Standard Model

Lecture 3
Strong force

Marcin Kucharczyk

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Lecture 3

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- Chiral symmetry breaking
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Strong force

- Quantum electrodynamics (QED)

  → QFT action for electromagnetic force:

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi - m\bar{\psi}\psi \right)$$

  $F_{\mu\nu}$ - field strength of electromagnetism *(excitations - photons)*
  ψ - Dirac spinor describing electron

  → possible to add fields for electrically charged particles *(e.g. muon)*

  upon quantisation → quantum electrodynamics

- QED - perturbative quantum field theory

  → massless photon

  → charged fermion of mass *m*

Strong force

- Quantum chromodynamics (QCD)
- *U(1)* group of electromagnetism replaced by gauge group: $G = SU(3)$

  → strong force action: $S = \int d^4x \left(-\frac{1}{2} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu} + i \sum_i \bar{q}_i \not{D} q_i - m_i \bar{q}_i q_i \right)$

Strong force

- Gauge field G_μ

  → Yang-Mills field with Lie-algebra (G_μ - 3×3 Hermitian matrix)

  → gauge indices: $(G_\mu)^a{}_b$ $a,b = 1,2,3$ *(additional index -* ***colour****)*

  → 8 gauge bosons (***gluons***) contained within G_μ $\quad \dim SU(N) = N^2 - 1$

- Decompose G_μ into gluon fields:

$$G_\mu = G_\mu^A T^A \qquad T^A \text{ - generators of } SU(3)$$

$$\mathrm{Tr}(T^A T^B) = \frac{1}{2}\delta^{AB} \qquad \text{in anotrher basis:} \quad T^A = \frac{1}{2}\lambda^A$$

- 3×3 Gell-Mann matrices λ^A :

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

  → *SU(2)* - sub-group of *SU(3)*

  ⇒ Pauli matrices in λ^1, λ^2, λ^3

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

  → 2 diagonal λ^3 and λ^8 *(SU(3) rank = 2)*

- Associated field strength: $\quad G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - i g_s [G_\mu, G_\nu]$

Strong force

- Strong coupling constant g_s

  → dimensionless coupling

  → characterises strength of strong force

- Quarks

  → Dirac spinors q_α α = 1, 2, 3, 4

  → transform in fundamental 3D representation of *SU(3)* *(carry colour index a = 1, 2, 3)*

  colour - complex normalised 3-vector rotated by *SU(3)* *(red, green, blue)*

- Covariant derivative for each quark *q* *(suppressing spinor index)*

$$\mathcal{D}_\mu q^a = \partial_\mu q^a - i g_s (G_\mu)^a{}_b q^b$$

- Flavour index $i = 1, \ldots, N_f$ $(N_f = 6)$

$$\begin{aligned} m_{\text{down}} &= 5 \text{ MeV} \quad \text{and} \quad m_{\text{up}} = 2 \text{ MeV} \\ m_{\text{strange}} &= 93 \text{ MeV} \quad \text{and} \quad m_{\text{charm}} = 1.3 \text{ GeV} \\ m_{\text{bottom}} &= 4.2 \text{ GeV} \quad \text{and} \quad m_{\text{top}} = 173 \text{ GeV} \end{aligned}$$

  → masses span 5 orders of magnitude

  → measured experimentally

Strong force

- 3 generations of quarks:

  (1) **down & up** (2) **strange & charm** (3) **bottom & top**

- **Electric charge**

  → quarks carry electric charge

  → fractional charges relative to electron

$$Q_{\text{down}} = -\frac{1}{3}e \quad \text{and} \quad Q_{\text{up}} = \frac{2}{3}e$$

  → strange & bottom: $Q = -\frac{1}{3}e$

  → charm & top: $Q = +\frac{2}{3}e$

Charge +2/3 quark is lighter than charge −1/3 quark

Strong coupling

Strong coupling

Why particles we see in the world not directly related to fields in fundamental Lagrangian?

→ don't see free quarks

→ strong force don't extend beyond nucleus

Asymptotic freedom

- Strong coupling constant g_s

→ not constant! *(depends on distance scale / energy scale)*

Assumption: Theory with $G = SU(N_c)$ gauge group coupled to N_f massless quarks

→ N_c - number of colours, N_f - number of flavours

→ gauge coupling g^2_s depends on energy scale μ:

$$\frac{1}{g_s^2(\mu)} = \frac{1}{g_0^2} - \frac{b_0}{(4\pi)^2}\log\frac{\Lambda_{UV}^2}{\mu^2}$$

g^2_0 - bare coupling in Lagrangian

$$g_s^2(\Lambda_{UV}) = g_0^2 \qquad b_0 = \frac{11}{3}N_c - \frac{2}{3}N_f$$

Running of coupling constant in terms of the one-loop β function

$$\beta(g) \equiv \mu\frac{dg_s}{d\mu} = -\frac{b_0}{(4\pi)^2}g_s^3$$

Asymptotic freedom

- Running of coupling constant

  → overall minus sign in β function

  → flow of coupling ⇒ theory weakly coupled at high energies *(strongly at low energies)*

