_{{\rm eq},\mu\nu}(x,k) &= \pm\frac{1}{m}\int dP\, e^{-\beta\cdot p \pm \xi}\;\delta^{(4)}(k \mp p)\left[\left(p_\mu\omega_{\nu\alpha} - p_\nu\omega_{\mu\alpha}\right)p^\alpha + m^2\omega_{\mu\nu}\right].
\end{aligned}
$$

Coefficient functions in equilibrium Wigner function expansion satisfies

$$
\begin{aligned}
&k^\mu\,\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) - m\,\mathcal{F}^{\pm}_{\rm eq}(x,k) = 0,\\
&k_\mu\,\mathcal{F}^{\pm}_{\rm eq}(x,k) - m\,\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) = 0,\\
&\mathcal{P}^{\pm}_{\rm eq}(x,k) = 0,\\
&k^\mu\,\mathcal{A}^{\pm}_{{\rm eq},\,\mu}(x,k) = 0,\\
&k_\mu\,\mathcal{V}^{\pm}_{{\rm eq},\nu}(x,k) - k_\nu\,\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) = 0,\\
&k^\mu\,\mathcal{S}^{\pm}_{{\rm eq},\,\mu\nu}(x,k) = 0,\\
&k^\beta\,\tilde{\mathcal{S}}^{\pm}_{{\rm eq},\mu\beta}(x,k) + m\,\mathcal{A}^{\pm}_{{\rm eq},\,\mu}(x,k) = 0,\\
&\epsilon_{\mu\nu\alpha\beta}\,k^\alpha\,\mathcal{A}^{\pm\,\beta}_{\rm eq}(x,k) + m\,\mathcal{S}^{\pm}_{{\rm eq},\,\mu\nu}(x,k) = 0.
\end{aligned}
$$

Note that such constraint are also fulfilled by the total Wigner function given by the sum of the particle and antiparticle contributions.

Also note that the above relationship holds for any form of $\beta_\mu(x)$, $\xi(x)$ and $\omega_{\mu\nu}(x)$.

Semi-classical expansion

$$\mathcal{W}(x,k) = \frac{1}{4}\left[\mathcal{F}(x,k) + i\gamma_5\mathcal{P}(x,k) + \gamma^\mu\mathcal{V}_\mu(x,k) + \gamma_5\gamma^\mu\mathcal{A}_\mu(x,k) + \Sigma^{\mu\nu}\mathcal{S}_{\mu\nu}(x,k)\right].$$

The Wigner function satisfies the equation of the form

$$(\gamma_\mu K^\mu - m)\,\mathcal{W}(x,k) = 0; \qquad K^\mu = k^\mu + \frac{i\hbar}{2}\,\partial^\mu.$$

The real parts:

$k^\mu\mathcal{V}_\mu - m\mathcal{F} = 0,$
$\frac{\hbar}{2}\partial^\mu\mathcal{A}_\mu + m\mathcal{P} = 0,$
$k_\mu\mathcal{F} - \frac{\hbar}{2}\partial^\nu\mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu = 0,$
$-\frac{\hbar}{2}\partial_\mu\mathcal{P} + k^\beta\tilde{\mathcal{S}}_{\mu\beta} + m\mathcal{A}_\mu = 0,$
$\frac{\hbar}{2}\left(\partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu\right) - \epsilon_{\mu\nu\alpha\beta}k^\alpha\mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} = 0,$

The imaginary parts:

$\hbar\partial^\mu\mathcal{V}_\mu = 0,$
$k^\mu\mathcal{A}_\mu = 0,$
$\frac{\hbar}{2}\partial_\mu\mathcal{F} + k^\nu\mathcal{S}_{\nu\mu} = 0,$
$k_\mu\mathcal{P} + \frac{\hbar}{2}\partial^\beta\tilde{\mathcal{S}}_{\mu\beta} = 0,$
$(k_\mu\mathcal{V}_\nu - k_\nu\mathcal{V}_\mu) + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha\mathcal{A}^\beta = 0.$

