Standard Model

Lecture 4
Anomalies

Marcin Kucharczyk

IFJ PAN, Kraków
05 November 2026

Contents

Lecture 4

- Gauge anomalies
- Chiral anomalies
- ’t Hooft anomalies

Anomalies

Anomalies - understand structure of Standard Model

- QED & QCD

  → Lagrangians with gauge field coupled to Dirac fermions

  → Dirac fermions not irreducible representations of Lorentz group

  ⇒ decomposed to 2 Weyl fermions

  Yang-Mills coupled to single Weyl fermion ⇒ inconsistent quantum theory

  → write classical Lagrangian & classical Hamiltonian

  → no corresponding quantum theory ⇒ **anomaly**

- Massless QCD → classical $U(1)_A$ axial symmetry

  → spontaneously broken *(like non-Abelian chiral symmetry)*

  → no associated light meson (η' m ~ 1 GeV)

  Axial $U(1)_A$ symmetry in massless QED & QCD → symmetry of classical theory

  → not symmetry of quantum theory ⇒ **anomaly**

Number of different manifestations of anomalies in QFT

Anomaly related to fermions

- **Assume:** Dirac fermion

  → decomposed to 2 Weyl fermions: $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

- Single Weyl fermion - left-handed spinor ψ_L

  → action for massless Weyl spinor: $S = \int d^4x \; i\bar{\psi}_L \bar{\sigma}^\mu \partial_\mu \psi_L \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$

  → invariant under *U(1)* global symmetry $\psi_L \to e^{i\alpha}\psi_L \;\Rightarrow\; j^\mu = \psi_L^\dagger \bar{\sigma}^\mu \psi_L$

  Couple current to gauge field A_μ: $S = \int d^4x \; i\bar{\psi}_L \bar{\sigma}^\mu \mathcal{D}_\mu \psi_L \qquad \mathcal{D}_\mu \psi_L = \partial_\mu \psi_L - ieqA_\mu \psi_L$

  → invariant under gauge symmetry:

$$\psi \to e^{ieq\alpha(x)}\psi \quad \text{and} \quad A_\mu \to A_\mu + \partial_\mu \alpha$$

- A_μ as *dynamical gauge field*

  → classical theory: dynamical variable

  → quantum theory: integrate over A_μ in path integral

- A_μ as *background gauge field*

  → fixed parameter of theory *(breaks Lorentz symmetry)*

Anomaly related to fermions

- **Assume:** A_μ - background gauge field

Is quantum theory invariant under gauge transformation?

→ **Assume:** path integral *with partition function in Euclidean space:*

$$Z[A] = \int D\psi_L D\bar{\psi}_L \ \exp\left(-\int d^4x \ i\bar{\psi}_L \bar{\sigma}^\mu \mathcal{D}_\mu \psi_L\right)$$

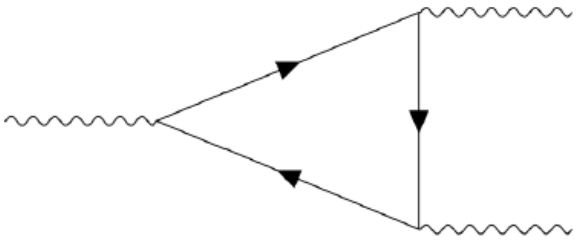
→ action *(exponent)* is invariant under gauge transformations

Anomaly - measure in path integral not invariant under gauge transformations

→ changes by phase: $Z[A] \to \exp\left(\frac{ie^3q^3}{32\pi^2}\int d^4x \ \alpha F_{\mu\nu}{}^\star F^{\mu\nu}\right) \ Z[A] \qquad {}^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$

Only happens for fermions

→ masses for fermions break symmetries



Anomaly associated to triangle diagram

→ external legs - currents of *U(1)*

→ fermion in the loop

3 legs reflected in q^3 prefactor of anomaly

Gauge anomalies

Gauge anomalies

Anomaly - prevents from gauging

→ action invariant under gauge symmetry

→ measure / partition function not invariant

⇒ gauge field not dynamical field *(path integral)*

- *U(1)* gauge theory coupled to single Weyl fermion

→ not consistent gauge theory

→ multiple Weyl fermions ⇒ **cancels anomaly**

- Left-handed & right-handed fermions

→ charge q under *U(1)* (right-handed) → complex conjugation → $-q$ (left-handed)

→ left-handed Weyl fermions $(\psi_L)_i$ with q_i under *U(1)*

⇒ phase proportional to sum of q^3_i

→ theory consistent (*anomaly cancellation*) if: $\sum_i q_i^3 = 0$ or $\sum_{\text{left}} q_i^3 = \sum_{\text{right}} q_i^3$

