A new family of exact solutions of relativistic hydrodynamics

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Introduction and motivation

A New Family of Exact Solutions of Relativistic Hydro

Rapidity and pseudorapidity distributions

R_{long} HBT radius

Outlook to other presentations

Conclusions, summary

Partially supported by NKTIH FK 123842 and FK123959
and EFOP 3.6.1-16-2016-00001

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arXiv.org:1805.01427 +

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Context

Renowned exact solutions, reviewed in arXiv:1805.01427

- Landau-Khalatnikov solution: dn/dy ~ Gaussian
- Hwa solution (1974) – Bjorken: same solution + ε_0 (1983)
- Chiu, Sudarshan and Wang: plateaux, Wong: Landau revisited
- Revival of interest:
  - Zimányi, Bondorf, Garpman (1978)
  - Buda-Lund model + exact solutions (1994-96)
  - Biró, Karpenko, Sinyukov, Pratt (2007)
  - Bialas, Janik, Peschanski, Borsch+Zhdanov (2007)
  - CsT, Csanád, Nagy (2007-2008)
  - CsT, Csernai, Grassi, Hama, Kodama (2004)
  - Gubser (2010-11)
  - Hatta, Noronha, Xiao (2014-16)

New simple solutions → Evaluation of measurables (Jiang's talk)

Rapidity distribution → Advanced initial energy density (Kasza)

HBT radii → Advanced life-time estimation (here)

Goal

Need for solutions that are:

- **explicit**
- **simple**
- **accelerating**
- **relativistic**
- **realistic / compatible with the data:**
  - **lattice QCD EoS**
  - **ellipsoidal symmetry (spectra, v_2, v_4, HBT)**
  - **finite dn/dy**
- **Generelization of a class that satisfies each of these criteria**
  - **but not simultaneously**
  - **T. Cs, M. I. Nagy, M. Csanád,** arXiv:nucl-th/0605070, PLB (2008)
  - **M.I. Nagy, T. Cs., M. Csanád,** arXiv:0709.3677 , PRC77:024908 (2008)
  - **M. Csanád, M. I. Nagy, T. Cs,** arXiv:0710.0327 [nucl-th] EPJ A (2008)

New family of exact solutions:

CsT, Kasza, Csanád, Jiang, arXiv.org:1805.01427

Perfect fluid hydrodynamics

Energy-momentum tensor:

$$T_{\mu\nu} = w u_\mu u_\nu - p g_{\mu\nu}$$

$$w = \varepsilon + p$$

$$\partial_\nu T^{\mu\nu} = 0$$

Relativistic

Euler equation:

$$w u^\nu \partial_\nu u^\mu = \left(g^{\mu\rho} - u^\mu u^\rho\right) \partial_\rho p$$

Energy conservation:

$$w \partial_\mu u^\mu = -u^\mu \partial_\mu \varepsilon$$

Charge conservation:

$$\sum \mu_i \partial_\mu \left(n_i u^\mu\right) = 0$$

Consequence is entropy conservation:

$$\partial_\mu \left(\sigma u^\mu\right) = 0.$$

Self-similar, ellipsoidal solutions

Publication (for example):

T. Cs, L.P.Csernai, Y. Hama, T. Kodama, Heavy Ion Phys. A 21 (2004) 73

3D spherically symmetric HUBBLE flow:

$$u^\mu = \frac{x^\mu}{\tau}$$

No acceleration:

$$u^\mu \partial_\mu u_\nu = 0.$$

Define a scaling variable for self-similarly expanding ellipsoids:

$$s = \frac{r_x^2}{\dot{X}_0^2 t^2} + \frac{r_y^2}{\dot{Y}_0^2 t^2} + \frac{r_z^2}{\dot{Z}_0^2 t^2}$$

EoS: (massive)
ideal gas

$$\epsilon = mn + \kappa p\,, \qquad p = nT.$$

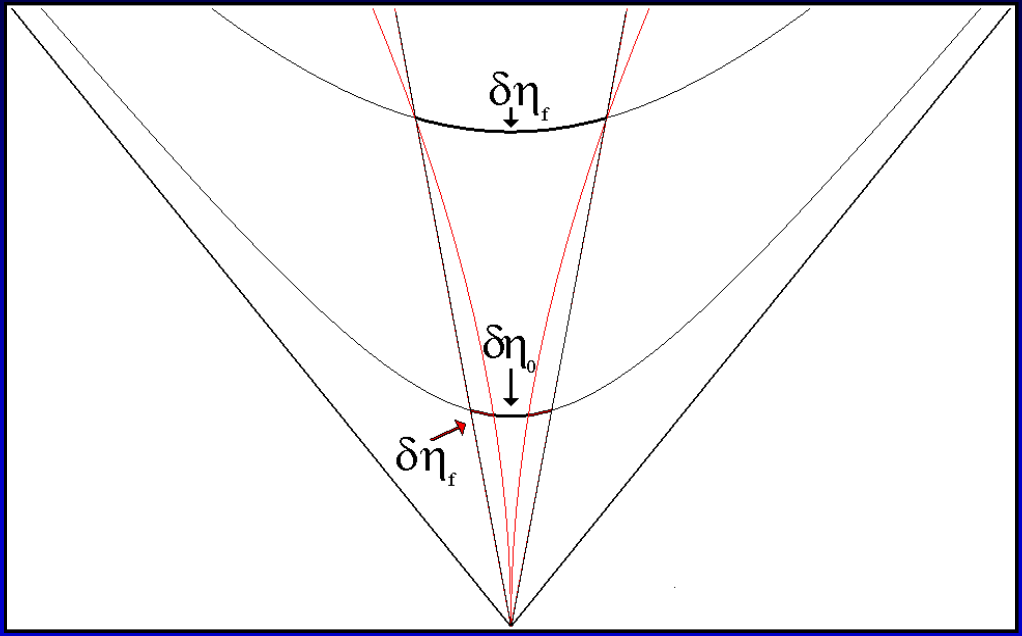
$$\varepsilon_Q = m_Q n_Q + \lambda_\varepsilon n_Q T + B, \qquad p_Q = \lambda_p n_Q T - B,$$

$$n(t,\mathbf{r}) = n_0 \left(\frac{\tau_0}{\tau}\right)^3 \mathcal{V}(s) \qquad T(t,\mathbf{r}) = T_0 \left(\frac{\tau_0}{\tau}\right)^{3/\kappa} \frac{1}{\mathcal{V}(s)} \quad \Rightarrow \quad p(t,\mathbf{r}) = p_0 \left(\frac{\tau_0}{\tau}\right)^{3+3/\kappa}$$

Scaling function $\mathcal{V}(s)$ can be chosen freely.

Shear and bulk viscous corrections in NR limit: known analytically.