$$\mathcal{F} = \mathcal{F}^{(0)} + \hbar\mathcal{F}^{(1)} + \hbar^2\mathcal{F}^{(2)} + \cdots, \quad \mathcal{P} = \mathcal{P}^{(0)} + \hbar\mathcal{P}^{(1)} + \hbar^2\mathcal{P}^{(2)} + \cdots,$$
$$\mathcal{V}_\mu = \mathcal{V}_\mu^{(0)} + \hbar\mathcal{V}_\mu^{(1)} + \hbar^2\mathcal{V}_\mu^{(2)} + \cdots, \quad \mathcal{A}_\mu = \mathcal{A}_\mu^{(0)} + \hbar\mathcal{A}_\mu^{(1)} + \hbar^2\mathcal{A}_\mu^{(2)} + \cdots,$$
$$\mathcal{S}_{\mu\nu} = \mathcal{S}_{\mu\nu}^{(0)} + \hbar\mathcal{S}_{\mu\nu}^{(1)} + \hbar^2\mathcal{S}_{\mu\nu}^{(2)} + \cdots.$$

Zeroth and first order real parts

Zeroth order real part:

$$k^\mu \mathcal{V}^{(0)}_\mu - m\mathcal{F}^{(0)} = 0,$$
$$\mathcal{P}^{(0)} = 0,$$
$$k_\mu \mathcal{F}^{(0)} - m\mathcal{V}^{(0)}_\mu = 0,$$
$$k^\beta \tilde{\mathcal{S}}^{(0)}_{\mu\beta} + m\mathcal{A}^{(0)}_\mu = 0,$$
$$\epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}^\beta_{(0)} + m\mathcal{S}^{(0)}_{\mu\nu} = 0,$$

Zeroth order imaginary part:

$$k^\mu \mathcal{A}^{(0)}_\mu = 0,$$
$$k^\nu \mathcal{S}^{(0)}_{\nu\mu} = 0,$$
$$k_\mu \mathcal{V}^{(0)}_\nu - k_\nu \mathcal{V}^{(0)}_\mu = 0.$$

First order real part:

$$k^\mu \mathcal{V}^{(1)}_\mu - m\mathcal{F}^{(1)} = 0,$$
$$\frac{1}{2}\partial^\mu \mathcal{A}^{(0)}_\mu + m\mathcal{P}^{(1)} = 0,$$
$$k_\mu \mathcal{F}^{(1)} - \frac{1}{2}\partial^\nu \mathcal{S}^{(0)}_{\nu\mu} - m\mathcal{V}^{(1)}_\mu = 0,$$
$$-\frac{1}{2}\partial_\mu \mathcal{P}_{(0)} + k^\beta \tilde{\mathcal{S}}^{(1)}_{\mu\beta} + m\mathcal{A}^{(1)}_\mu = 0,$$
$$\frac{1}{2}\left(\partial_\mu \mathcal{V}^{(0)}_\nu - \partial_\nu \mathcal{V}^{(0)}_\mu\right) - \epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}^\beta_{(1)} - m\mathcal{S}^{(1)}_{\mu\nu} = 0.$$

First order imaginary part:

$$\partial^\mu \mathcal{V}^{(0)}_\mu = 0,$$
$$k^\mu \mathcal{A}^{(1)}_\mu = 0,$$
$$\frac{1}{2}\partial_\mu \mathcal{F}^{(0)} + k^\nu \mathcal{S}^{(1)}_{\nu\mu} = 0,$$
$$k_\mu \mathcal{P}^{(1)} + \frac{1}{2}\partial^\beta \tilde{\mathcal{S}}^{(0)}_{\mu\beta} = 0,$$
$$k_\mu \mathcal{V}^{(1)}_\nu - k_\nu \mathcal{V}^{(1)}_\mu + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\partial^\alpha \mathcal{A}^\beta_{(0)} = 0.$$

Kinetic equations for coefficient functions

By solving the coupled equations for zeroth and first order one get the following equations for the coefficient functions

$$k^\mu \partial_\mu \mathcal{F}_{(0)}(x,k) = 0.$$

$$k^\mu \partial_\mu \, \mathcal{A}^\nu_{(0)}(x,k) = 0, \quad k_\nu \, \mathcal{A}^\nu_{(0)}(x,k) = 0.$$