Vector-like theories: pairs of Weyl fermions with $\pm q$ *(Dirac fermion with q)*

Abelian chiral gauge theories

- Can we write consistent Abelian chiral gauge theory with $q_i \in \mathbb{Z}$

→ *N* = 3 fermions, e.g. q_i = *(x,y,-z)*

Anomaly cancellation condition: $x^3 + y^3 = z^3$

⇒ no positive integer solutions

→ *N* = 4 fermions, e.g. q_i = *(x,y,z,-w)*

Anomaly cancellation condition: $x^3 + y^3 + z^3 = w^3$

⇒ solution, e.g.: 3, 4, 5, 6

Chiral gauge theory with *N* = 4 Weyl fermions of 2 positive & 2 negative charge

$$x^3 + y^3 = z^3 + w^3$$

→ solution, e.g.: 1, 12, 9, 10

Add gravity: $\sum_i q_i^3 = 0$ and $\sum_i q_i = 0$

→ simplest Abelian chiral gauge coupled to gravity ⇒ *N* = 5 Weyl fermions

$$q_i = \{1, 5, -7, -8, 9\}$$

Non-Abelian gauge anomalies

- Non-Abelian gauge symmetry *G*

→ **Assumption:** single Weyl fermion in representation *R* of group *G*

⇒ generator : T_R^A

→ gauge transformation: $\psi_L \to e^{ig\alpha^A(x)T_R^A}\psi_L \quad \text{and} \quad A_\mu \to \Omega A_\mu \Omega^{-1} + \frac{i}{g}\Omega\partial_\mu\Omega^{-1} \qquad \Omega = e^{i\alpha^A T^A}$

T^A - fundamental representation

→ partition function changes by phase *(fields non-Abelian)*

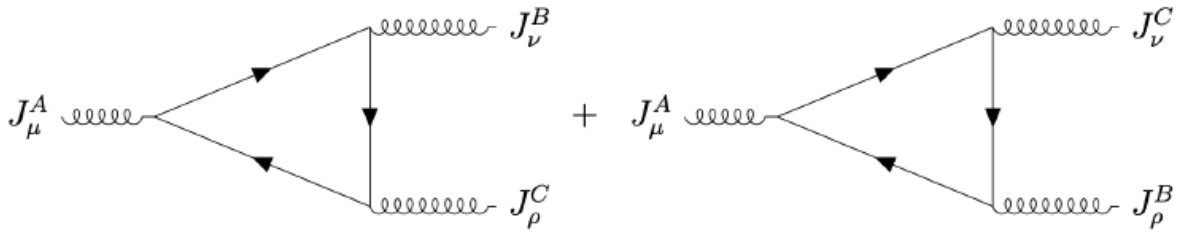
$$Z[A] \to \exp\left(\frac{ig^3 A(R)}{16\pi^2}\int d^4x\ \mathrm{Tr}\,(\alpha F_{\mu\nu}{}^\star F^{\mu\nu})\right)\ Z[A]$$

A(R) - group theoretic factor

→ fundamental representation: *A(R)* = 1

→ other representations: $\mathrm{Tr}\, T_R^A \{T_R^B, T_R^C\} = A(R)\,\mathrm{Tr}\, T^A \{T^B, T^C\}$

emergence of anti-commutator ⇒ sum over indices in triangle diagrams



Non-Abelian gauge anomalies

- Non-Abelian gauge theory coupled to left-handed Weyl fermions

$$\sum_i A(R_i) = 0$$

→ for non-Abelian (*and Abelian*): anomally cancellation ⇒ fermions in pairs (±*q*)

If *R* - complex representation ⇒ conjugate representation $A(\bar{R}) = -A(R)$

→ cancel anomaly - coupling gauge field to left-handed fermions in *R* and $\bar{R}$ pairs

→ cancel anomaly - Dirac fermion

A(R) = 0 for real representation

→ couple single Weyl fermion in real representation to non-Abelian gauge group

e.g. *SU(N)* coupled to single adjoint Weyl fermion

⇒ *super-Yang-Mills*

SU(N) coupled to single fundamental Weyl fermion ⇒ not quantum theory

Non-Abelian gauge anomalies

- Property of anomalies

  Only massless fermions contribute to anomalies

  $\Rightarrow$ contributions from massive fermions always cancel

- **Assumption:** Dirac mass for fermion in complex representation preserving symmetry

