Standard Model

Lecture 1
Symmetries

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Lecture 1

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Introduction

Standard Model (SM) describes a bunch of particles interacting with 3 forces:

- strong nuclear force
- weak nuclear force
- electromagnetic force

force of gravity is not part of Standard Model

Each force in Standard Model is associated to a Lie group

- Standard Model is built around the group

$$G = U(1) \times SU(2) \times SU(3)$$

→ $SU(3)$ is associated to strong force

→ $SU(2)$ is associated to weak force

→ $U(1)$ is not associated to electromagnetism but to electromagnetic-like force - hypercharge (it also plays role in weak force)

Introduction

Despite similarities of each force, resulting physics is different

- 3 forces interact with matter
- in SM the matter comes in form of 15 Weyl fermions
- collectively go by the name of the electron, up quark, down quark, and neutrino

Quantum field theory (QFT)

- quartet of particles - electron, neutrino, up & down quarks - must come together as a collective
- e.g. electron and up quark and no companions makes no sense

Matter is not made of 15 fermions with four different names

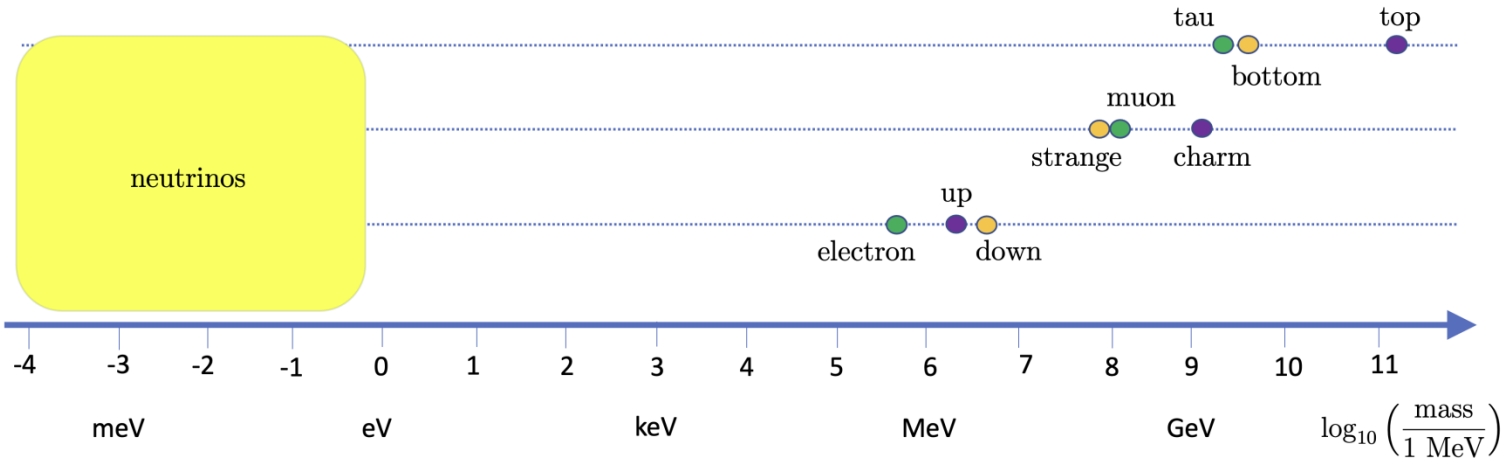
- **45 fermions with twelve different names**

electron	down quark	up quark	electron neutrino
1	9	4	$\sim 10^{-6}$
muon	strange quark	charm quark	muon neutrino
207	186	2495	$\sim 10^{-6}$
tau	bottom quark	top quark	tau neutrino
3483	8180	340,000	$\sim 10^{-6}$

masses relative to electron mass

$m_e \approx 0.51$ MeV

Introduction



3 particle generations

- 1st generation: down, up, electron, electron neutrino
- 2nd generation: strange, charm, muon, muon neutrino
- 3rd generation: bottom, top, tau, tau neutrino

Between generations particles differ by flavour quantum number & mass

- particles in each generation experience identical forces
- all the masses can be traced to interactions of fermions with Higgs field

Why these particular masses? Why this ordering of masses?

Spacetime symmetries

Symmetries

Structure of Standard Model follows various symmetries

- Poincaré symmetries of spacetime
  - → scalars, fermions and gauge fields
  - → basic building blocks of SM
- *(Local)* gauge symmetries
  - → interactions of the spin 1 fields
  - → Standard Model has specifying gauge group

$$G = U(1) \times SU(2) \times SU(3)$$

- Global symmetries
  - → act on fermions
  - → include baryon and lepton number + flavour symmetries
- Discrete symmetries
  - → e.g. parity (P), time-reversal (T), and charge conjugation (C)
  - → important in structure of Standard Model

Spacetime symmetries

Spacetime is effectively flat on length scales for particle physics

- Minkowski space $\mathbb{R}^{1,3}$ with Minkowski metric:

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$$

→ point in Minkowski space: $x^\mu = (x^0, x^1, x^2, x^3)$

- Symmetries of Minkowski space include Lorentz transformations

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \qquad \Lambda^T \eta \Lambda = \eta$$

- Discrete transformations

  → parity with Λ = diag(1,−1,−1,−1)

  → time reversal with Λ = diag(−1, 1, 1, 1)

Transformations connected to identity

→ det Λ = 1

→ $\Lambda^0{}_0 > 0$

→ form Lorentz group *SO*(1,3)