To get the equation for the first order coefficient functions one has to go beyond first order (upto second order) if one does so one can easily show,

$$k^\mu \partial_\mu \mathcal{F}_{(1)}(x,k) = 0,$$

$$k^\mu \partial_\mu \mathcal{A}^\nu_{(1)}(x,k) = 0, \quad k_\nu \mathcal{A}^\nu_{(1)}(x,k) = 0.$$

One can define explicitly the two cases:

CASE 1:

$$\begin{aligned} \mathcal{F}^{(0)} &= \mathcal{F}_{\rm eq}, \\ \mathcal{P}^{(0)} &= 0, \\ \mathcal{V}^{(0)}_\mu &= \mathcal{V}_{{\rm eq},\mu}, \\ \mathcal{A}^{(0)}_\mu &= \mathcal{A}_{{\rm eq},\mu}, \\ \mathcal{S}^{(0)}_{\mu\nu} &= \mathcal{S}_{{\rm eq},\mu\nu}. \end{aligned}$$

CASE 2:

$$\begin{aligned} \mathcal{F}^{(0)} &= F_{\rm eq}, \quad \mathcal{F}^{(1)} = 0, \\ \mathcal{P}^{(0)} &= 0, \quad \mathcal{P}^{(1)} = 0, \\ \mathcal{V}^{(0)}_\mu &= V_{{\rm eq},\mu}, \quad \mathcal{V}^{(1)}_\mu = 0, \\ \mathcal{A}^{(0)}_\mu &= 0, \quad \mathcal{A}^{(1)}_\mu = A_{{\rm eq},\mu}, \\ \mathcal{S}^{(0)}_{\mu\nu} &= 0, \quad \mathcal{S}^{(1)}_{\mu\nu} = S_{{\rm eq},\mu\nu}. \end{aligned}$$

Extended global equilibrium

In the CASE 1 we find:

$$k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0,$$

$$k^\mu \partial_\mu \, \mathcal{A}^\nu_{\rm eq}(x,k) = 0, \quad k_\nu \, \mathcal{A}^\nu_{\rm eq}(x,k) = 0.$$

These equations will be exactly fulfilled if the β_μ field satisfies

$$\partial_\mu \beta_\nu(x) + \partial_\nu \beta_\mu(x) = 0 \quad \text{(Killing equation)}$$

while the parameter ξ and $\omega_{\mu\nu}$ are constant (this also means ζ is constant).
Solution to the Killing equaton

$$\beta_\mu(x) = \beta^0_\mu + \varpi^0_{\mu\nu} x^\nu, \qquad \beta^0_\mu\text{=constt}, \;\; \varpi^0_{\mu\nu} = -\tfrac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right) = \varpi_{\mu\nu} \text{ =constt}$$

- It does not constrain that the spin polarization tensor is equal to thermal vorticity.

In the CASE 2 we find:

$$k^\mu \partial_\mu F_{\rm eq}(x,k) = 0,$$

$$k^\mu \partial_\mu \, A^\nu_{\rm eq}(x,k) = 0, \quad k_\nu \, A^\nu_{\rm eq}(x,k) = 0,$$

which leads to the same constraints on β_μ, ξ, and $\omega_{\mu\nu}$, as in the CASE 1.

Charge current

We use the definition of the charge current introduced in Ref. [de-Groot:1980] to obtain

$$N^\alpha_{\rm eq}(x) = {\rm tr} \int {\rm d}^4{\rm k}\, \gamma^\alpha\, \mathcal{W}_{\rm eq}({\rm x},{\rm k}) = \int {\rm d}^4{\rm k}\, \mathcal{V}^\alpha_{\rm eq}({\rm x},{\rm k}) = \frac{1}{\rm m} \int {\rm d}^4{\rm k}\, {\rm k}^\alpha \mathcal{F}_{\rm eq}({\rm x},{\rm k})$$

$$N^\alpha_{\rm eq} = 4 \cosh(\zeta) \sinh(\xi) \int \frac{d^3p}{(2\pi)^3 E_p}\, p^\alpha\, e^{-\beta\cdot p}.$$