  $\rightarrow$ left-handed ψ_L + right-handed ψ_R

  $\rightarrow$ mass term: $\bar{\psi}_L \psi_R$

  $\rightarrow$ contribution to anomaly from 2 Weyl fermions cancels

  Only fermions that cannot get mass preserving *G* contribute to anomaly for *G*

  $\Rightarrow$ only gauge groups with anomalies $\rightarrow$ with complex representations

  $\rightarrow$ gauge anomalies in groups: $G = \begin{cases} SU(N) \text{ with } N \geq 3 \\ SO(4N+2) \\ E_6 \end{cases}$ and $G = U(1)$

  $N \geq 2$: anomaly vanishes for E_6 & $SO(4N+2)$

  $G = SU(N)$ with $N \geq 3$

Non-Abelian chiral gauge theories

- Chiral non-Abelian gauge theories

  → **Assume:** left-handed & right-handed fermions transform in different representations

- Gauge group $G = SU(N)$

  Anomaly coefficients: $A = N + 4$ (*symmetric*) $A = N - 4$ (*anti-symmetric*)

  anti-fundamental representation $\bar{N}$: $A = -1$

  → construct chiral gauge $G = SU(N)$ with $N - 4\bar{N}$ left-handed Weyl fermions

  ⇒ e.g. $G = SU(5)$ with **10** and $\bar{\mathbf{5}}$

- Chiral gauge theory E_6 or $SO(4N+2)$ with complex representations

  → anomaly coefficients vanish

  e.g. $SO(10)$ with single Weyl fermion in **16** representation

  ⇒ spinor representation of $SO(10)$

Chiral gauge theories (*non-Abelian & Abelian*)

→ not possible to have mass terms for fermions

→ $\chi_L \psi_L$ or $\bar{\chi}_R \psi_L$ not gauge invariant

Mixed anomalies

- **Assumption:** Single Weyl fermion

→ coupled to background non-Abelian gauge field A_μ in representation R of $G = SU(N)$

→ + Abelian gauge field a_μ

Partition function:

$$Z[A;a] = \int D\psi_L D\bar{\psi}_L \ \exp\left(-\int d^4x \ i\bar{\psi}_L \bar{\sigma}^\mu \mathcal{D}_\mu \psi_L\right) \qquad \mathcal{D}_\mu \psi_L = \partial_\mu \psi_L - igA_\mu^A T_R^A \psi_L - ieqa_\mu \psi_L$$

→ *U(1)* gauge transformation: $\psi_L \to e^{ieq\alpha}\psi_L$

phase (*depends on U(1) field strength*) + *I(R)* (*depends on the SU(N) field strength*)

$$Z[A;a] \to \exp\left(\frac{ie^3q^3}{32\pi^2}\int d^4x \ \alpha f_{\mu\nu}{}^\star f^{\mu\nu} + \frac{ieg^2qI(R)}{16\pi^2}\int d^4x \ \alpha \mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}\right) Z[A;a]$$

→ *I(R)* - Dynkin index: $\mathrm{Tr}\, T_R^A T_R^B = \frac{1}{2} I(R)\,\delta^{AB} \qquad T_R^A T_R^A = C(R)\,\mathbb{1}$

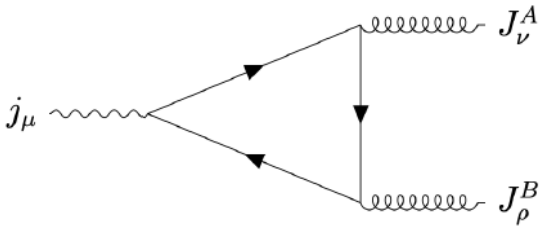
conjugate representation: $I(\bar{R}) = I(R)$

Second term → **mixed anomaly**

⇒ single *U(1)* charge *q* + 2 non-Abelian charges

Mixed anomalies

- Perturbation theory



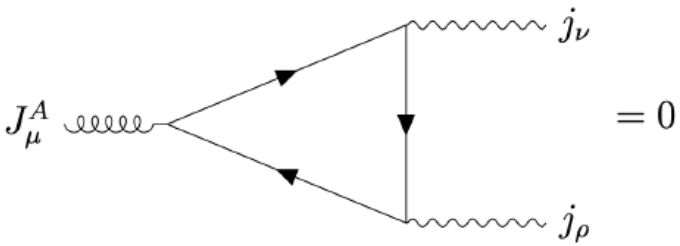
- Consistent gauge theory

  → mixed anomalies must cancel

  → left-handed fermions with *U(1)* & charge q_i in *SU(N)* representations R_i

  Anomaly cancellation: $\sum_i q_i I(R_i) = 0$

- **Assume:** Single non-Abelian current + 2 Abelian currents



  → vanishes automatically

  ⇒ proportional to trace of generator: Tr $T^A = 0$

Witten anomaly

- *G = SU(2)* - smallest *G = SU(N)* gauge group

  → representations real or pseudoreal *(representation isomorphic to complex conjugate)*