Lorentz group

Group *SO*(1,3) doesn't have any spinor representations

- Related group Spin(1,3) that admits spinors:

$$SO(1,3) \cong \mathrm{Spin}(1,3)/\mathbb{Z}_2$$

  → $\mathbb{Z}_2$ related to minus sign that spinors get under a 2π rotation

- *Spin*(1,3) and *SO*(1,3) share same Lie algebra *so*(1,3)

  → Lorentz transformation acting on 4-vector: $\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$

- Matrices $M^{\mu\nu}$ generate the algebra so(1,3):

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma}\right)$$

  → six different Lorentz transformations decompose into 3 rotations J_i and 3 boosts K_i

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} \quad \text{and} \quad K_i = M_{0i} \qquad j,k = 1,2,3 \quad \epsilon_{123} = +1$$

  → rotation matrices are Hermitian $J_i^\dagger = J_i$ opposite to boost matrices $K_i^\dagger = -K_i$

- Such generators fulfill: $[J_i, J_j] = i\epsilon_{ijk}J_k\ , \quad [J_i, K_j] = i\epsilon_{ijk}K_k\ , \quad [K_i, K_j] = -i\epsilon_{ijk}J_k$

  → rotations form *su*(2) sub-algebra

  → two mutually commuting *su*(2) algebras sitting inside *so*(1, 3) as linear combinations:

$$A_i = \frac{1}{2}(J_i + iK_i) \quad \text{and} \quad B_i = \frac{1}{2}(J_i - iK_i)$$

Lorentz group

- Both A_i and B_i are Hermitian, thus:

$$[A_i, A_j] = i\epsilon_{ijk}A_k \ , \quad [B_i, B_j] = i\epsilon_{ijk}B_k \ , \quad [A_i, B_j] = 0$$

→ representations of *SU*(2), integer or half-integer $j \in \frac{1}{2}\mathbb{Z}$ ⇒ SPIN

→ dimension of representation is $2j + 1$

→ can find two su(2) sub-algebras ⇒ (j_1, j_2) with $j_1, j_2 \in \frac{1}{2}\mathbb{Z}$

→ this representation must have dimension $(2j_1 + 1)(2j_2 + 1)$

- Physical spin of a particle

→ quantum number under rotations

→ $j = j_1 + j_2$

- Spin-statistics theorem

→ particles with $j \in Z$ are bosons

→ those with $j \in Z + ½$ are fermions

$$\begin{aligned}
(0,0): &\quad \text{scalar} \\
(\tfrac{1}{2},0): &\quad \text{left-handed Weyl spinor} \\
(0,\tfrac{1}{2}): &\quad \text{right-handed Weyl spinor} \\
(\tfrac{1}{2},\tfrac{1}{2}): &\quad \text{vector} \\
(1,0): &\quad \text{self-dual 2-form} \\
(0,1): &\quad \text{anti-self-dual 2-form}
\end{aligned}$$

Poincaré group

Continuous symmetries of Minkowski space comprise of Lorentz transformations together with spacetime translations

- combined, form Poincaré group
- spacetime translations generated by momentum 4-vector P^μ

$$[P^\mu, P^\nu] = 0 \quad \text{and} \quad [M^{\mu\nu}, P^\sigma] = i\left(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}\right)$$

- Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho} M^{\mu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma}\right)$$

Algebra of Poincaré group = spacetime translations + Lorentz algebra

Representations of Poincaré group

→ Casimirs (Casimir operators) - operators that commute with all generators of the group

→ irreducible representations ⇒ eigenvalue of the Casimir

Representations of Poincaré group

Irreducible representations of Poincaré group are called 'particles'

- Characterised by the Casimirs

  relativistic version of angular momentum - *Pauli-Lubański vector*

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \qquad [W_\mu, P_\nu] = 0$$

- Two Casimirs of the Poincaré group are formed from momentum P_μ and Pauli-Lubański vector W_μ

$$C_1 = P_\mu P^\mu \quad \text{and} \quad C_2 = W_\mu W^\mu$$

Momentum operator P^μ

→ all states labelled by eigenvalue p^μ which is 4-momentum of the particle

→ first Casimir is the rest mass of particle: $C_1 = p_\mu p^\mu = m^2$

Opposite to rotations representations of Poincaré group must be infinite dimensional

→ labelled by the continuous variable p^μ

Poincaré group - massive representations

- **Assume that** $C_1 = m^2 \neq 0$
- Simplest choice is to boost to rest frame of particle → $p^\mu = (m, 0, 0, 0)$

  Pauli-Lubański vector: $W^0 = 0 \quad \text{and} \quad W^i = -mJ^i$ J^i - generators of rotations
- J^i (Lorentz algebra) don't change value of $p^\mu = (m, 0, 0, 0)$

  → find a representation of Lie algebra $su(2)$, generated by J^i

  → second Casimir of Poincaré group $C_2 = -m^2J^2$

  → eigenvalue of J^2 is $j(j+1)$ for some $j \in \frac{1}{2}\mathbb{Z}$ (representations of rotation group)
- For different (fixed) p^μ we have same result ($p_\mu p^\mu = m^2$)

  → elements of Lorentz group that don't change p^μ → form a little group

  → for massive particles little group is $SU(2)$

  induced representation

  → $SU(2)$ representation uplift to representation of full Poincaré group
- Massive particles characterised by their mass m and spin j