This agrees with [W. Florkowski et. al. 2017]. Doing the integral over the momentum one finds that the charge current is proportional to the flow vector,

$$N^\alpha_{\rm eq} = n u^\alpha,$$

where

$$n = 4\, \cosh(\zeta) \sinh(\xi)\, n_{(0)}(T)$$

Here $n_{(0)}(T) = \langle (u\cdot p)\rangle_0$ is the number density of spin-0, neutral Boltzmann particles,

$$\langle \cdots \rangle_0 \equiv \int \frac{d^3p}{(2\pi)^3 E_p} (\cdots)\, e^{-\beta\cdot p}.$$

We note that the form of $N^\alpha_{\rm eq}(x)$ holds generally for CASE 1, the result for CASE 2 is obtained by taking the limit $\zeta \to 0$, hence, by replacing $\cosh(\zeta)$ with unity.

$$\partial_\alpha N^\alpha_{\rm eq}(x) = 0.$$

Energy-momentum tensor

The energy-momentum tensor defined in Ref.[de-Groot:1980] has the form

$$T^{\mu\nu}_{\rm eq}(x) = \frac{1}{m}{\rm tr}\int {\rm d}^4{\rm k}\,{\rm k}^\mu\,{\rm k}^\nu \mathcal{W}_{\rm eq}({\rm x},{\rm k}) = \frac{1}{\rm m}\int {\rm d}^4{\rm k}\,{\rm k}^\mu\,{\rm k}^\nu \mathcal{F}_{\rm eq}({\rm x},{\rm k}).$$

From above equation one can easily obtain

$$T^{\mu\nu}_{\rm eq}(x) = 4\cosh(\zeta)\cosh(\xi)\int \frac{d^3p}{(2\pi)^3E_p}p^\mu p^\nu e^{-\beta\cdot p}.$$

This agrees again with that given in [W. Florkowski et al. 2017].

$$T^{\mu\nu}_{\rm eq}(x) = (\varepsilon+P)u^\mu u^\nu - Pg^{\mu\nu}.$$

The energy density and pressure are given by the expressions

$$\varepsilon = 4\,\cosh(\zeta)\cosh(\xi)\,\varepsilon_{(0)}(T), \quad P = 4\,\cosh(\zeta)\cosh(\xi)\,P_{(0)}(T),$$

where, $\varepsilon_{(0)}(T) = \langle (u\cdot p)^2\rangle_0$ and $P_{(0)}(T) = -(1/3)\langle [p\cdot p-(u\cdot p)^2]\rangle_0$.
Similarly to the case of the charge current, the form of $T^{\mu\nu}_{\rm eq}(x)$ holds in general for CASE 1, while the result for CASE 2 is obtained by the limit $\cosh(\zeta)\to 1$.
The energy-momentum tensor should be conserved,

$$\partial_\alpha T^{\alpha\beta}_{\rm eq}(x) = 0.$$

1. Canonical form

$$S^{\lambda,\mu\nu}_{\rm can}(x) = \frac{1}{4}\int d^4k \,\mathrm{tr}\left[\left\{\sigma^{\mu\nu},\gamma^{\lambda}\right\}\mathcal{W}_{\rm eq}(x,k)\right] = \frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}\int d^4k\,\mathcal{A}_{{\rm eq},\kappa}(x,k)$$

$$\begin{aligned} S^{\lambda,\mu\nu}_{\rm can} &= \frac{\sinh(\zeta)\cosh(\xi)}{\zeta}\int dP\, e^{-\beta\cdot p}\left(\omega^{\mu\nu}p^{\lambda} + \omega^{\nu\lambda}p^{\mu} + \omega^{\lambda\mu}p^{\nu}\right) \\ &= \frac{w}{4\zeta}\left(u^{\lambda}\omega^{\mu\nu} + u^{\mu}\omega^{\nu\lambda} + u^{\nu}\omega^{\lambda\mu}\right), \end{aligned}$$

where we have introduced the spin density w defined by the expression

$$w = 4\sinh(\zeta)\cosh(\xi)n_{(0)}(T).$$

Limit $\frac{\sinh(\zeta)}{\zeta} \to 1$ will give canonical spin tensor for the CASE 2.
Total angular momentum

$$\hat{J}^{\mu,\alpha\beta}(x) = L^{\mu,\alpha\beta}(x) + S^{\mu,\alpha\beta}(x) = x^{\alpha}T^{\mu\beta}(x) - x^{\beta}T^{\mu\alpha} + S^{\mu,\alpha\beta}(x).$$