Auxiliary variables: η_x, τ, Ω , η_p, y



$$\tau = \sqrt{t^2 - r_z^2},$$

$$\eta_x = \frac{1}{2}\ln\left(\frac{t+r_z}{t-r_z}\right),$$

$$\Omega = \frac{1}{2}\ln\left(\frac{1+v_z}{1-v_z}\right).$$

$$u^\mu = (\cosh(\Omega), \sinh(\Omega)),$$

$$v_z = \tanh(\Omega).$$

$$\eta_p = \frac{1}{2}\ln\left(\frac{p+p_z}{p-p_z}\right),$$

$$y = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right),$$

Consider a 1+1 dimensional, finite, expanding fireball

Assume: $\Omega = \Omega(\eta_x)$

Notation T. Cs., G. Kasza, M. Csanád, Z. Jiang, arXiv.org:1805.01427

Hydro in Rindler coordinates, new sol

$$\partial_\nu T^{\mu\nu} = 0,$$
$$\partial_\mu(\sigma u^\mu) = 0,$$

Assumptions of TCs, Kasza, Csanád and Jiang, arXiv.org:1805.01427 :

$$\Omega = \Omega(\eta_x),$$ $$\varepsilon = \kappa p,$$ $$p = \frac{T\sigma}{1+\kappa}.$$

For the entropy density, the continuity equation is solved.

From energy-momentum conservation, the
Euler and temperature equations are obtained:

$$\begin{aligned}
\partial_{\eta_x}\Omega + \kappa\left(\tau\partial_\tau + \tanh(\Omega-\eta_x)\partial_{\eta_x}\right)\ln(T) &= 0,\\
\partial_{\eta_x}\ln(T) + \tanh(\Omega-\eta_x)\left(\tau\partial_\tau\ln(T) + \partial_{\eta_x}\Omega\right) &= 0.
\end{aligned}$$

A New Family of Exact Solutions of Hydro

$$\eta_x(H) = \Omega(H) - H,$$

$$\Omega(H) = \frac{\lambda}{\sqrt{\lambda-1}\sqrt{\kappa-\lambda}} \arctan\left(\sqrt{\frac{\kappa-\lambda}{\lambda-1}}\tanh(H)\right),$$

$$\sigma(\tau,H) = \sigma_0\left(\frac{\tau_0}{\tau}\right)^{\lambda}\mathcal{V}_\sigma(s)\left[1+\frac{\kappa-1}{\lambda-1}\sinh^2(H)\right]^{-\frac{\lambda}{2}},$$

$$T(\tau,H) = T_0\left(\frac{\tau_0}{\tau}\right)^{\frac{\lambda}{\kappa}}\mathcal{T}(s)\left[1+\frac{\kappa-1}{\lambda-1}\sinh^2(H)\right]^{-\frac{\lambda}{2\kappa}},$$

$$\mathcal{T}(s) = \frac{1}{\mathcal{V}_\sigma(s)},$$

$$u_\mu \partial^\mu s = 0.$$

$$s(\tau,H) = \left(\frac{\tau_0}{\tau}\right)^{\lambda-1}\sinh(H)\left[1+\frac{\kappa-1}{\lambda-1}\sinh^2(H)\right]^{-\lambda/2}.$$

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$$s(\tau,H) = \left(\frac{\tau_0}{\tau}\right)^{\lambda-1}\sinh(H)\left[1+\frac{\kappa-1}{\lambda-1}\sinh^2(H)\right]^{-\lambda/2}.$$

New feature:
Solution is given
as *parametric curves of H* in η_x :
$(\eta_x(H),\Omega(H,\tau))$

Simlification, for now:
limit the solution in η_x
where parametric curves
correspond to *functions*

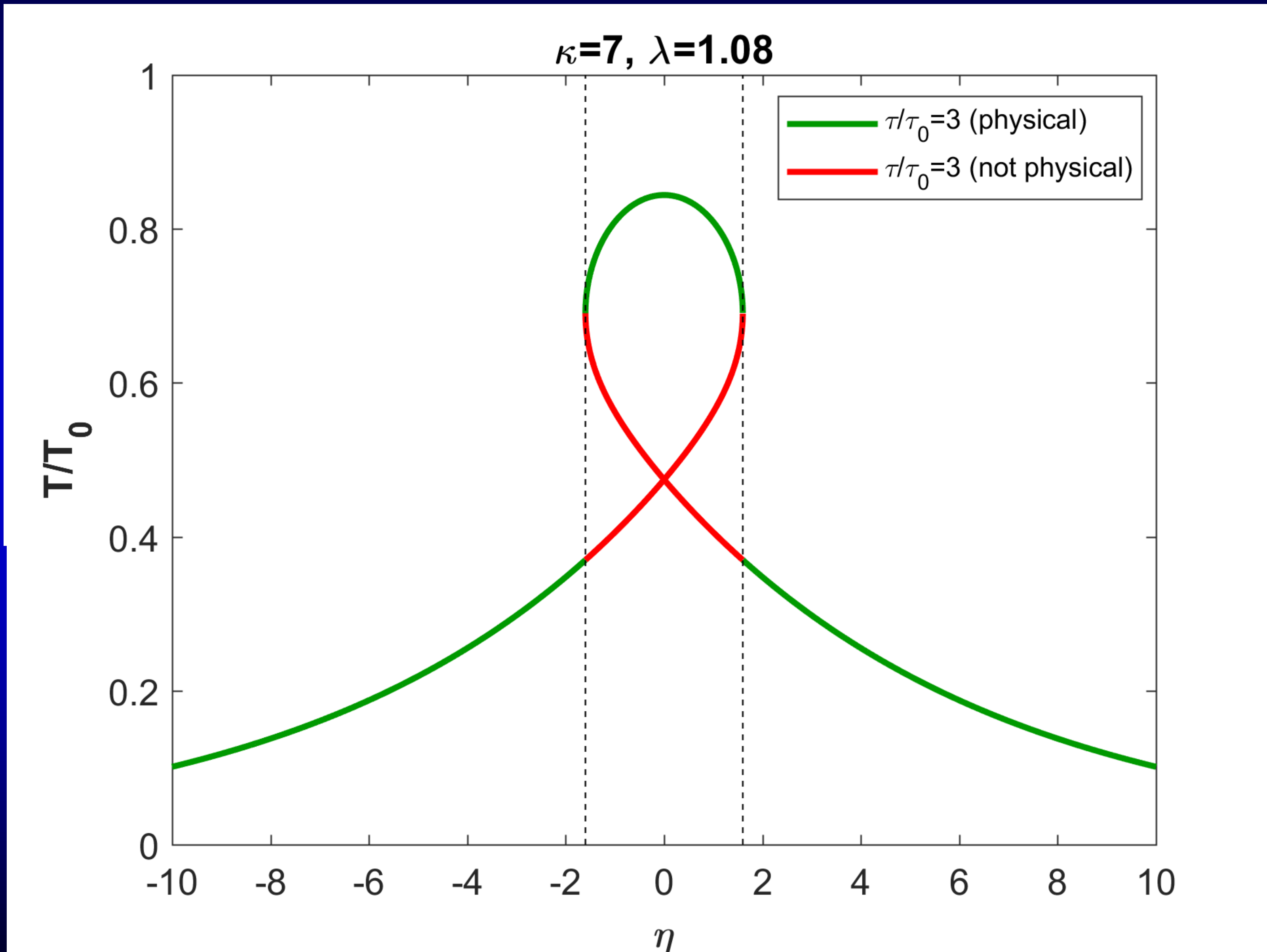
New: not discovered before,
as far as we know …

Family:
For each positive scaling
function $\mathcal{T}(s)$, a different
solution, with same T_0, s_0, κ, λ

Not self-similar:
Coordinate dependenc NOT on
scaling variable s ONLY,
butadditional dependence on
$H = H(\eta_x)$ too.