  ⇒ no perturbative gauge anomalies

  → fundamental representation - generators: $T^A = \frac{1}{2}\sigma^A$

  → little group ⇒ $\mathrm{Tr}\,\sigma^A\{\sigma^B, \sigma^C\} = 0 \qquad A, B, C = 1, 2, 3$

  ⇒ no anomaly

SU(N) coupled to single fundamental Weyl fermion

→ inconsistent for $N \geq 3$

→ consistent for $N = 2$

Witten anomaly *(SU(2) anomaly)*

→ not appear in perturbation theory

→ field configurations to sum over in path integral that wind around Euclidean spacetime

Winding characterised by homotopy theory - homotopy group element:

$$\Pi_4(SU(2)) = \mathbb{Z}_2$$

⇒ cancel anomaly ⇒ *SU(2)* gauge theory with even nr of fundamental Weyl fermions

Chiral anomalies

Chiral anomalies

Anomaly for symmetry group - depends on symmetry is global or gauged

→ consistency conditions on any gauge theory *(charges, representations etc.)*

- Global quantum theory

  Chiral anomaly (ABJ anomaly - Adler, Bell, Jackiw)

  → mixed anomaly of *U(1)* global symmetry & gauge symmetry *G*

- **Assumption:** Left-handed fermions

  → transforming in representations R_i under $G = SU(N)$ gauge symmetry

  → **Assume:** q_i under global *U(1)* symmetry of classical action

Euclidean partition function:

$$\mathcal{Z} = \int \mathcal{D}A \ \exp\left(-\frac{1}{2}\int d^4x \ \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\right) \ Z[A]$$

A - non-Abelian gauge field

$Z[A]$ - partition function for fermions

Chiral anomalies

$$Z[A] = \int D\psi_{L\,i} D\bar{\psi}_{L\,i}\ \exp\left(-\int d^4x\ i \sum_i \bar{\psi}_L \bar{\sigma}^\mu \mathcal{D}_\mu \psi_L\right)$$

Global *U(1)* transformation:

$$\psi_{L\,i} \to e^{i\alpha q_i} \psi_{L\,i} \qquad \alpha \in \mathbb{R}$$

mixed anomaly $\Rightarrow$ partition function is not invariant

Fermionic part of partition function

$$Z[A] \to \exp\left(\frac{i\alpha}{16\pi^2} \sum_i q_i I(R_i) \int d^4x\ \mathrm{Tr}\, F_{\mu\nu} {}^\star F^{\mu\nu}\right) Z[A]$$

$\to$ classical action invariant under global *U(1)* symmetry

$\to$ symmetry of quantum theory $\Rightarrow$ fermionic measure must be invariant

Cancellation condition:

$$\sum_i q_i I(R_i) = 0$$

Axial anomaly in QCD

- **Example:** Anomaly for $U(1)_A$ axial symmetry of QCD

→ **Assume:** $G = SU(N_c)$ coupled to N_f massless Dirac fermions

Classical action: $$S = \int d^4x \left(-\frac{1}{2}\mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu} + i\sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu \mathcal{D}_\mu \psi_i \right)$$

→ $U(N_f)_L \times U(N_f)_R$ global symmetry - each rotating q_L & q_R independently

$U(1)$ subgroups: $$U(1)_V : \ \psi_{Li} \to e^{i\alpha}\psi_{Li} \quad \text{and} \quad \psi_{Ri} \to e^{i\alpha}\psi_{Ri}$$
$$U(1)_A : \ \psi_{Li} \to e^{i\alpha}\psi_{Li} \quad \text{and} \quad \psi_{Ri} \to e^{-i\alpha}\psi_{Ri}$$

$U(1)_V$ - vector-like symmetry *(same on left- & right-handed spinors)* - baryon number in SM

$U(1)_A$ - axial symmetry *(different on left & right-handed spinors)*

→ ψ_L transforms in $\mathbf{N_c}$ *of* $SU(N_c)$ conjugated right-handed $\bar{\psi}_R$ in $\bar{\mathbf{N}}_\mathbf{c}$ ⇒ Dynkin index:

$$I(\mathbf{N_c}) = I(\overline{\mathbf{N}}_\mathbf{c}) = 1$$

→ **under $U(1)_V$**: ψ_L charge +1 $\bar{\psi}_R$ charge -1

⇒ anomaly vanishes → $U(1)_V$ - good symmetry of quantum theory

→ **under $U(1)_A$**: ψ_L charge +1 $\bar{\psi}_R$ charge +1

⇒ anomaly not vanish → $U(1)_A$ - not symmetry of quantum theory

no associated light Goldstone boson *(η' much heavier than pions)*

Noether's theorem - anomalous symmetries

- Theory with continuous symmetry

  → Noether's theorem ⇒ corresponding conserved current J^μ

  continuity: $\partial_\mu J^\mu = 0$

- **Assume:** Anomalous symmetry

  → symmetry of the classical action

  → no symmetry of full quantum theory

  What about conservation of current?