  → states in this representation of the Poincaré group $|p_\mu, j_3\rangle$

Poincaré group - massless representations

For massless particles first Casimir vanishes $C_1 = m^2 = 0$

- for fixed momentum we can't boost to the rest frame
- we can only choose momentum to take the form $p^\mu = (E, 0, 0, E)$

Pauli-Lubański vector:

(replaced $M^{\mu\nu}$ with rotation generator J_i or boost gen. K_i)

$$W_\mu = E \begin{pmatrix} -M_{12} \\ M_{23} - M_{02} \\ M_{31} + M_{01} \\ M_{12} \end{pmatrix} = E \begin{pmatrix} -J_3 \\ J_1 - K_2 \\ J_2 + K_1 \\ J_3 \end{pmatrix}$$

Components of W^μ (little group): $[W_1, W_2] = 0\ , \quad [W_3, W_2] = -iEW_1\ , \quad [W_3, W_1] = iEW_2$

→ Euclidean group in $\mathbb{R}^2$ - *ISO(2)*

→ W_1 and W_2 - generators of translations, W_3 - generator of rotations

- little group - just *U(1)* of rotations generated by J_3

→ representations of *U(1)* - single eigenvalue *h* *(helicity - analog of spin for massless particles)*

→ single state in this representation of the Poincaré group $|p_\mu, h\rangle$ (*U(1)* is Abelian)

→ helicity tells us eigenvalue of state under rotation along the direction of motion

U(1) $\subset$ *SU(2)* → under a rotation of $\theta = 2\pi$, the states are either left same (for $h \in Z$) or pick up minus sign (for $h \in \mathbb{Z} + \frac{1}{2}$)

γ: $h = 1$ (2 polarisation states)

Coleman-Mandula theorem

Not unusual for quantum field theories to exhibit continuous symmetries

→ e.g. global *U(1)* symmetry rotating phase of complex field, non-Abelian *SU(N)* symmetry

- Generators *T* always Lorentz scalars → commute with the Poincaré generators

$$[P^\mu, T] = [M^{\mu\nu}, T] = 0$$

Coleman-Mandula theorem

In any spacetime dimension greater than d = 1+1, the symmetry group of any interacting quantum field theory must factorise as:

Poincaré × Internal

Restricts possibility that Poincaré group is subsumed into a larger group

Consequences

- Conformal invariance: Coleman-Mandula theorem assumes that theory has a mass gap

  → all particles are massive

  → massless theories ⇒ scale invariance ($x^\mu \to \lambda\, x^\mu$) ⇒ special conformal transformations

- Supersymmetry: symmetry that relates bosons to fermions

  → generators form so-called *super-Lie algebra*, with commutation with Poincaré group

Spinors

Dirac vs Weyl spinors

The interest now in spin ½ particles - spinors

Clifford algebra admits unique irreducible representation

- 4-component Dirac spinor ψ

  → gamma matrices $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

  → chiral basis of gamma matrices $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ and $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  → 2 collections of 2 × 2 matrices $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$

  → σ^i with i = 1, 2,3 are Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- Generators of Lorentz transformations for Dirac spinor

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \qquad \sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$$
$$\bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$$

  → both anti-symmetrized in μ and ν, each is a collection of six 2 × 2 matrices

Dirac vs Weyl spinors

- Generators $S^{\mu\nu}$ are box diagonal

  → are not irreducible representation of Lorentz algebra

  → formed of 2 distinct representations generated by $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$

- 4-component Dirac spinor also decomposes into two 2-component spinors

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

  → left-handed and right-handed spinors

  → ψ_L is in (½, 0) representation, ψ_R in (0, ½) (Weyl spinors)

- Dirac spinor is a combination of both representations (½, 0) ⊕ (0, ½)

- Under a Lorentz transformation left-handed Weyl spinor transforms:

$$\psi_L \rightarrow S\psi_L \quad \text{with} \quad S = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)$$

$\omega_{\mu\nu}$ - six numbers specifying Lorentz transformation

(Left-Handed)* = Right-Handed

Two representations - for left-handed Weyl spinor and for right-handed Weyl spinor - are related by complex conjugation

→ but $(\sigma^{\mu\nu})^\star \neq \bar{\sigma}^{\mu\nu}$

- Additional conjugation by the anti-symmetric tensor

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \epsilon^T (\sigma^{\mu\nu})^\star \epsilon = -\bar{\sigma}^{\mu\nu}$$

→ complex conjugation flips sign of $(\sigma^2)^* = -\sigma^2$ leaving other Pauli matrices alone: $(\sigma^i)^* = \sigma^i$ for $i = 1,3$

→ but conjugation by $\epsilon = i\sigma^2$ flips sign of σ^i with $i = 1,3$ leaving σ^2 alone

- Physical implication ⇒ if left-handed particle described by Weyl spinor ψ_L, its anti-particle is right-handed and is described by the conjugate spinor $\psi_L^\star$ $(\psi_L^\dagger)$

Complex conjugate of the transformation law

$$\psi_L^\star \to \exp\left(+\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})^\star\right)\psi_L^\star \quad \Longrightarrow \quad \epsilon\psi_L^\star \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)\epsilon\psi_L^\star$$

$\epsilon\psi_L^\star$ transforms as a right-handed spinor

Build scalars from spinors

Given 2 left-handed spinors ψ_L and χ_L we can build a scalar

→ indices: $(\psi_L)_\alpha$, $(\chi_L)_\alpha$ with $\alpha = 1,2$

Anti-symmetric matrix: $\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- Define the scalar quantity

$$\psi_L \chi_L := \epsilon^{\alpha\beta}(\psi_L)_\beta(\chi_L)_\alpha = (\psi_L)_2(\chi_L)_1 - (\psi_L)_1(\chi_L)_2$$