Conservation of energy momentum and total angular momentum implies

$$\partial_{\mu}T^{\mu\nu}(x) = 0, \quad \partial_{\lambda}J^{\lambda,\mu\nu}(x) = 0, \Rightarrow \partial_{\lambda}S^{\lambda,\mu\nu}(x) = T^{\mu\nu}(x) - T^{\nu\mu}(x) = 0.$$

We expect

$$\partial_{\lambda}S^{\lambda,\mu\nu}_{\rm can}(x) = 0.$$

Case of global equilibrium with rigid rotation

In this case, the flow four-vector u^μ can be taken in the form

$$u^0 = \gamma, \quad u^1 = -2\,\Omega_0 \gamma\, y, \quad u^2 = 2\,\Omega_0 \gamma\, x, \quad u^3 = 0,$$

$\gamma = 1/\sqrt{1 - 4\Omega_0^2 r^2}$ is the Lorentz gamma factor and $r = \sqrt{x^2 + y^2}$ is distance from the center of the vortex in the transverse plane.

One can check that such a hydrodynamic configuration leads to the conserved charge current and conserved energy-momentum tensor.

if,

$$T = T_0 \gamma, \quad \mu = \mu_0 \gamma, \Omega = \Omega_0 \gamma$$

where, T_0, μ_0 and Ω_0 are constants.

In this case all components of the spin polarization tensor $\omega^{\mu\nu}$ vanish, except for $\omega^{12} = -\omega^{21} = 2\Omega_0/T_0$.

The canonical spin tensor $S_{\rm can}^{\lambda,\mu\nu}(x)$ is not conserved,

$$\partial_\lambda S_{\rm can}^{\lambda,01}(r) = -\partial_y \left[\frac{w(r)}{4\zeta_0} u^0(r) \right] \omega^{21} \neq 0,$$

2. Phenomenological form

A phenomenological form of the spin tensor has been used [W. Florkowski et al. 2017],

$$S^{\lambda,\mu\nu}_{\rm ph}(x) = \frac{1}{2}\int dP\, e^{-\beta\cdot p}\, p^{\lambda}\,{\rm tr}[(X^{+} - X^{-})\Sigma^{\mu\nu}].$$

Carrying out the trace calculation we get

$$S^{\lambda,\mu\nu}_{\rm ph}(x) = \frac{\sinh(\zeta)\cosh(\xi)}{\zeta}\int dP\, e^{-\beta\cdot p}\, p^{\lambda}\,\omega^{\mu\nu} = \frac{w}{4\zeta}u^{\lambda}\omega^{\mu\nu},$$

which agrees with the expression used in Ref. [Becattini 2009].

One can easily check that in global equilibrium unlike canonical the phenomenological spin tensor satisfies the conservation law,

$$\partial_{\lambda} S^{\lambda,\mu\nu}_{\rm ph}(x) = 0$$

Which is consistent with the conservation laws of charge current and energy momentum tensor and with the concept of the global equilibrium.

3. de Groot - van Leeuwen - van Weert formulation

The spin tensor introduced by de Groot, van Leeuwen, and van Weert has the form

$$S^{\lambda,\mu\nu}_{\rm GLW} = \frac{1}{4}\int d^4k\, \mathrm{tr}\left[\left(\left\{\sigma^{\mu\nu},\gamma^{\lambda}\right\} + \frac{2\mathrm{i}}{\mathrm{m}}\left(\gamma^{[\mu}\mathrm{k}^{\nu]}\gamma^{\lambda} - \gamma^{\lambda}\gamma^{[\mu}\mathrm{k}^{\nu]}\right)\right)\mathcal{W}_{\rm eq}(\mathrm{x},\mathrm{k})\right].$$