  ⇒ **asymptotic freedom**

Asymptotic freedom only if nr of flavours sufficiently small:

$$N_f < \frac{11}{2} N_c$$

→ in QCD: $N_c = 3$, $N_f = 6$

- Non-Abelian gauge theories

  → asymptotic freedom rare in $d = 3+1$ dimensions

  → only happens for non-Abelian gauge theories

  ⇒ coupling bigger going to low energies

  opposite in e.g. QED fine structure constant, Yukawa couplings, self-interactions of scalars

QED: at $r \geq 10^{-12}$ m fine structure constant stops running → plateau at $\alpha \approx 1/137$

going to higher energies / shorter distances → α increases

Asymptotic freedom

- Yang-Mills theory

  → asymptotic freedom ⇒ Yang-Mills theory clear at high energies / short distance

  → theory of massless interacting gluon fields

  dynamics described by classical equations of motion + perturbative quantum corrections

  → not clear at large distances *(strong coupling)*

  β function computed in perturbation theory ⇒ valid only for: $g_s^2(\mu) \ll 1$

- Strong coupling scale

  At what scale does $g^2_s(\mu)$ diverges?

$$\Lambda_{\rm QCD} = \mu \exp\left(-\frac{8\pi^2}{b_0 g_s^2(\mu)}\right)$$ - **strong coupling scale *(QCD scale)***

  scale at which gauge coupling becomes strong

  → Renormalization Group invariant: $d\Lambda/d\mu = 0$

$$\Lambda_{\rm QCD} = \Lambda_{UV}\, e^{-8\pi^2/b_0 g_0^2}$$

  $g_0 \ll 1$ ⇒ small Λ_{UV} ⇒ physical scale $\Lambda_{\rm QCD}$ exponentially suppressed

  (Λ_{UV} cut-off: $\Lambda_{\rm QCD} \ll \Lambda_{UV}$)

  → low-energy scale naturally generated from high energy scale

QCD scale

- QCD scale for massive particles

  → quarks in QCD have mass

  → quark of mass m contributes to β function *(massless for scales $\mu \gg m$)*

  → decouples for scales $\mu \ll$ m

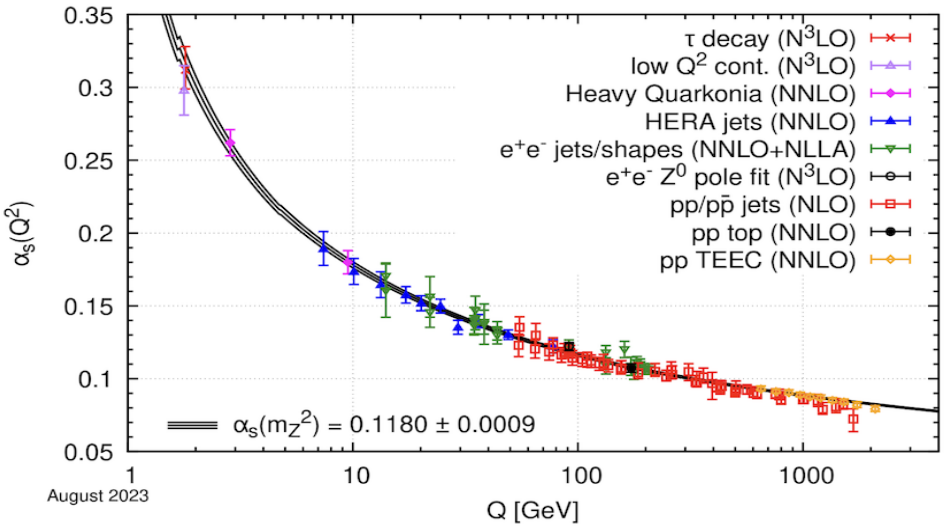
- Different masses

  → β function acts as having $N_f = 6$ massless quarks for $\mu \gg 173$ GeV

  → β function acts as having $N_f = 5$ massless quarks for $4.2\ \text{GeV} \ll \mu \ll 173\ \text{GeV}$

  .....

Combined experimental data for running of $\alpha_s = g_s^2/4\pi$



QCD scale

What is strong coupling scale Λ_{QCD}?

For strong force:

$$\Lambda_{\mathrm{QCD}} \approx 200\ \mathrm{MeV}$$

→ can vary *(different regularisation schemes)*

→ give energy scale at which physics of QCD takes place

Strength of coupling g_s

→ specify energy scale at which measurement is done

→ mass of Z-boson ($M_Z \approx 90$ GeV)

Measured strong coupling constant

$$\alpha_s(M_Z) = \frac{g_s^2(M_Z)}{4\pi} = 0.1184 \pm 0.0007$$

Mass gap

When coupling is small $\Rightarrow$ QFT similar to classical theory

$\rightarrow$ e.g. QED vs Maxwell theory

When coupling is large QFT different from classical

- Yang-Mills vs QCD

$\rightarrow$ Yang-Mills action *(ignoring quarks)*:

$$S = \int d^4x \; -\frac{1}{2}\mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu}$$