→ transforms as a scalar:

$$\psi_L \chi_L \to S_\alpha{}^\gamma S_\beta{}^\delta \epsilon^{\alpha\beta}(\psi_L)_\delta(\chi_L)_\gamma = (\det S)\epsilon^{\gamma\delta}(\psi_L)_\delta(\chi_L)_\gamma = \psi_L \chi_L \qquad S_\alpha{}^\gamma S_\beta{}^\delta \epsilon^{\alpha\beta} = \det S\, \epsilon^{\gamma\delta}$$

- Important: can form a scalar from 2 left-handed spinors

- Spinors are anti-commuting variables whose components are Grassmann-valued

$$\psi_L \chi_L = (\psi_L)_2(\chi_L)_1 - (\psi_L)_1(\chi_L)_2 = -(\chi_L)_1(\psi_L)_2 + (\chi_L)_2(\psi_L)_1 = \chi_L \psi_L$$

→ this means that we can form a scalar from just a single left-handed Weyl spinor

$$\psi_L \psi_L = (\psi_L)_2(\psi_L)_1 - (\psi_L)_1(\psi_L)_2 = 2(\psi_L)_2(\psi_L)_1$$

→ similar expressions for right-handed spinors

Lagrangians for spinors

How to construct Lagrangians for spinors

- For Dirac spinor Lorentz invariant action:

$$S_{\text{Dirac}} = -\int d^4x \left(i\bar{\psi}\gamma^\mu \partial_\mu \psi - M\bar{\psi}\psi \right) \qquad \bar{\psi} = \psi^\dagger \gamma^0$$

- Decomposed in terms of Weyl fermions

$$S_{\text{Dirac}} = -\int d^4x \left(i\bar{\psi}_L \bar{\sigma}^\mu \partial_\mu \psi_L + i\bar{\psi}_R \sigma^\mu \partial_\mu \psi_R - M(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) \right) \qquad \bar{\psi}_L = \psi_L^\dagger \qquad M \in \mathbb{R}$$

→ mass term couples the left- and right-handed Weyl spinors

→ complex conjugate $\bar{\psi}_R$ is a left-handed spinor

⇒ writing $\bar{\psi}_R \psi_L$ two left-handed spinors are combined to a scalar

Mass term *M* which combines two different spinors (left and right) is called **Dirac mass**

- Possible to write down Lorentz invariant action with mass term:

$$\mathcal{S}_{\text{Weyl}} = -\int d^4x \left(i\bar{\psi}_L \bar{\sigma}^\mu \partial_\mu \psi_L + \frac{m}{2}\psi_L \psi_L + \frac{m^\star}{2}\bar{\psi}_L \bar{\psi}_L \right) \qquad m \in \mathbb{C}$$

Mass term *m* is known as a **Majorana mass**

Gauge invariance

Gauge invariance - Maxwell theory

In SM forces associated to *(massless)* spin 1 particles - gauge bosons *(but $W^{\pm}$, Z^0 are massive)*

→ much of dynamics of these forces is fixed by gauge invariance

- In electromagnetism

  → fundamental field is 4-vector $A_\mu(x)$ → gauge potential

  → gauge transformation: $A_\mu \to A_\mu + \partial_\mu \alpha$ → gauge symmetry

  'symmetry' - two configurations related by gauge transformation describe same physical configuration

- Physical information in A_μ can be found in field strength (being gauge invariant):

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

  → includes electric field **E** and the magnetic field **B**

  - For $A_\mu = (\varphi, A)$: $\mathrm{E} = -\nabla\phi - \frac{\partial \mathrm{A}}{\partial t}$ and $\mathrm{B} = \nabla \times \mathrm{A}$
  - Dynamics of gauge field is described by the action: $S_{\mathrm{Maxwell}} = -\frac{1}{4}\int d^4x \; F_{\mu\nu}F^{\mu\nu}$
  - Equations of motion: $\partial_\mu F^{\mu\nu} = 0$

    → Two of Maxwell equations: Gauss' law $\nabla\cdot\mathbf{E} = 0$ and Ampére's law $\nabla\times\mathbf{B} = \partial\mathbf{E}/\partial t$

Gauge invariance - Maxwell theory

Other two Maxwell equations

→ follow immediately from constructing $F^{\mu\nu}$ in terms of the gauge potential

→ dual field strength

$$^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

→ **E** and **B** swapped (one of them with minus sign)

Using anti-symmetry of $\epsilon^{\mu\nu\rho\sigma}$ $\Rightarrow$ Bianchi identity

$$\partial_\mu {}^\star F^{\mu\nu} = 0$$

Expanding → remaining two Maxwell equations:

- magnetic monopoles don't exist $\nabla \cdot \mathbf{B} = 0$

- law of induction $\nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0$

Keep gauge invariance → not possible to introduce mass term $m^2 A_\mu A^\mu$

Coupling to matter

- Electromagnetism - *U(1)* gauge group
- Charged fields under electromagnetism (complex)

  → complex scalar field *φ(x)* of charge *e*

  → gauge transformation of scalar field: $\phi \to e^{ie\alpha}\phi \qquad e^{ie\alpha(x)} \in U(1)$
- Theories invariant under gauge transformation

  → replacing partial derivatives with covariant derivative: $\mathcal{D}_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$