Performing the traces, and then carrying out the integration over k we get

$$\begin{aligned} S^{\lambda,\mu\nu}_{\rm GLW} &= \frac{\sinh(\zeta)\cosh(\xi)}{m^2\zeta}\int dP\, e^{-\beta\cdot p} p^{\lambda}\left(m^2\omega^{\mu\nu} + 2p^{\alpha}p^{[\mu}\omega^{\nu]}{}_{\alpha}\right) \\ &= \frac{w}{4\zeta}u^{\lambda}\omega^{\mu\nu} + \frac{2\sinh(\zeta)\cosh(\xi)}{m^2\zeta} s^{\lambda,\mu\nu}_{\rm GLW} \end{aligned}$$

where

$$s^{\lambda,\mu\nu}_{\rm GLW} = A u^{\lambda}u^{\alpha}u^{[\mu}\omega^{\nu]}{}_{\alpha} + B\left(\Delta^{\lambda\alpha}u^{[\mu}\omega^{\nu]}{}_{\alpha} + u^{\lambda}\Delta^{\alpha[\mu}\omega^{\nu]}{}_{\alpha} + u^{\alpha}\Delta^{\lambda[\mu}\omega^{\nu]}{}_{\alpha}\right)$$

and

$$B = -\frac{1}{\beta}\left(\epsilon_{(0)} + P_{(0)}\right), \quad A = \frac{1}{\beta}\left[3\epsilon_{(0)} + \left(3 + \frac{m^2}{T^2}\right)P_{(0)}\right] = -3B + \frac{m^2}{T}P_{(0)}. \qquad (1)$$

Limit $\frac{\sinh(\zeta)}{\zeta} \rightarrow 1$ will give the GLW spin tensor for the CASE 2.

Conservation laws

One can get the conservation law for charge and energy-momentum by taking zeroth and first moments of the kinetic equations,

$$k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0$$

or,

$$k^\mu \partial_\mu F_{\rm eq}(x,k) = 0.$$

The main difference between the two is that the moment of later does not involves the vorticity therefore will lead to perfect fluid description. Spin dynamics gets decoupled in this case.
In any case, the conservation laws for charge, energy, and momentum are not sufficient to determine the dynamics of spin and they should be supplemented by information coming from the equations for the axial coefficients of the equilibrium Wigner function.

$$k^\alpha \partial_\alpha \mathcal{A}^\mu_{\rm eq}(x,k) = 0,$$

$$k^\alpha \partial_\alpha A^\mu_{\rm eq}(x,k) = 0.$$

Conservation laws

Considering the first kinetic equation for the axial coefficient function, we can obtain the conservation laws for spin tensor in the way as follows,

$$k^\alpha \partial_\alpha \int dP\, e^{-\beta\cdot p} \frac{\sinh(\zeta)}{\zeta} \left[\delta^{(4)}(k-p)e^{\xi} + \delta^{(4)}(k+p)e^{-\xi}\right] \tilde{\omega}_{\mu\nu}\, p^\nu = 0$$

$$\frac{k^\alpha}{2}\epsilon_{\mu\nu\rho\sigma}\left\{\partial_\alpha \int dP\, e^{-\beta\cdot p} \frac{\sinh(\zeta)}{\zeta} \left[\delta^{(4)}(k-p)e^{\xi} + \delta^{(4)}(k+p)e^{-\xi}\right] p^\nu\, \omega^{\rho\sigma}\right\} = 0.$$

If we multiply above equation by $k^\eta \epsilon^{\mu\gamma\lambda\eta}$ and then take the moment of above equation. We will get the conservation of GLW spin tensor.

This suggest that GLW spin tensor, in fact, a more natural choice.

Limit $\frac{\sinh(\zeta)}{\zeta} \to 1$ will give conservation of GLW spin tensor for the second kinetic equation for the axial coefficient function (CASE 2).