Explicit and Exact:
Fluid rapidity, temperature,
entropy density explicitly given
by formulas

Limited in space-time rapidity η_x



$h\left(\frac{1}{\sqrt{\kappa}}\right),$

$\left(\frac{1}{\sqrt{\kappa}}\right),$

$|\eta_{max}|)\Big),$

Illustration: results for T

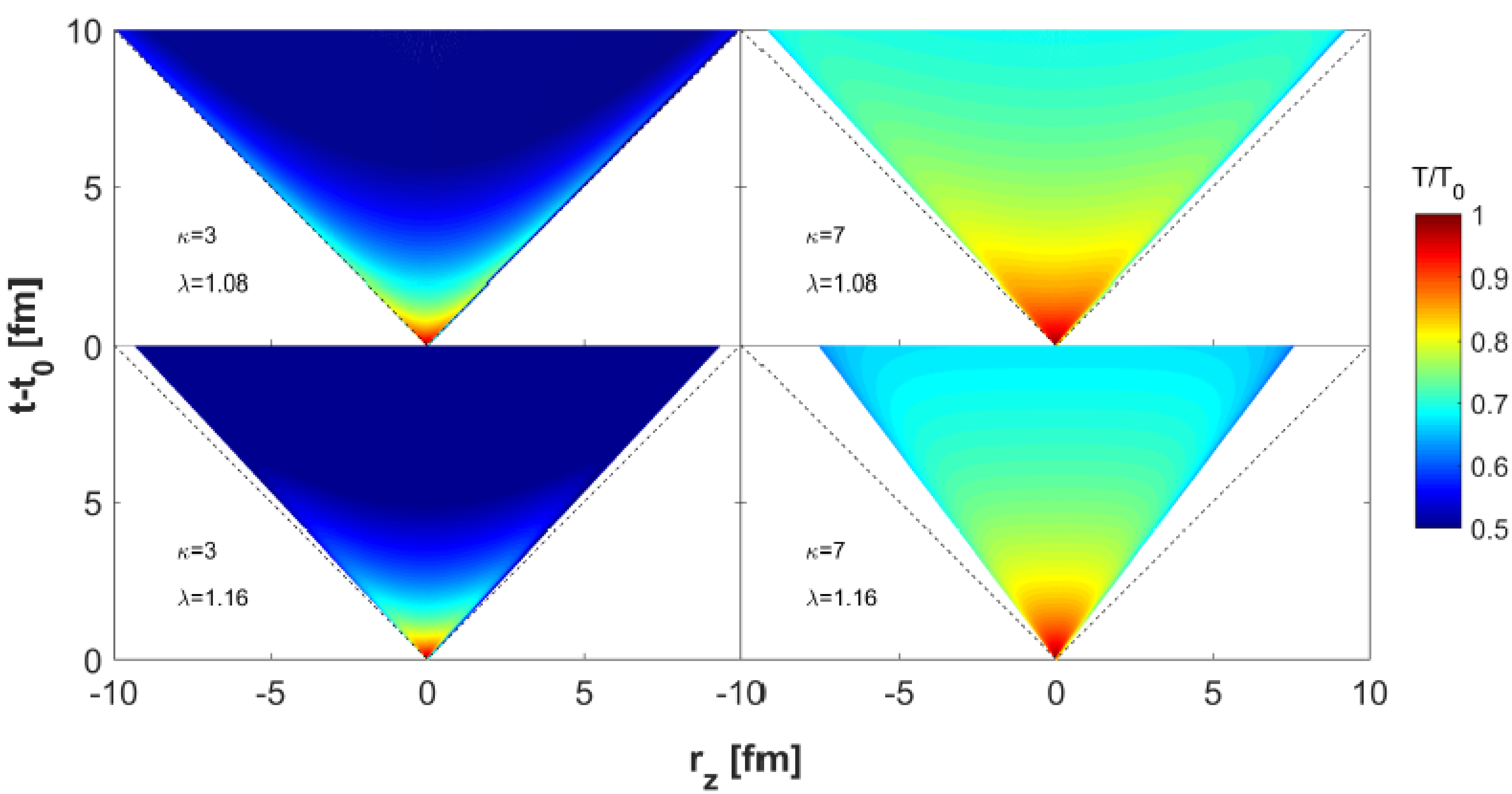


Figure 1. Temperature maps in the forward light cone from our new, longitudinally finite solutions for $\kappa = \varepsilon/p = 3$ (left column) and for $\kappa = 7$ (right column) evaluated for the acceleration parameters $\lambda = 1.08$ (top row) corresponding to a broader rapidity distribution and for $\lambda = 1.16$ (bottom row) corresponding to a narrower rapidity distribution, approximately corresponding to heavy ion collisions at $\sqrt{s_{NN}} = 200$ GeV RHIC and $\sqrt{s_{NN}} = 2.76$ TeV LHC energies.