  $\mathcal{D}_\mu\phi$ transforms covariantly under a gauge transformation

  → for complex conjugate: $\mathcal{D}_\mu\phi^\dagger = (\partial_\mu + ieA_\mu)\phi^\dagger$

  D_μ is $-ieA_\mu$ for scalar, but $+ieA_\mu$ for conjugate scalar
- Gauge invariant action: $S_{\text{scalar}} = \int d^4x \left(\mathcal{D}_\mu\phi^\dagger\mathcal{D}^\mu\phi - V(|\phi|)\right) \qquad |\phi|^2 = \phi^\dagger\phi$
- Multiple scalar fields

  → they can carry different charges

  → if gauge group is *U*(1) charges should be integer multiples of each other

  → each field transforms as $\phi \to e^{ieq\alpha}\phi \quad \text{with} \quad q \in \mathbb{Z}$

More on Lie algebras

- Important extension of Maxwell theory

  → gauge group *U(1)* is replaced by a compact Lie group *G*

- Lie group - differentiable manifold

  → group element is labelled by continuous parameters *(e.g. rotation group, Poicaré group)*

  → for group elements continuously connected to the identity: $U \in G \qquad U = e^{i\theta^A T^A}$

  θ^A - numbers that tell us which group element we're working with; T^A - generators of the group

  → T^A - infinitesimal action of the group, when θ small: $U \approx \mathbb{1} + i\theta^A T^A + \mathcal{O}(\theta^2)$

- Lie group captured in behaviour of infinitesimal generators T^A

  → form associated Lie algebra: $[T^A, T^B] = i f^{ABC} T^C \qquad (T^A)^\dagger = T^A$

  A,B,C = 1, . . . , dim *G* and f^{ABC} are fully anti-symmetric structure constants

  → commutator have same properties as matrix commutation → Jacobi identity:

  $$[T^A, [T^B, T^C]] + [T^B, [T^C, T^A]] + [T^C, [T^A, T^B]] = 0$$

  → constraints on structure constants: $f^{ADE} f^{BCD} + f^{BDE} f^{CAD} + f^{CDE} f^{ABD} = 0$

- Simple, compact Lie groups

  → '*simple*' - no trivial *U(1)* factors floating around that commute with everything else

  → '*compact*' - if continue to rotate in the group → come back to where started from

More on Lie algebras

- Classification of simple compact Lie algebras

  → All of such groups referred to as non-Abelian *(elements don't commute with each other)*

G	$SU(N)$	$SO(N)$	$Sp(N)$	E_6	E_7	E_8	F_4	G_2
$\dim G$	N^2-1	$\frac{1}{2}N(N-1)$	$N(2N+1)$	78	133	248	52	14
$\dim F$	N	N	$2N$	27	56	248	6	7

dim *F* - dimension of the fundamental representation *F* $E_{6,7,8}$, F_4, G_2 - *exceptional Lie algebras (cyclic groups)*

- In SM - we will only need two different representations: fundamental & adjoint

  → *F* - fundamental representation → smallest (non-trivial) matrix

  → *adj* - representation that has *dim(adj)* = *dimG* with generators: $T^A(\mathrm{adj})_{BC} = -if^{ABC}$

- The generators of any simple Lie algebra obey Tr T^A = 0

  → in the fundamental representation: $\mathrm{Tr}\, T^A T^B = \frac{1}{2}\delta^{AB}$

- Simplest non-Abelian Lie group - *SU(2)*

  → dim(*SU(2)*) = 3

  → structure constants $f^{ABC} = \epsilon^{ABC}$

  → fundamental representation - 2 × 2 Pauli matrices $T^A = \frac{1}{2}\sigma^A$

Yang-Mills theory

Yang-Mills theory - generalisation of Maxwell theory

→ group $U(1)$ is replaced by simple, compact Lie group G

→ specify Yang-Mills theory: need to specify choice of G + coupling constant $g > 0$

→ coupling constant g - same role as charge e in Maxwell theory

- Lie algebra-valued gauge potential: $A_\mu = A_\mu^A T^A$

  → gauge field $A^A{}_\mu$ with $A = 1, \ldots, \dim G$

The fields $A^A{}_\mu$ are referred to as gauge bosons

- The action of gauge symmetry is associated to a Lie group: $\Omega(x) \in G$ *(gauge group)*

  → action on the gauge field: $A_\mu \to \Omega A_\mu \Omega^{-1} + \frac{i}{g} \Omega \, \partial_\mu \Omega^{-1}$

- Connect to gauge transformations in electromagnetism: $G = U(1) \quad \Omega(x) = e^{ie\alpha(x)}$

  → using that *U(1)* is 'simple' (everything commutes): $\Omega A_\mu \Omega^{-1} + \frac{i}{e} \Omega \partial_\mu \Omega^{-1} = A_\mu + \partial_\mu \alpha$

  gauge transformation in Yang-Mills reproduces gauge transformation in Maxwell

- Field strength in Yang-Mills theory: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \qquad F_{\mu\nu} = F_{\mu\nu}^A T^A$

  → field strength transforms under gauge transformation: $F_{\mu\nu} \to \Omega \, F_{\mu\nu} \, \Omega^{-1}$

Yang-Mills theory

- In transformation of field strength (in contrast to electromagnetism)

  → Yang-Mills 'electric field' $E_i = F_{0i}$ and 'magnetic field' $B_i = -\frac{1}{2}\,\epsilon_{ijk}F_{jk}$