Summary

- We have discussed about the Wigner function (constructed from the local equilibrium phase space distribution functions for spin-1/2) and it's spinor decomposition.
- We have analyzed in more detail the case where the Wigner function satisfy kinetic equation with a vanishing collision term.
- We have found, in contrast to many earlier claims found in the literature, Wigner function approach does not imply a direct relation between the thermal vorticity and spin polarization, except for the fact that the two should be constant in global equilibrium.
- We have also outlined procedures to formulate hydrodynamics with spin from the kinetic equations derived from Wigner function.
- We have found that it would be useful to construct the hydrodynamics with spin with the help of the spin tensors derived by de Groot, van Leeuwen, and van Weert.

Future Plan: Our next task is to incorporate electromagnetic fields and collision term in the present framework.

THANK YOU

Back up slides

Global thermodynamic equilibrium (Zubarev, Becattini)

$$\hat{\rho}(t) = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x) - \xi(x)\hat{N}^\mu\right)\right].$$

Here $d^3\Sigma_\mu$ is an element of a space-like, three-dimensional hypersurface Σ_μ. We can take it as, $d^3\Sigma_\mu = (dV, 0, 0, 0)$.
The operators $\hat{T}^{\mu\nu}(x)$, $\hat{J}^{\mu,\alpha\beta}(x)$ and $\hat{N}^\mu(x)$ are the energy-momentum, angular momentum and charge operators respectively.
In global thermodynamic equilibrium the operator $\hat{\rho}(t)$ should be independent of time.

$$\partial_\mu\left(\hat{T}^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x) - \xi(x)\hat{N}^\mu\right)$$
$$= \hat{T}^{\mu\nu}(x)\left(\partial_\mu b_\nu(x)\right) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\left(\partial_\mu\omega_{\alpha\beta}(x)\right) - \partial_\mu\xi(x) = 0.$$

For asymmetric energy momentum tensor, $b_\nu = b^0_\nu$, $\omega_{\alpha\beta} = \omega^0_{\alpha\beta}$, $\xi = \xi^0$.
For symmetric energy momentum tensor, $b_\nu = b^0_\nu + \delta\omega^0_{\nu\rho}x^\rho$, $\omega_{\alpha\beta} = \omega^0_{\alpha\beta}$, $\xi = \xi^0$.

Global and local thermodynamic equilibrium (Zubarev, Becattini)

$$\hat{J}^{\mu,\alpha\beta}(x) = \hat{L}^{\mu,\alpha\beta}(x) + \hat{S}^{\mu,\alpha\beta}(x).$$

Using above equation, we can write two cases discussed above can be expressed by a single form of the density operator

$$\hat{\rho}_{\rm EQ} = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)\beta_\nu(x) - \frac{1}{2}\hat{S}^{\mu,\alpha\beta}(x)\omega^0_{\alpha\beta} - \xi^0\hat{N}^\mu\right)\right].$$

For asymmetric energy-momentum tensor $\beta_\mu(x) = b^0_\mu + \omega^0_{\mu\gamma}x^\gamma$.
$\beta_\mu(x)$ is a Killing vector, $\omega_{\mu\gamma} = \omega^0_{\mu\gamma}$.
For symmetric energy-momentum tensor $\beta_\mu(x) = b^0_\mu + (\delta\omega^0_{\mu\gamma} + \omega^0_{\mu\gamma})x^\gamma$.
$\beta_\mu(x)$ is again a Killing vector, $\omega_{\mu\gamma} \neq \omega^0_{\mu\gamma}$.

1. **global equilibrium —** β_μ field is a Killing vector, $\varpi_{\mu\nu} = \omega_{\mu\nu} =$ constt, in addition $\xi =$ constt.
2. **extended global equilibrium —** β_μ field is a Killing vector, $\varpi_{\mu\nu} =$ constt, $\omega_{\mu\nu} =$ constt but $\varpi_{\mu\nu} \neq \omega_{\mu\nu}$, in addition $\xi =$ constt.
3. **local equilibrium —** β_μ field is not a Killing vector but we still have $\omega_{\mu\nu}(x) = \varpi_{\mu\nu}(x)$, $\xi = \xi(x)$,
4. **extended local equilibrium —** β_μ field is not a Killing vector and $\omega_{\mu\nu}(x) \neq \varpi_{\mu\nu}(x)$, $\xi = \xi(x)$.