Limited in space-time rapidity η_x

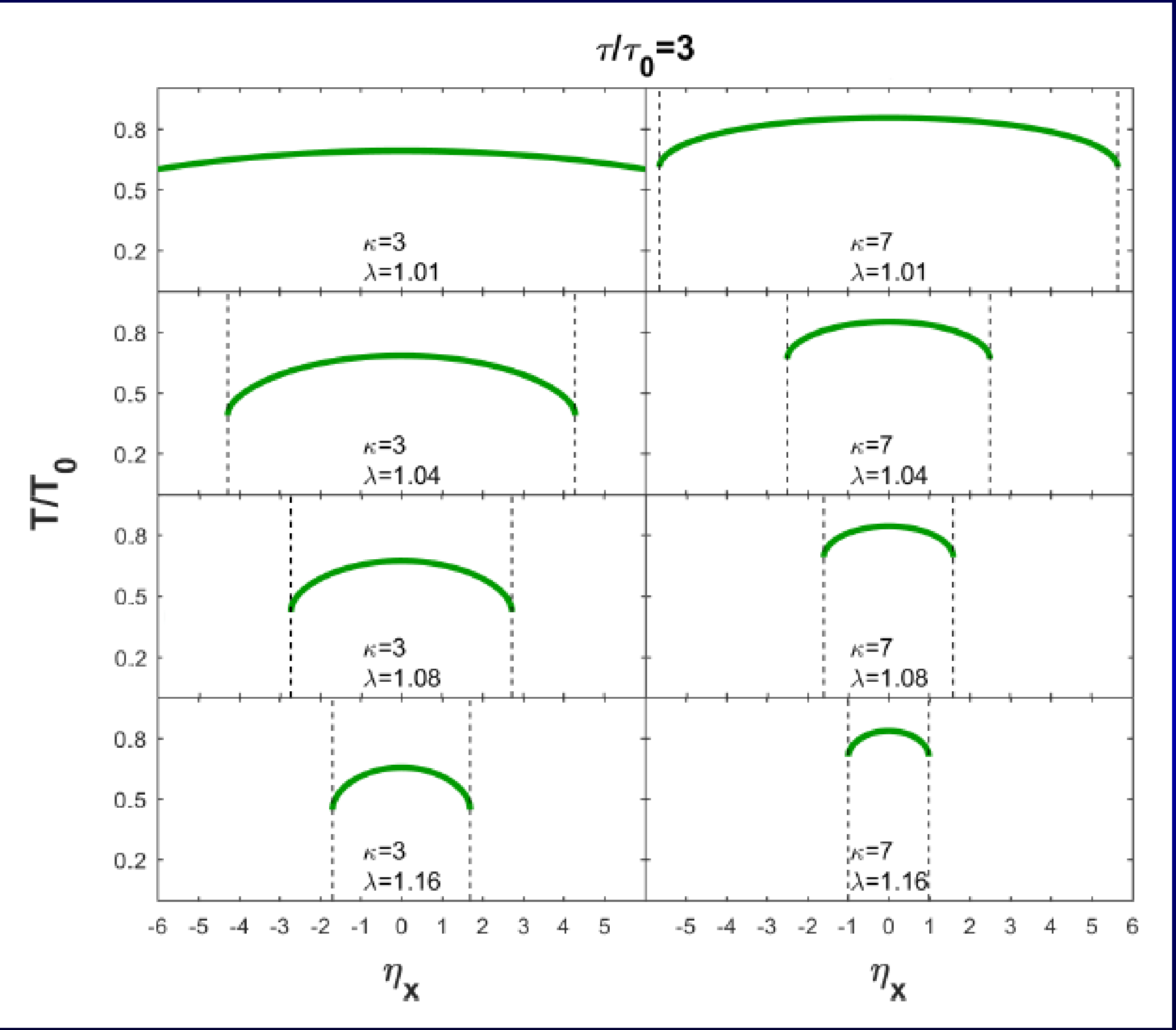


Illustration: results for fluid rapidity Ω

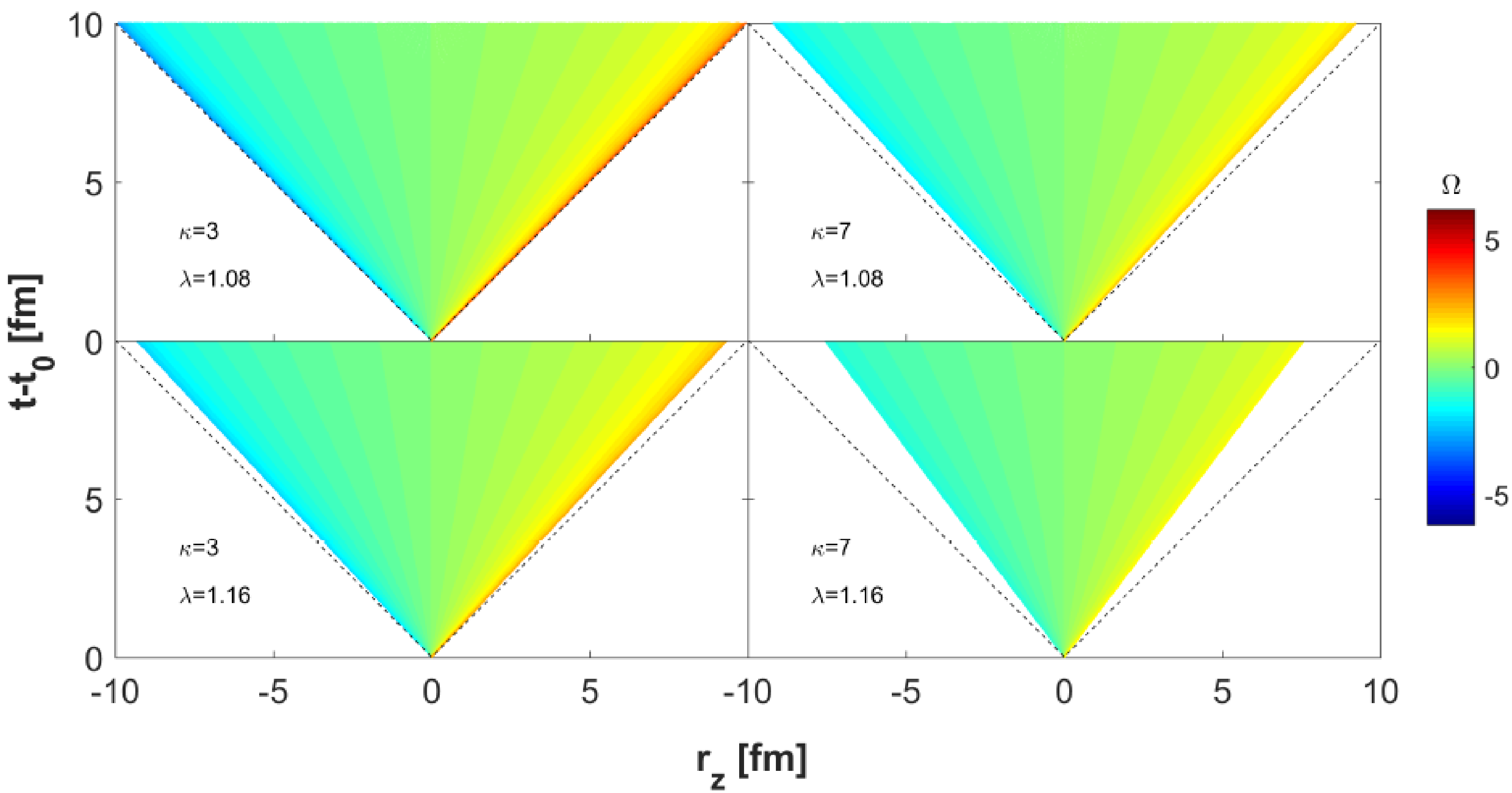
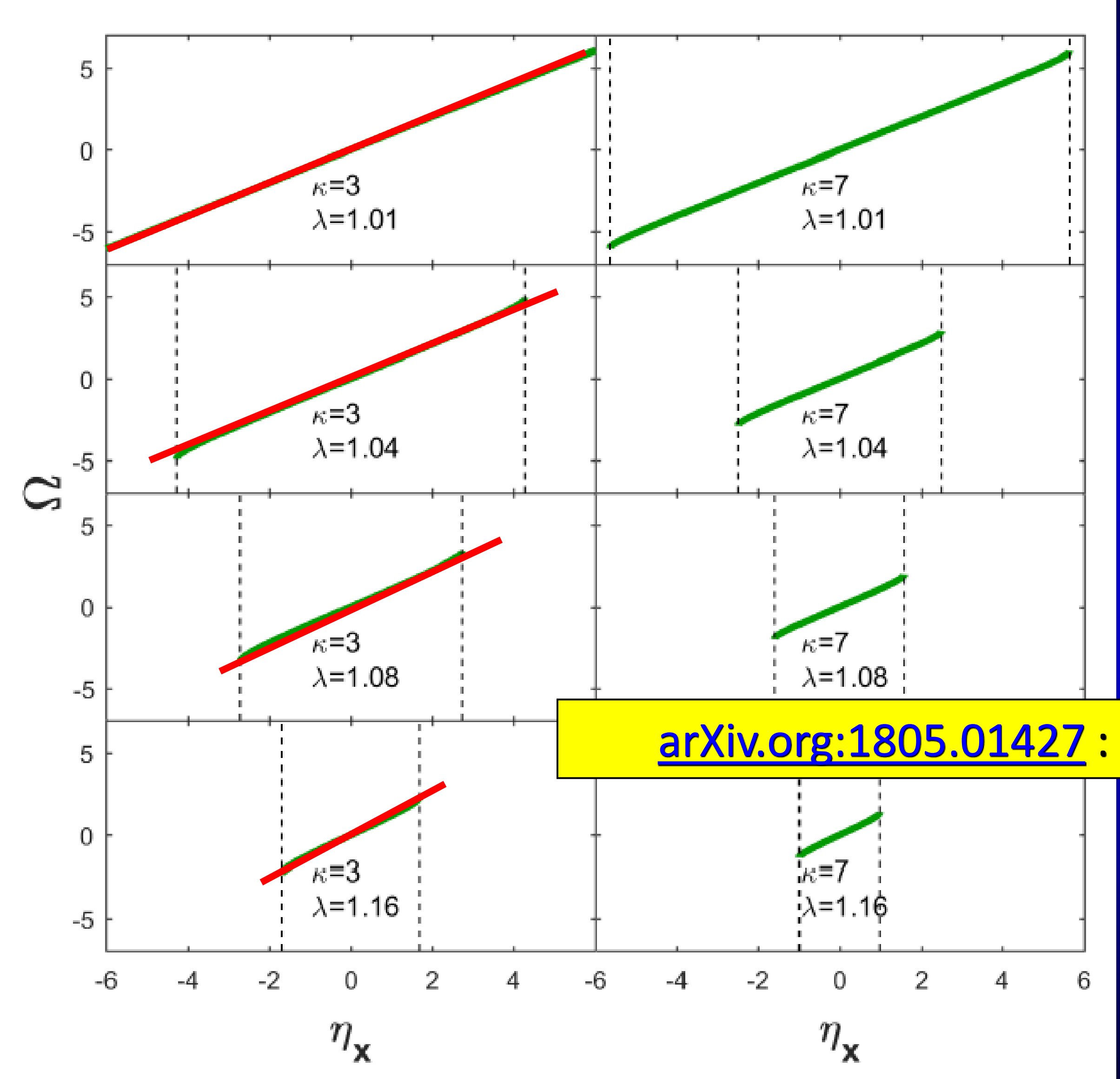