  → they are not gauge invariant

- Lie groups structure → determined by elements infinitesimally close to the identity

  → look at gauge transformations that are everywhere close to identity:

$$\Omega(x) \approx 1 + ig\alpha^A(x)T^A + \ldots \qquad \alpha^A \text{ are taken to be everywhere small}$$

- Infinitesimal transformation of gauge field: $A_\mu \to A_\mu + \delta A_\mu \qquad \delta A_\mu = \partial_\mu \alpha - ig[A_\mu, \alpha]$

  $\alpha = \alpha^A T^A$ is Lie algebra-valued infinitesimal transformation

  → covariant derivative: $\mathcal{D}_\mu \alpha = \partial_\mu \alpha - ig[A_\mu, \alpha]$ where $\delta A_\mu = \mathcal{D}_\mu \alpha$

  → covariant derivative acting on Lie algebra-valued (adjoint) field α.

- How infinitesimal gauge transformations act on the field strength:

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu - ig[A_\mu, \delta A_\nu] - ig[\delta A_\mu, A_\nu] = [\mathcal{D}_\mu, \mathcal{D}_\nu]\alpha$$

  → commutator of two covariant derivatives: $[\mathcal{D}_\mu, \mathcal{D}_\nu]\alpha = -ig[F_{\mu\nu}, \alpha]$

  → expected infinitesimal gauge transformation: $\delta F_{\mu\nu} = ig[\alpha, F_{\mu\nu}]$

Yang-Mills action

Dynamics of Yang-Mills field is generalisation of Maxwell action

$$S_{\mathrm{YM}} = -\frac{1}{2}\int d^4x \ \mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu}$$

- Only difference is overall trace

  → which ensures that action is invariant under gauge transformations $F_{\mu\nu} \to \Omega\, F_{\mu\nu}\, \Omega^{-1}$

- Classical equations of motion derived by minimizing action wrt each gauge field A^a_μ

$$\mathcal{D}_\mu F^{\mu\nu} = 0$$

  Young-Mills equations → covariant derivative

$$\mathcal{D}_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}]$$

  non-linear → Yang-Mills fields interact with themselves

- Bianchi identity: dual field strength ${}^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ → identity (not linear): $\mathcal{D}_\mu {}^\star F^{\mu\nu} = 0$

- Field configurations sitting within subgroup *U(1)* $\in$ *G*

  → commutators vanish

  → equations reduce to those of Maxwell theory

- Yang-Mills (like Maxwell theory) - theory of massless particles

  → keep gauge invariance ⇒ can't add mass term

Coupling to matter

- As in electromagnetism we can couple Yang-Mills field to matter

  → require matter fields in some representation *R* of gauge group

  matter fields come in some vector of dimension dim *R*

  → for each representation → generators $T^A(R)$ - square matrices of dimension dim *R*

$$T^A(R)^a{}_b \quad \text{with } a,b = 1,\ldots,\dim R \quad \text{and} \quad A = 1,\ldots,\dim G$$

- **Assumption:** consider scalar field in representation *R*

  → Under gauge transformation $\Omega(x) = e^{ig\alpha^A(x)T^A}$ scalar transforms:

  some representations *R* are real, and some are complex $\phi^a \to (\Omega_R)^a{}_b \phi^b \quad \text{with} \quad (\Omega_R)^a{}_b = \left(\exp\left(ig\alpha^A T^A(R)\right)\right)^a{}_b$

- Action for φ that is invariant under the gauge transformation

  → construct covariant derivative $\mathcal{D}_\mu \phi^a = \partial_\mu \phi^a - igA^A_\mu T^A(R)^a{}_b \phi^b$

  → covariant derivative transforms covariantly $\mathcal{D}_\mu \phi^a \to (\Omega_R)^a{}_b \mathcal{D}_\mu \phi^b$

- Covariant derivative in terms of $N \times N$ matrix-valued ($SU(N)$) gauge field $A_\mu = A^A{}_\mu T^A$

$$\mathcal{D}_\mu \phi^a = \partial_\mu \phi^a - ig(A_\mu)^a{}_b \phi^b$$ acting on fundamental fields: $[\mathcal{D}_\mu, \mathcal{D}_\nu] = -igF_{\mu\nu}$

action for φ - need to replace partial derivatives in action with covariant derivatives

→ it is gauge invariant

C,P, and T symmetries

Parity *P*

Discrete symmetries → crucial to understand structure of SM

→ 3 important: parity (*P*), charge conjugation (*C*), time reversal (*T*)

Parity *P* - inversion of spatial coordinates *(interpreted as Lorentz transformation)*

$$P : (t, \mathbf{x}) \mapsto (t, -\mathbf{x})$$

→ action of *P* mimics what system looks like reflected in the mirror

→ reflection implemented by $R : (x, y, z) \mapsto (x, y, -z)$

- Electromagnetic fields transformation under parity

→ covariant derivative: $\mathcal{D}_\mu = \partial_\mu - iA_\mu$

→ under *P* transformation ∂_0 is left unaffected, spatial derivatives ∂_i change sign

$$P : A_0(t, \mathbf{x}) \mapsto +A_0(t, -\mathbf{x}) \quad \text{and} \quad P : A_i(t, \mathbf{x}) \mapsto -A_i(t, -\mathbf{x})$$