  Transformation of scalar field ϕ :

  $$\delta\phi = \alpha X(\phi) \qquad \alpha \text{ - infinitesimally small parameter}$$

  → symmetry if:

  $$\delta\mathcal{L} = 0$$

  Prove Noether's theorem

  → allow constant α to depend on spacetime: $\alpha = \alpha(x)$

Noether's theorem - anomalous symmetries

- Lagrangian no longer invariant

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu(\delta\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu(\alpha X(\phi)) + \frac{\partial\mathcal{L}}{\partial\phi}\,\alpha X(\phi)$$

$$= (\partial_\mu\alpha)\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,X(\phi) + \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu X(\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\,X(\phi)\right]\alpha$$

$\delta\mathcal{L} = 0$ when α is constant $\Rightarrow$ must vanish

$$\delta\mathcal{L} = (\partial_\mu\alpha)J^\mu \quad \text{with} \quad J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,X(\phi)$$

$\rightarrow$ action: $S = \int d^4x\ \mathcal{L}$

$$\delta S = \int d^4x\ \delta\mathcal{L} = \int d^4x\ (\partial_\mu\alpha)J^\mu = -\int d^4x\ \alpha\,\partial_\mu J^\mu$$

$\alpha(x)$ decay asymptotically

$\rightarrow$ for any field ϕ with change $\delta\phi$

ϕ in classical equations of motion $\Rightarrow$ $\delta S = 0$ for *any* $\delta\phi$ *(also for transformation with $\alpha(x)$)*

$\Rightarrow$ equations of motion satisfied $\rightarrow$ conservation law (**Noether's theorem**)

$$\partial_\mu J^\mu = 0$$

Free fermion

Assume: Free, massless Dirac fermion ψ with action: $S = -\int d^4x\, i\bar{\psi}\gamma^\mu\partial_\mu\psi \qquad \bar{\psi} = \psi^\dagger\gamma^0$

→ 2 symmetries: vector & axial

- Vector symmetry in terms Dirac fermion: $\psi \to e^{i\alpha}\psi$

  → infinitesimally: $U(1)_V : \delta\psi = i\alpha\psi \quad \text{and} \quad \delta\bar{\psi} = -i\alpha\bar{\psi}$

  → associated current: $J_V^\mu = \bar{\psi}\gamma^\mu\psi$

- Axial symmetry: $\psi \to e^{i\alpha\gamma^5}\psi$

  → infinitesimally: $U(1)_A : \delta\psi = i\alpha\gamma^5\psi \quad \text{and} \quad \delta\bar{\psi} = i\alpha\bar{\psi}\gamma^5$

  → associated current: $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$

Turn on mass term for fermion: $S = -\int d^4x\, i\bar{\psi}\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi$

→ action invariant under *U(1)*$_V$: $\partial_\mu J_V^\mu = 0$

→ mass term is not invariant under *U(1)*$_A$

Noether's theorem $\alpha = \alpha(x)$ → extra term to $\delta S = \int d^4x\, (\partial_\mu\alpha) J_A^\mu + 2im\alpha\bar{\psi}\gamma^5\psi$

→ δS =0 for all transformations ⇒ axial current must satisfy: $\partial_\mu J_A^\mu = 2im\bar{\psi}\gamma^5\psi$

Conservation of axial charge fails when fermion has mass

Conservation law - anomalous symmetries

How is conservation of axial charge affected by anomaly?

- **Assume:** N_f massless Dirac fermions coupled to Yang-Mills

→ action: $S_\theta = \int d^4x \left(-\frac{1}{2g^2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \frac{\theta g_s^2}{16\pi^2} \mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu} - i \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu \mathcal{D}_\mu \psi_i \right)$

→ shift θ angle *(effect of anomaly)*: $U(1)_A : \theta \to \theta + 2N_f \alpha$

→ axial current: $\partial_\mu J^\mu_A = \frac{N_f g^2}{8\pi^2} \mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}$ ⇒ anomaly equation of classical action

Operator equation in quantum field theory - **Ward identity**

→ anomaly equation ⇒ axial symmetry not conserved

but $\mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}$ - total derivative:

$$\mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu} = 2\partial_\mu K^\mu \quad \text{with} \quad K^\mu = \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr} \left(A_\nu \partial_\rho A_\sigma - \frac{2i}{3} A_\nu A_\rho A_\sigma \right)$$

→ define combination of $J^\mu{}_A$ and K^μ to construct conserved current

but K^μ not gauge invariant

Conservation law - anomalous symmetries

Under what conditions axial charge change?