Figure 3. Fluid rapidity Ω maps in the forward light cone from our new, longitudinally finite solutions are shown for for $\kappa = \varepsilon/p = 3$ (left column) and for $\kappa = 7$ (right column) evaluated for the acceleration parameters $\lambda = 1.01$, 1.02, 1.04 and 1.08 (from top to bottom rows) corresponding to nearly flat and with increasing λ, gradually narrowing rapidity distributions.

Limited in space-time rapidity η_x



arXiv.org:1805.01427 : $\Omega \approx \lambda \eta_x$

Approximations near midrapidity

$$\frac{1-\kappa\tanh^2(\Omega-\eta_s)}{1-\tanh^2(\Omega-\eta_s)}\Omega' = \lambda,$$

$$\Rightarrow \quad \Omega' = \lambda\frac{1-\tanh^2(\Omega-\eta_s)}{1-\kappa\tanh^2(\Omega-\eta_s)},$$

$$\Rightarrow \quad \eta_s = -H + \frac{\lambda}{\sqrt{\lambda-1}\sqrt{\kappa-\lambda}}\mathrm{Arctan}\left(\sqrt{\frac{\kappa-\lambda}{\lambda-1}}\tanh(H)\right).$$

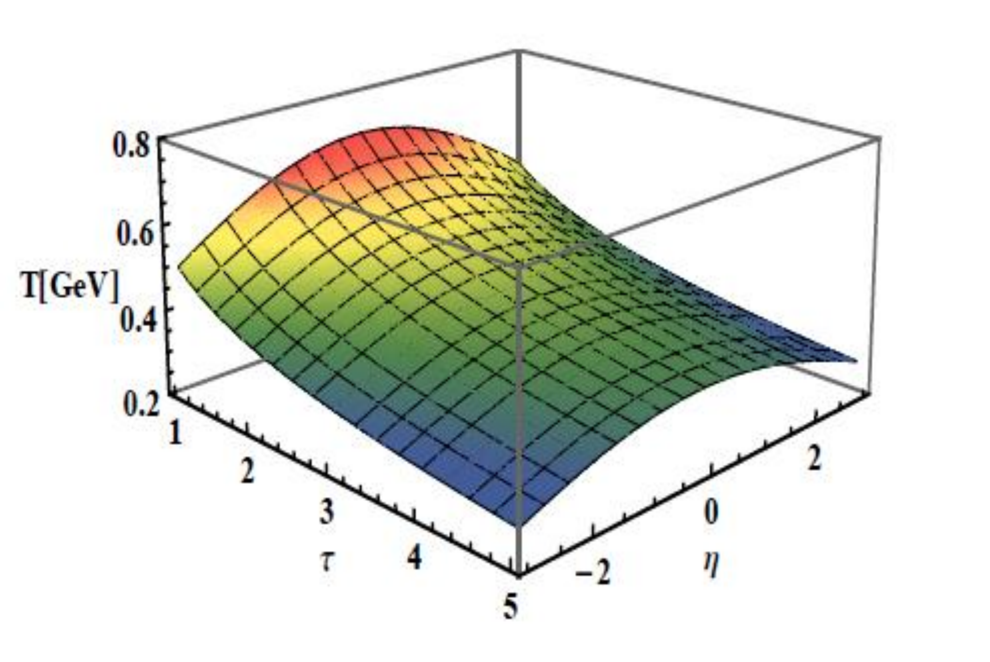
$$H = \Omega-\eta_s \ll 1,\ \kappa > 1,\ \lambda > 1,$$

$$T = T_i\left(\frac{\tilde{\tau}_0}{\tau}\right)^{\frac{\lambda}{\kappa}}\exp\left[-\frac{\lambda(\lambda-1)(\kappa-1)}{2\kappa}\eta_s^2\right],$$

$$H = \Omega-\eta_s \ll 1,$$

$$\mathrm{Arctan}\left(\sqrt{\frac{\kappa-\lambda}{\lambda-1}}\tanh(H)\right) \simeq \sqrt{\frac{\kappa-\lambda}{\lambda-1}}H,$$