- Tracing to definitions of electric and magnetic fields: $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t \quad \mathbf{B} = \nabla \times \mathbf{A}$

$$P : \mathbf{E}(t, \mathbf{x}) \mapsto -\mathbf{E}(t, -\mathbf{x}) \quad \text{and} \quad P : \mathbf{B}(t, \mathbf{x}) \mapsto +\mathbf{B}(t, -\mathbf{x})$$

→ vectors: vectors like **E** which transform under parity in same way as *x*

→ pseudovectors: vectors like **B** which don't get a minus sign under parity

Parity *P*

Quantum theory → parity transformation - unitary operator on Hilbert space *P*

- Fields $A_\mu(x)$ are now also operators → transformations:

$$PA_0(t,\mathbf{x})P^\dagger = A_0(t,-\mathbf{x}) \quad \text{and} \quad PA_i(t,\mathbf{x})P^\dagger = -A_i(t,-\mathbf{x})$$

- Spinor tranformation

→ equation of motion for left-handed massless Weyl spinor ψ_L: $\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \quad \bar{\sigma} = (\mathbb{1}, -\sigma^i)$

→ under *P* transformation spatial derivative changes sign ⇒ Weyl equation not invariant

- For single left-handed Weyl spinor theory is not invariant under parity

→ in addition to ψ_L, provide right-handed Weyl spinor ψ_R: $\sigma^\mu \partial_\mu \psi_R = 0 \quad \sigma^\mu = (\mathbb{1}, \sigma^i)$

→ different minus signs in σ^μ and $\bar{\sigma}^\mu$ ⇒ compensate for a parity transformation if exchange left- and right-handed spinors

$$P\psi_L(t,\mathbf{x})P^\dagger = \psi_R(t,-\mathbf{x}) \quad \text{and} \quad P\psi_R(t,\mathbf{x})P^\dagger = \psi_L(t,-\mathbf{x})$$

→ 2 spinors ψ_L and ψ_R ⇒ Dirac spinor $\psi = (\psi_L, \psi_R)^T$

→ action on Dirac spinor: $P\psi(t,\mathbf{x})P^\dagger = \gamma^0\psi(t,-\mathbf{x}) \quad \text{with} \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

In Quantum Field Theory: terms in action always constructed of even number of fermions

Parity P

- Various fermion bilinears under parity, e.g.

$$P : \bar{\psi}\psi \mapsto \bar{\psi}\psi \quad \text{and} \quad P : \bar{\psi}\gamma^5\psi \mapsto -\bar{\psi}\gamma^5\psi$$

$\bar{\psi}\psi$ transforms as *scalar*

$\bar{\psi}\gamma^5\psi$ transforms as *pseudoscalar*

- Action of parity on scalar field ϕ

→ kinetic terms invariant under: $P\phi(t,\mathbf{x})P^\dagger = \pm\phi(t,-\mathbf{x})$

→ kinetic terms don't distinguish between scalar (plus sign) or pseudoscalar (minus sign)

- Theory is invariant under parity

→ can assign transformation laws to various fields

→ after quantisation ⇒ fields give rise to particles

Different species of particles thought as parity even or parity odd

Parity violated in weak interactions *(Lee & Yang, 1956)*

Charge conjugation

Charge conjugation - operation that switches particles with their anti-particles

- Theory invariant under charge conjugation

  → laws of physics for particles same for anti-particles

- Charge conjugation acting on complex scalar field ϕ coupled to electromagnetism

$$C : \phi \mapsto \pm\phi^\dagger$$

  ± ambiguity is like ambiguity in action of parity

- Action on gauge field → no ± ambiguity in action on gauge field

  → covariant derivatives: $\mathcal{D}_\mu\phi = (\partial_\mu - ieA_\mu)\phi$ and $\mathcal{D}_\mu\phi^\dagger = (\partial_\mu + ieA_\mu)\phi^\dagger$

  → transformation: $C : A_\mu \mapsto -A_\mu$

- Charge conjugation - quantum operator *C:* $C\phi C^\dagger = \pm\phi^\dagger$ and $CA_\mu C^\dagger = -A_\mu$

  → non-Abelian gauge fields (generators are Hermitian): $CA_\mu C^\dagger = -A_\mu^\star$

- Dirac spinors: $C : \psi \mapsto C\psi^\star$ (*C* - 4×4 matrix allowing components of spinor to be mixed under *C*)

  → Dirac equation of motion: $i\gamma^\mu(\partial_\mu - ieA_\mu)\psi - M\psi = 0$

  → its complex conjugate: $-i(\gamma^\mu)^\star(\partial_\mu + ieA_\mu)\psi^\star - M\psi^\star = 0$

Charge conjugation

- Compare results for action on those two Dirac equations:

$$C^\dagger \gamma^\mu C = -(\gamma^\mu)^\star$$

→ charge conjugation matrix depends on chosen basis of gamma matrices

→ chiral basis: all γ matrices are real except γ^2 (imaginary)

→ action of charge conjugation: $C : \psi \mapsto \pm i\gamma^2\psi^\star \quad \text{with} \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$

- Theories invariant under charge conjugation

→ assign eigenvalue $C = \pm 1$ to each particle (*C*-parity)

- Weyl spinors

→ decompose Dirac spinor : $\psi = (\psi_L, \psi_R)^T$

→ action of charge conjugation on Weyl spinors: $C : \psi_L \mapsto \pm i\sigma^2\psi_R^\star \quad \text{and} \quad C : \psi_R \mapsto \mp i\sigma^2\psi_L^\star$

Theory with single Weyl fermion is not invariant under parity or charge conjugation