- Measure axial charge $\Rightarrow$ integrate over spatial slice

$$Q_A = \int d^3x \; J_A^0$$

$\rightarrow$ change in axial charge from $t \rightarrow -\infty$ to $t \rightarrow +\infty$

$$\Delta Q_A = \int dt d^3x \; \frac{\partial J_A^0}{\partial t} = \int d^4x \; \partial_\mu J_A^\mu = \frac{N_f g_s^2}{8\pi^2} \int d^4x \; \mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}$$

Integral of $\mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}$ is quantised

$\Rightarrow$ Q_A - various integer amounts

Weak coupling $\Rightarrow$ violation of axial charge mediated by instantons

Neutral pion decay

$$\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$$

- Lifetime ~ 10^{-16} *s*

$$\pi^0 \to \gamma\gamma$$

- **Assume:** Axial anomaly also in QED coupled to massless fermions

  → N_f Dirac fermions ψ_i with charge Q_i under *U(1)* gauge symmetry

  → chiral anomaly in axial symmetry: $\psi_i \to e^{i\alpha\gamma^5 q_i}\psi_i$

  → associated current: $\partial_\mu J^\mu_A = \left(\sum_i q_i Q_i^2\right) \frac{1}{16\pi^2} F_{\mu\nu}{}^\star F^{\mu\nu}$ (from triangle diagram with 1 $J^\mu{}_A$ leg + 2 γ legs)

  charges: linear in axial q_i & quadratic in gauge Q_i

- **Assume:** $N_f = 2$ light quarks *(up & down)*

  → assume they are massless

  ⇒ QCD action with $U(1)_V \times SU(2)_L \times SU(2)_R$ symmetry

  → coupling to γ with quark charges: $Q_1 = \frac{2}{3}$ and $Q_2 = -\frac{1}{3}$

  ⇒ breaks flavour symmetry to: $U(1)_L \times U(1)_R \subset SU(2)_L \times SU(2)_R$

Neutral pion decay

- Combine into new vector & axial symmetries:

$$U(1)'_V: \quad u \to e^{i\alpha} u \quad \text{and} \quad d \to e^{-i\alpha} d$$
$$U(1)'_A: \quad u \to e^{i\alpha\gamma^5} u \quad \text{and} \quad d \to e^{-i\alpha\gamma^5} d$$

→ *U(1)'$_V$* - anomaly-free

→ *U(1)'$_A$* - anomaly-free with QCD gauge field *(cancellation in charges)*

U(1)'$_A$ - anomaly with QED gauge field *(each quark N_c = 3 colours - SU(3) gauge group)*

⇒ chiral anomaly: $\partial_\mu J'^{\mu}_A = N_c \left(\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 \right) \frac{1}{16\pi^2} F_{\mu\nu}{}^\star F^{\mu\nu} = \frac{N_c}{48\pi^2} F_{\mu\nu}{}^\star F^{\mu\nu} \qquad N_c = 3$

- Additional axial current: $J'^{\mu}_A = \bar{u}\gamma^\mu\gamma^5 u - \bar{d}\gamma^\mu\gamma^5 d$

→ creates neutral pion: $\langle 0|J'^{\mu}_A(x)|\pi^0(p)\rangle = -i f_\pi \delta^{ab} p^\mu e^{-ix\cdot p}$

→ anomaly equation ⇒ amplitude for $\pi^0 \to \gamma\gamma$

- proportional to N_c

- experimental method to determine $N_c = 3$

Surviving discrete symmetries

- Anomalous symmetry → shifting θ angle

  → $\theta \in [0, 2\pi)$ ⇒ anomalous *U(1)* symmetry transformation: $\theta \to \theta+2\pi$

  ⇒ θ not changed → discrete subgroup of *U(1)* remains

- $U(1)_A : \theta \to \theta + 2N_f\alpha$

  → under $U(1)_A$ transformation $e^{-i\alpha} = e^{2\pi i/2N_f}$: $\theta \to \theta+2\pi$