$$\Rightarrow \quad H = (\lambda-1)\eta_s,$$



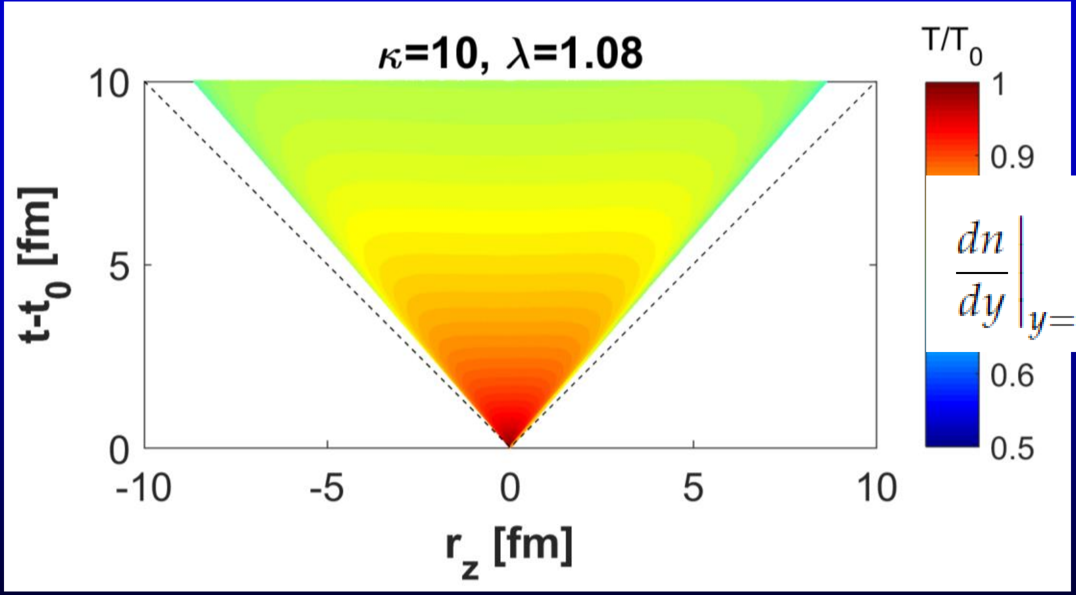
Observables: rapidity distribution

$$d\sigma^\mu = \frac{1}{A(\eta_x)} (\partial_{\eta_x} r_z, \partial_{\eta_x} t) d\eta_x,$$

$$\frac{\tau(H)}{\tau_f} = \cosh^{\frac{\kappa}{\lambda-\kappa}}(H) \left[1 + \frac{\kappa-1}{\lambda-1}\sinh^2(H)\right]^{\frac{\lambda}{2(\kappa-\lambda)}}.$$

dn/dy evaluated analytically, in a saddle-point approximation

$$\frac{dn}{dy} \approx \left.\frac{dn}{dy}\right|_{y=0} \cosh^{-\frac{1}{2}\alpha(\kappa)-1}\left(\frac{y}{\alpha(1)}\right) \exp\left(-\frac{m}{T_f}\left[\cosh^{\alpha(\kappa)}\left(\frac{y}{\alpha(1)}\right) - 1\right]\right),$$



$$\alpha(\kappa) = \frac{2\lambda-\kappa}{\lambda-\kappa}.$$

$$\left.\frac{dn}{dy}\right|_{y=0} = \frac{R^2\pi\tau_f}{(2\pi\hbar)^3}\sqrt{\frac{(2\pi T_f m)^3}{\lambda(2\lambda-1)}} \exp\left(-\frac{m}{T_f}\right),$$

Pseudorapidity distribution

dn/dη evaluated analytically, in an advanced saddle-point approximation

$$\frac{dn}{dy} \approx \left.\frac{dn}{dy}\right|_{y=0} \cosh^{-\frac{1}{2}\alpha(\kappa)-1}\left(\frac{y}{\alpha(1)}\right) \exp\left(-\frac{m}{T_f}\left[\cosh^{\alpha(\kappa)}\left(\frac{y}{\alpha(1)}\right)-1\right]\right),$$

$$\left.\frac{dn}{dy}\right|_{y=0} = \frac{R^2\pi\tau_f}{(2\pi\hbar)^3}\sqrt{\frac{(2\pi T_f m)^3}{\lambda(2\lambda-1)}} \exp\left(-\frac{m}{T_f}\right),$$

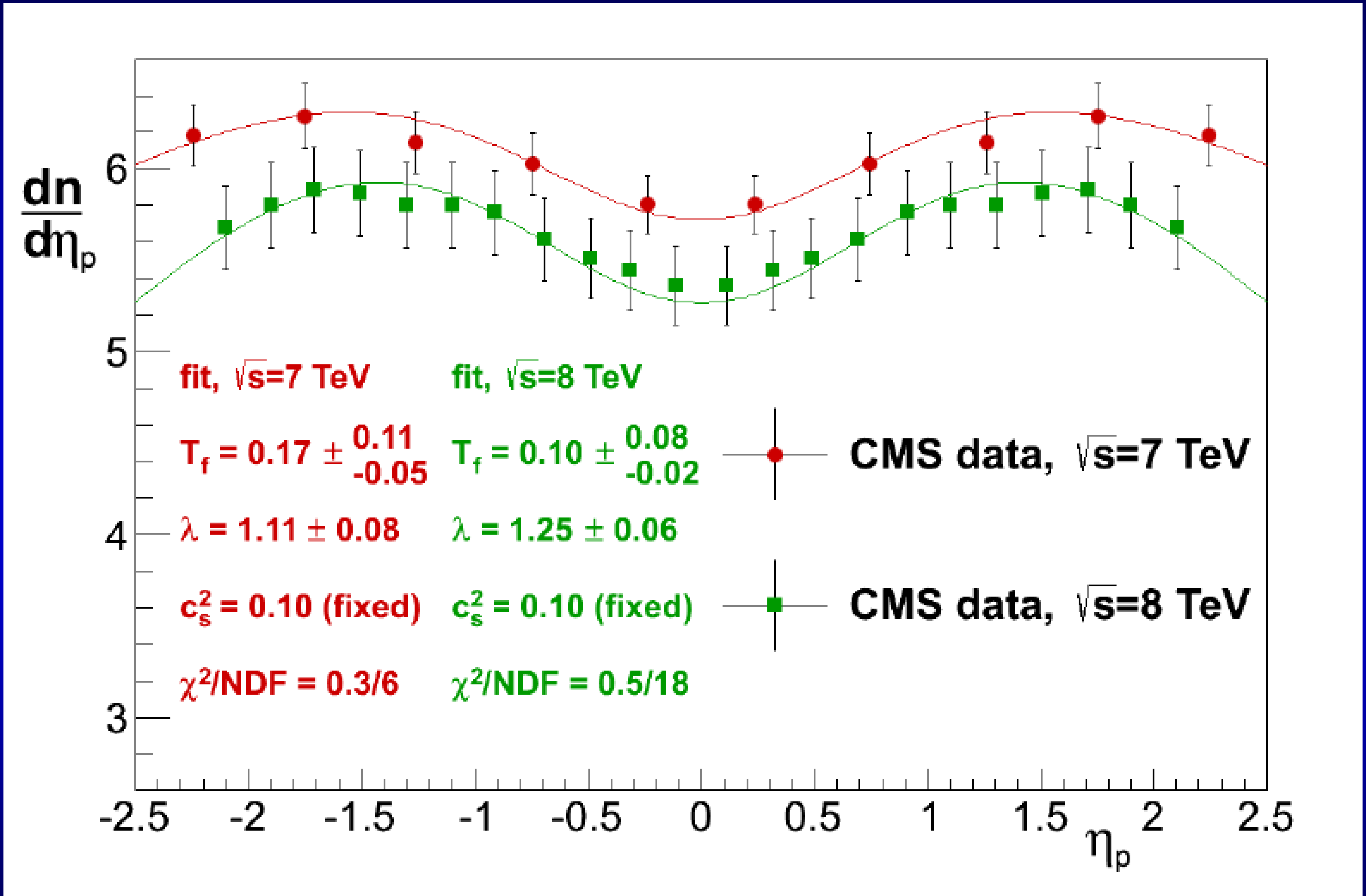
$$\alpha(\kappa) = \frac{2\lambda-\kappa}{\lambda-\kappa}.$$