→ combine two symmetries *P* and *C:* $CP : \psi_L(t, \mathrm{x}) \mapsto \mp i\sigma^2\psi_L^\star(t, -\mathrm{x}) \quad \text{and} \quad CP : \psi_R(t, \mathrm{x}) \mapsto \pm i\sigma^2\psi_R^\star$

→ Weyl fermion coupled to gauge field invariant under CP

→ symmetry to be violated by weak interaction terms *(β decays)*

Time reversal

Time reversal $T : (t, \mathbf{x}) \mapsto (-t, \mathbf{x})$

- Implement time reversal symmetry in quantum theories

  → free particle moving in $\mathbb{R}^3$

  → Schrödinger equation for wavefunction Ψ: $i\frac{\partial \Psi}{\partial t} = -\nabla^2 \Psi$

  → heat equation for temperature *T* diffuseing in a system: $\frac{\partial T}{\partial t} = \nabla^2 T$

  heat equation isn't time reversal invariant → left-hand side gets '-' sign, right-hand side not

- Schrödinger equation is time reversal invariant

  → crucial factor of *i* in Schrödinger equation

  → time reversal in quantum mechanics: $T : \Psi(t) \mapsto \Psi^\star(-t)$

  → complex conjugation ⇒ *T* is anti-unitary operator

Acting on states: $T(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha^\star T|\psi_1\rangle + \beta^\star T|\psi_2\rangle$

→ *T* operator obeys: $\langle T\psi_1|T\psi_2\rangle = \langle \psi_1|\psi_2\rangle^\star$

→ anti-linear behaviour changes transformation properties of fields

Time reversal

- Action of time reversal on gauge field:

$$T : A_0(t,\mathbf{x}) \mapsto +A_0(-t,\mathbf{x}) \quad \text{and} \quad T : A_i(t,\mathbf{x}) \mapsto -A_i(-t,\mathbf{x})$$

- Tracing to definitions of electric and magnetic fields: $\mathbf{E} = -\nabla A_0 - \partial \mathbf{A}/\partial t \quad \mathbf{B} = \nabla \times \mathbf{B}$

$$T : \mathbf{E}(t,\mathbf{x}) \mapsto +\mathbf{E}(-t,\mathbf{x}) \quad \text{and} \quad T : \mathbf{B}(t,\mathbf{x}) \mapsto -\mathbf{B}(-t,\mathbf{x})$$

- Action of time reversal can mix components of a Dirac spinor:

  → transformation for chiral basis γ matrices: $T : \psi(t,\mathbf{x}) \mapsto \Theta\psi(-t,\mathbf{x}) \quad \text{where} \quad \Theta = \gamma^1\gamma^3$

  → time reversal also acts by complex conjugation

  ⇒ gives back original Dirac equation if $\Theta^{-1}(\gamma^0)^\star\Theta = \gamma^0 \quad \text{and} \quad \Theta^{-1}(\gamma^i)^\star\Theta = -\gamma^i$

- Action on the component Weyl spinors $\psi = (\psi_L, \psi_R)^T$

$$T : \psi_L(t,\mathbf{x}) \mapsto i\sigma^2\psi_L(-t,\mathbf{x}) \quad \text{and} \quad T : \psi_R(t,\mathbf{x}) \mapsto i\sigma^2\psi_R(-t,\mathbf{x})$$

  → time reversal (like *CP*) does not mix the left- and right-handed Weyl spinors

Break time-reversal invariance

→ example - presence of electric dipole moment (?)

→ spin and dipole moment transform differently under *P* & *T*

$$P : \mathbf{S} \mapsto \mathbf{S} \quad \text{and} \quad T : \mathbf{S} \mapsto -\mathbf{S}$$
$$P : \mathbf{E} \mapsto -\mathbf{E} \quad \text{and} \quad T : \mathbf{E} \mapsto \mathbf{E}$$

CPT

There are theories invariant under *C, P* and *T*

Other theories break them - e.g. in SM all three symmetries are broken

- Quantum field theories must be invariant under combined *CPT*

  → proof of CPT theorem ⇒ check if all Lorentz invariant terms are invariant under *CPT*

- Example: spinors

  → Dirac spinor transformed by anti-unitary operation

$$CPT : \psi(x) \mapsto -\gamma^5 \psi^\star(-x) \quad \text{with} \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  → all fermion bilinears are invariant under this transformation, e.g.:

$$\bar{\psi}\psi = \psi^\dagger \gamma^0 \psi \mapsto \psi^T \gamma^5 \gamma^0 \gamma^5 \psi^\star = -\psi^T \gamma^0 \psi^\star = \bar{\psi}\psi$$

- Example: neutrino physics

  → neutrinos oscillate from one flavour to another travelling through space

  → muon neutrino converts into electron neutrino: $\nu^\mu \longrightarrow \nu^e$

  → CP conjugate process: $\bar{\nu}^\mu \longrightarrow \bar{\nu}^e$

CPT

- No reason for amplitudes for two processes to be equal if *CP* is broken
- Time reversed process of $\nu^\mu \longrightarrow \nu^e$:

$$\nu^e \longrightarrow \nu^\mu$$

→ if time reversal is broken

⇒ may have different amplitude wrt muon neutrino → electron neutrino

- CPT

→ amplitudes for $\bar{\nu}^\mu \longrightarrow \bar{\nu}^e$ and $\nu^e \longrightarrow \nu^\mu$ must be equal

All experimental tests so far have failed to find any violation of CPT