  → compensating $U(1)_V$ transformation ⇒ surviving subgroup:

$$\mathbb{Z}_{N_f} : \quad \psi_{Li} \to e^{2\pi i/N_f}\psi_{Li} \quad \text{and} \quad \psi_{Ri} \to \psi_{Ri}$$

  ⇒ $SU(N_f)_L$ global symmetry

- *SU(N)* Yang-Mills coupled to single massless Weyl spinor λ in adjoint representation

  → adjoint representation real ⇒ no gauge anomaly

  ⇒ **super Yang-Mills** *(classically: global U(1) symmetry rotating phase of λ:* $U(1) : \quad \lambda \to e^{i\alpha}\lambda$ *)*

  → quantum mechanically - anomaly under *U(1)*

  → $I(\text{adj}) = 2N$ ⇒ $U(1) : \quad \theta \to \theta + 2N\alpha$ (*U(1)* symmetry is anomalous)

  But: discrete $\mathbb{Z}_{2N}$ survives as: $\theta \to \theta+2\pi$

$$\mathbb{Z}_{2N} : \quad \lambda \mapsto e^{2\pi i/2N}\lambda$$

’t Hooft anomalies

’t Hooft anomalies

What if there is anomaly just for global symmetry?

→ mixed anomaly between global & gauge symmetries ⇒ no global symmetry

→ symmetry not conserved if couple to background gauge fields

- Anomaly → classify quantum field theories

2 QFTs with same global symmetry group can be deformed into each other if share same anomaly

→ anomalies provide constraints on theory

’t Hooft anomalies - anomalies in global symmetries

- Theory with global symmetry G_F

→ anomaly for non-Abelian symmetry G_F: $\sum A(R_i)$

→ symmetry not broken - anomaly remains unchanged if theory flows to strong coupling

- **Assume: Non-Abelian gauge theory weakly coupled in UV, strongly coupled in IR**

→ e.g.: QCD

→ no mixed anomalies with gauge symmetry

't Hooft anomalies

UV theory may have 't Hooft anomaly for G_F

- If G_F Abelian

  → anomaly $\mathcal{A}_{UV}$: $\sum Q_i^3$

- If G_F non-Abelian

  → anomaly $\mathcal{A}_{UV}$: $\sum A(R_i)$

Assume: UV theory flows under RG *(renormalisation group)* to IR theory

→ in QCD: theory of mesons & baryons

't Hooft anomaly matching

Either symmetry G_F is spontaneously broken or anomalies: $\mathcal{A}_{UV} = \mathcal{A}_{IR}$

→ G_F spontaneously broken ⇒ massless Goldstone bosons

→ G_F unbroken ⇒ massless fermions to reproduce anomaly

Confinement → chiral symmetry breaking

- Strong coupling effects in QCD

  → confinement

  → chiral symmetry breaking

't Hooft anomalies: **confinement ⇒ chiral symmetry breaking**

- **Assume:** $SU(N_c)$ gauge theory coupled to N_f massless Dirac fermions q_i in fundamental representation

  → vector-like theory ⇒ no gauge anomaly

  → $U(1)_A$ axial symmetry has chiral anomaly

  ⇒ global symmetry of the theory:

$$G_F = U(1)_V \times SU(N_f)_L \times SU(N_f)_R$$

- 't Hooft anomalies for G_F

  UV weakly coupled

  → fermions free

  → no 't Hooft anomaly for $U(1)^3_V$ *(vector-like symmetry)*

Confinement → chiral symmetry breaking

UV - two 't Hooft anomalies in chiral $SU(N_f)$ factors

1. non-Abelian anomaly: $[SU(N_f)_L]^3: \quad \mathcal{A} = \sum A(\square) = N_c$

   anomaly - left-handed quark q_L transforms in fundamental □ of $SU(N_f)_L$ & $A(\square) = 1$

2. mixed 't Hooft anomaly between $U(1)_V$ & $SU(N_f)$: $[SU(N_f)_L]^2 \times U(1)_V: \quad \mathcal{A}' = \sum qI(\square) = N_c$

   anomaly - number of quark colours → N_c fermions

IR strongly coupled

- Low N_f: G_F broken to $U(1)_V \times SU(N_f)_{diag}$

Assume: theory confines & IR physics described by weakly interacting mesons & baryons

→ 't Hooft anomaly matching ⇒ chiral symmetry broken

N_c is even: mesons & baryons are bosons

confining theory with no fermions at low-energies ⇒ cannot reproduce anomaly

N_c is odd: baryons are fermions

baryons are massive *(so no massless particles)* ⇒ cannot reproduce anomaly

If QCD confines into weakly interacting colour singlets ⇒ chiral symmetry is broken