$$\frac{dn}{d\eta_p} \approx \left.\frac{dn}{dy}\right|_{y=0} \frac{\langle p_T(y)\rangle \cosh(\eta_p)}{\sqrt{m^2+\langle p_T(y)\rangle^2\cosh(\eta_p)}} \cosh^{-\frac{1}{2}\alpha(\kappa)-1}\left(\frac{y}{\alpha(1)}\right) \exp\left(-\frac{m}{T_f}\left[\cosh^{\alpha(\kappa)}\left(\frac{y}{\alpha(1)}\right)-1\right]\right),$$

$$\langle p_T(y)\rangle \approx \frac{\sqrt{T_f^2+2mT_f}}{1+\frac{\alpha(\kappa)}{2\alpha(1)^2}\frac{T_f+m}{T_f+2m}y^2}.$$

An important by-product: $\langle p_T \rangle = \langle p_T(y)\rangle$ is rapidity dependent, a Lorentzian just as in Buda-Lund model

Pseudorapidity distribution



arXiv.org:1805.01427

Advanced life-time estimate

Life-time estimation: for Hwa-Bjorken type of flows

$$R_{long} = \sqrt{\frac{T_f}{m_t}}\tau_{Bj} \quad \Rightarrow \quad \tau_{Bj} = \sqrt{\frac{m_t}{T_f}}R_{long}.$$

Makhlin & Sinyukov, Z. Phys. C 39, 69 (1988)

Underestimates lifetime (Renk, CsT, Wiedemann, Pratt, ...)

Advanced CNC life-time estimate: arXiv:0710.0327

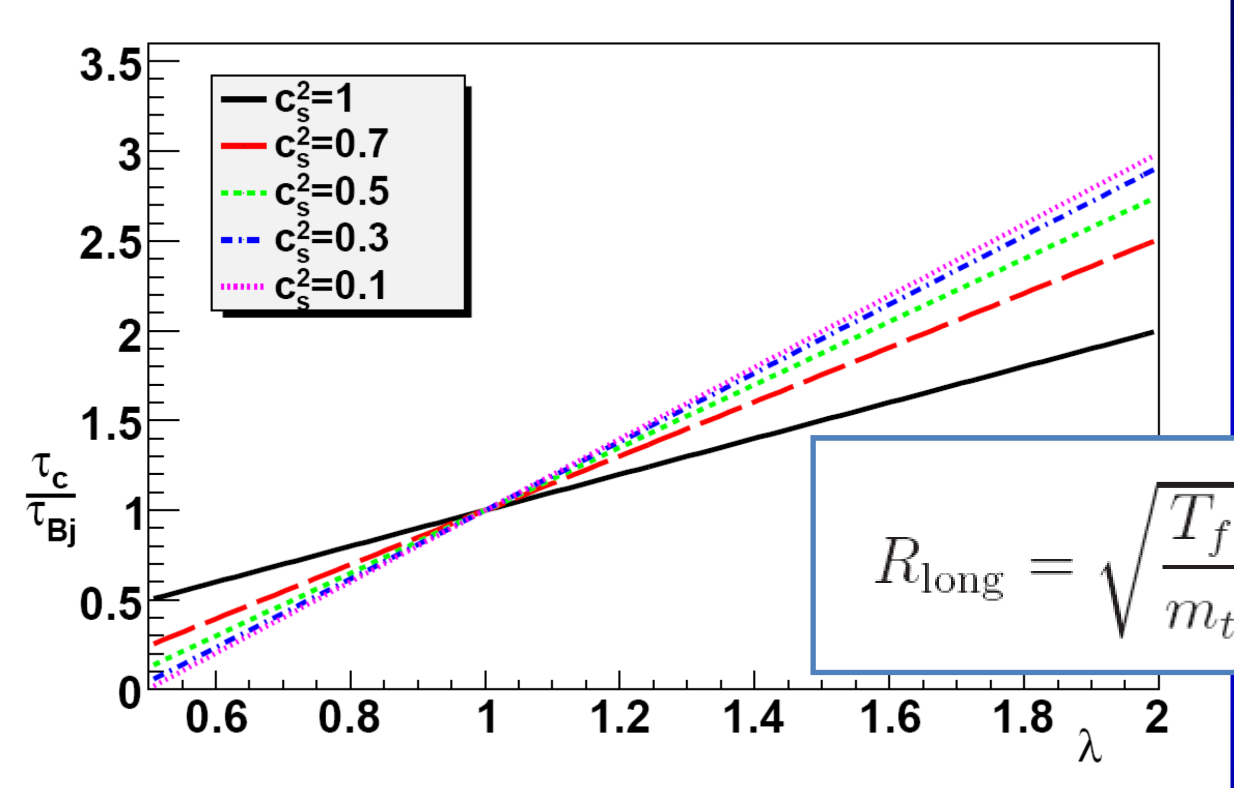
width of dn/dy related to acceleration and work

$$R_{\mathrm{long}} = \sqrt{\frac{T_f}{m_t}\frac{\tau_{\mathrm{c}}}{\lambda}} \quad \Rightarrow \quad \tau_{\mathrm{c}} = \lambda\tau_{Bj}.$$

At RHIC energies: correction is about +20%

EoS dependence = ?