Representations of massless baryons

- **Restrict to $N_c = 3$**

→ if *SU(3)* gauge group confines ⇒ massless fermion must be colour singlet
(baryons comprised of 3 quarks)

→ under $SU(N_f)_L \times SU(N_f)_R \subset G_F$: left- handed transform as $(\mathbf{N_f, 1})$ *(charge +1 under $U(1)_V$)*
right-handed transform as $(\mathbf{1, N_f})$ *(charge +1 under $U(1)_V$)*

→ baryons in QCD: spin = 1/2 or 3/2

Weinberg-Witten theorem

Cannot form massless bound states with helicity $\lambda \geq 1$

→ massless baryons must have $\lambda = \pm 1/2$

1. 3 left-handed spinors ⇒ 3 possible representations of $SU(N_f)_L$

$$\boxed{L}\boxed{L}\boxed{L} \quad , \quad \begin{array}{|c|}\hline L \\ \hline L \\ \hline L \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|}\hline L & L \\ \hline L \\ \cline{1-1} \end{array}$$

Young diagrams

2. 1 left-handed & 2 right-handed ⇒ baryons in representations of $SU(N_f)_L \times SU(N_f)_R$

$$\boxed{L} \otimes \boxed{R}\boxed{R} \quad , \quad \boxed{L} \otimes \begin{array}{|c|}\hline R \\ \hline R \\ \hline \end{array}$$

+ same options for massless right-handed baryons: $L \leftrightarrow R$

Representations of massless baryons

Which combination reproduces 't Hooft anomalies of UV theory?

1) assign index p_α (α = 1,...,5) to each of 5 baryons in **1.** and **2.**

$|p_\alpha|$ - number of baryon species in massless spectrum

left-handed: $p_\alpha > 0$ right-handed: $p_\alpha < 0$

$\rightarrow$ find p_α satisfying anomaly matching & reproduce $\mathcal{A}$ and $\mathcal{A}'$

2) define little group theory: representation **R** of $SU(N_f)$

$\rightarrow$ dim(R), anomaly coefficient $A(R)$, Dynkin index $I(R)$

IR anomalies for p_α massless baryons of each type

Anomaly for $SU(N_f)^3_L$ with $N_f \geq 3$

$$\mathcal{A} = \frac{1}{2}(N_f+3)(N_f+6)p_1 + \frac{1}{2}(N_f-3)(N_f-6)p_2 + (N_f^2-9)p_3 + \left(\frac{1}{2}N_f(N_f+1) - N_f(N_f+4)\right)p_4 + \left(\frac{1}{2}N_f(N_f-1) - N_f(N_f-4)\right)p_5$$

Anomaly for $SU(N_f)^2 \times U(1)_V$ *(charge 3 under $U(1)_V$)*

$$\frac{\mathcal{A}'}{3} = \frac{1}{2}(N_f+2)(N_f+3)p_1 + \frac{1}{2}(N_f-2)(N_f-3)p_2 + (N_f^2-3)p_3 + \left(\frac{1}{2}N_f(N_f+1) - N_f(N_f+2)\right)p_4 + \left(\frac{1}{2}N_f(N_f-1) - N_f(N_f-2)\right)p_5$$

Match anomalies: find p_α that $\mathcal{A} = \mathcal{A}' = 3$

No solution to 2nd anomaly: $\mathcal{A}'/3$ for IR is multiple of 3 $\Rightarrow$ cannot reproduce UV anomaly $\mathcal{A}'/3 = 1$

G = SU(3) gauge theory with 3 massless fermions must spontaneously break G_F flavour symmetry

Decoupling massive quarks

N_f is not multiple of 3

- $N_f = 4$

  $\rightarrow$ no solution to anomaly matching for $N_f = 4$

  $\Rightarrow$ consistent with decoupling of massive quarks

Extend to all N_f & to all N_c

't Hooft anomaly matching $\Rightarrow$ chiral symmetry broken for all $N_c \geq 2$ and all $N_f \geq 3$

- $N_f = 2$

  $\rightarrow$ only possible mixed $SU(2)^2_L \times U(1)_V$ 't Hooft anomaly

  $\rightarrow$ 't Hooft matching condition for gauge group $SU(3)$: $\frac{\mathcal{A}'}{3} = 10p_1 - 5p_4 + p_5 = 1$

  $\Rightarrow$ many solutions

  can match anomaly if massless baryons transforming under:

$$SU(2)_L \times SU(2)_R \times U(1)_V$$

  $\rightarrow$ cannot use 't Hooft anomaly matching to rule out existence of massless baryons

  Lattice calculations $\rightarrow$ for $N_f = 2$ chiral symmetry broken & no massless baryons