Conjectured CNC life-time estimate



CNC exact result (κ = 1):

$$R_{\rm long} = \sqrt{\frac{T_f}{m_t}\frac{\tau_{\rm c}}{\lambda}} \quad \Rightarrow \quad \tau_{\rm c} = \lambda \tau_{Bj}.$$

CNC conjecture (for $c_s^2 = 1/\kappa < 1$)

$$R_{\rm long} = \sqrt{\frac{T_f}{m_t}\frac{\tau_{\rm c}}{\lambda + (\lambda - 1)(1 - c_s^2)}} \quad \Rightarrow \quad \tau_{\rm c_s} = [\lambda + (\lambda - 1)(1 - c_s^2)]\tau_{Bj}.$$

New life-time estimate, exact result

Life-time estimation: for Hwa-Bjorken type of flows

$$R_{long} = \sqrt{\frac{T_f}{m_t}}\tau_{Bj} \quad \Rightarrow \quad \tau_{Bj} = \sqrt{\frac{m_t}{T_f}} R_{long}.$$

Makhlin & Sinyukov, Z. Phys. C 39, 69 (1988)

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New CKCJ solution → new life-time estimate

up to leading order, the EoS dependence cancels!

$$R_{\rm long} = \sqrt{\frac{T_f}{m_t}\frac{\tau_{\rm c}}{\lambda}} \quad \Rightarrow \quad \tau_{\rm c} = \lambda \tau_{Bj}.$$

Lack of κ dependence:

At mid-rapidity, the approximate flow profile is $v_z \sim \lambda\, \eta_x$

New, exact solutions: bulk viscosity

$$T^{\mu\nu} = e u^\mu u^\nu - p\Delta^{\mu\nu} + \Pi^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu},$$

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu}\Pi,$$

$$e = \kappa p, \qquad p = nT,$$

$$\partial_\mu (N^\mu) = 0.$$

$$De = -(e + p + \Pi)\theta + \sigma_{\mu\nu}\pi^{\mu\nu},$$

$$(e + p + \Pi)Du^\alpha = \nabla^\alpha (p + \Pi) - \Delta^\alpha_\nu u_\mu D\pi^{\mu\nu} - \Delta^\alpha_\nu \nabla_\mu \pi^{\mu\nu},$$

$$S = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2},$$

$$u^\mu = \frac{x^\mu}{\tau} = \gamma\left(1, \frac{\dot X}{X} r_x, \frac{\dot Y}{Y} r_y, \frac{\dot Z}{Z} r_z\right),$$

$$\theta = \partial_\mu u^\mu = \frac{d}{\tau},$$

$$D = u^\mu \partial_\mu = \frac{\partial}{\partial \tau}.$$

$$\tau = \sqrt{t^2 - r^2}$$

$$\sigma_{\mu\nu}\pi^{\mu\nu} = 0,$$

ζ/s η/s

$$\eta_s = \frac{1}{2}\ln\left(\frac{t + r_z}{t - r_z}\right)$$

•New, exact solution with bulk viscosity

$$u^\mu = \frac{x^\mu}{\tau} = \gamma\left(1, \frac{\dot{X}}{X}r_x, \frac{\dot{Y}}{Y}r_y, \frac{\dot{Z}}{Z}r_z\right),$$

$$\frac{de}{d\tau} + \frac{d(e+p)}{\tau} - \zeta(\frac{d}{\tau})^2 = 0.$$

$$e = e_0\left(\frac{\tau_0}{\tau}\right)^{(1+\frac{1}{\kappa})d} + \zeta\frac{d}{\tau}\frac{d}{\left(1+\frac{1}{\kappa}\right)d-1},$$

ζ/s η/s

$$\mathbf{v} = \frac{\mathbf{r}}{t} \quad \text{or} \quad u^\mu = \frac{x^\mu}{\tau},$$

$$n = n_0\left(\frac{\tau_0}{\tau}\right)^3\mathcal{V}(s),$$

$$p = p_0\left(\frac{\tau_0}{\tau}\right)^{3\left(1+\frac{1}{\kappa}\right)} + \frac{\zeta}{\kappa}\frac{3}{\tau}\frac{3}{\left(1+\frac{1}{\kappa}\right)3-1},$$

$$T = T_0\left(\frac{\tau_0}{\tau}\right)^{\frac{3}{\kappa}} + \frac{\zeta}{\kappa n_0}\frac{3}{\tau}\left(\frac{\tau}{\tau_0}\right)^3\frac{3}{\left(1+\frac{1}{\kappa}\right)3-1}\mathcal{T}(s),$$

$$\theta = \partial_\mu u^\mu = \frac{d}{\tau},$$

$$D = u^\mu\partial_\mu = \frac{\partial}{\partial\tau}.$$

$$\tau = \sqrt{t^2 - r^2}$$

$$\eta_s = \frac{1}{2}\ln\left(\frac{t+r_z}{t-r_z}\right)$$

New viscous solution in 1+3 dim

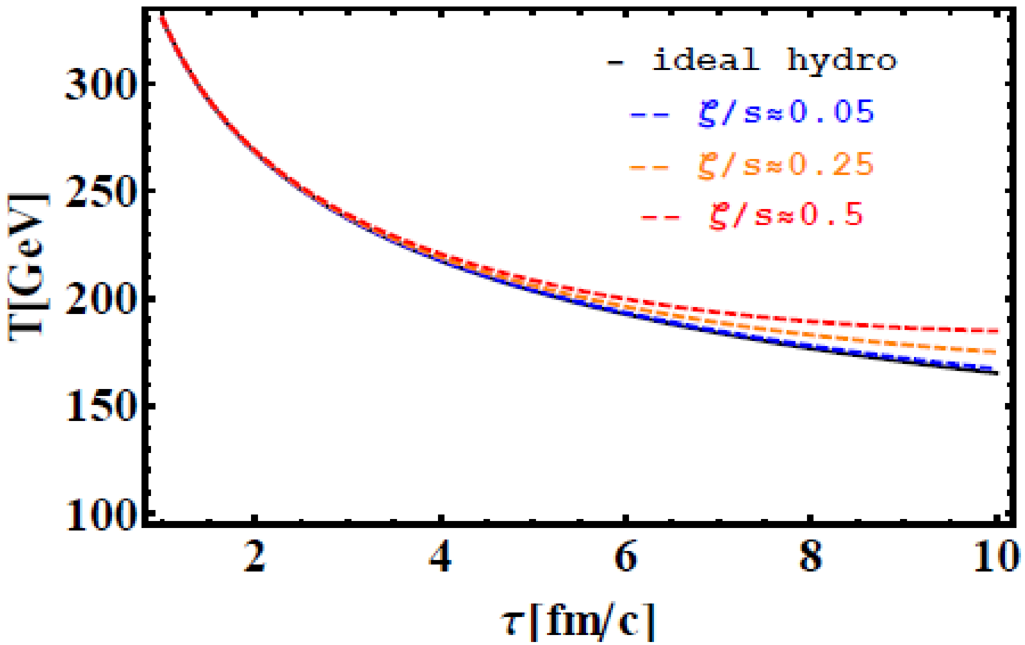

Fig. 1: The proper time τ and the ratio ζ/s of temperature for the new temperature solution for given primary initial condition $T_0 = 330$ MeV [], $\tau_0 = 1$ fm/c for RHIC, based on Eq.(21). (color online).

Bulk viscosity important at late stage, heats up

Shear viscosity effects cancel for asymptotically Hubble flows

Conclusions

Explicit solutions of a very difficult problem

New estimates of initial energy density
New exact solution
for arbitrary EOS with const e/p
after 10 years, finally

For asymptotically Hubble flows

shear effects cancel at late time
bulk viscosity heats up matter

A lot to do ...

more general EoS
less symmetry, ellipsoidal solutions
Rotating viscous solutions

Thank you for your attention

Questions and Comments ?