$\rightarrow$ massless interacting spin 1 fields

coupling strong at low energies $\Rightarrow$ cannot apply perturbative techniques

Quantum Yang-Mills is not theory of massless particles

$\rightarrow$ lightest particle of $m \sim \Lambda_{QCD}$

$\rightarrow$ particle called **glueball**

$\rightarrow$ theory is gapped

Gap between ground state and first excited state of energy $E = mc^2$

Short distance Coulomb force

Confinement

- **Assumption:** General case of *G* = *SU(N)* gauge theory

  2 test particles: (1) quark in fundamental representation $\mathbf{N}$

  (2) anti-quark in anti-fundamental $\overline{\mathbf{N}}$

  → if separated by short distance: $r \ll \Lambda_{\rm QCD}^{-1} \approx 5 \times 10^{-15}{\rm m}$

  ⇒ g^2_s is small → classical result

  → if separated by large distance: $r \gg \Lambda_{\rm QCD}^{-1}$

  ⇒ strongly coupled physics

- Short-distance regime: $r \ll \Lambda_{\rm QCD}^{-1}$

  → Compton wavelength of particle of mass *m:* λ ~ 1/*m*

  ⇒ distance separating 2 quantum particles *r* >> λ

Short-distance force by 2 quarks

→ quark mass $m \gg \Lambda_{\rm QCD}$

→ analysis valid only for *c*-, *b*- and *t*-quarks

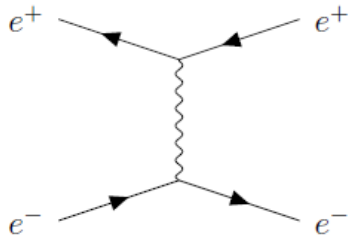
Short distance Coulomb force

- 2 particles of equal opposite charges ±*e* *(separated by r)*

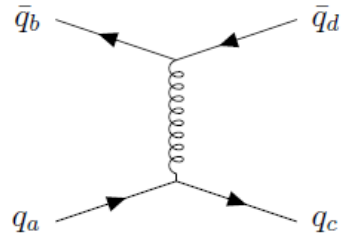
→ attractive Coulomb force:

$$V(r) = -\frac{e^2}{4\pi r}$$

→ **QED** - tree-level exchange of a single photon:



→ **QCD** - gluon as intermediary (colour indices: *a,b,c,d* = 1, . . . ,*N*)



Feynman rules for QCD

→ tree level potential between 2 quarks ⇒ same Coulomb force law:

$$V(r) = \frac{g_s^2}{4\pi r} T^A_{ca} T^{\star A}_{db}$$

Short distance Coulomb force

- N^2 possibilities for ingoing particles (a, b = 1,...,N) + N^2 for outgoing (c, d = 1,...,N)

  Reduces to simple group theory:

  $$\mathrm{N} \otimes \overline{\mathrm{N}} = 1 \oplus \mathrm{adj}$$

  adjoint representation - dimension $N^2 - 1$, $T^A T^{*A}$ - $N^2 \times N^2$ matrix

  → 2 eigenvalues for each representation

  ⇒ 2 different coefficients for forces

Result

- 2 particles in representations R_1 and R_2: $\mathrm{R} \subset \mathrm{R}_1 \otimes \mathrm{R}_2$

  → force by 2 particles proportional to:

  $$C(R) - C(R_1) - C(R_2)$$

  → *C(R)* - quadratic Casimir:

  $$T^A(R) T^A(R) = C(R)\,\mathbb{1}$$

In QCD with *G = SU(3)* leads to:

$$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}$$

→ 2 quarks attract in anti-symmetric $\overline{3}$ channel ⇒ add third quark

→ quarks experience mutual attraction in QCD

→ classical force ⇒ falls off with 1/r potential

Energetically preferable for 3 quarks to form colour singlet bound state *(confinement)*

Long distance confining force

What happens at large distance scales where Yang-Mills coupling is strong?

- Yang-Mills has mass gap

  → distance $\gg 1/\Lambda_{QCD}$ ⇒ force due to exchange of massive particles

- **Assumption:** Quark and anti-quark in $\mathbf{N}$ and $\overline{\mathbf{N}}$ representations

  → potential energy between 2 quarks grows linearly with distance:

$$V(r) = \sigma r$$

  → ***σ* - string tension** *(dimensions of energy per length)*

  Potential energy now increases with separation

  ⇒ infinite energy to pull quark-anti-quark pair to infinity

  General property of quantum Yang-Mills: only finite energy states are gauge singlets

  ⇒ **theory is confining** → quarks cannot exist as free particles

  → **mesons:** quark and anti-quark form colour singlet

  → **baryons:** colour singlet using invariant tensor: ϵ^{abc} of $SU(3)$

Effect of light quarks

- **Assumption:** Pure Yang-Mills theory

  → non-Abelian gauge group $G = SU(3)$

  → mass gap *(strong coupling scale Λ_{QCD})* + confinement

  confinement: for quark & anti-quark long-distance force law with linear behaviour

- **Assumption:** Yang-Mills coupled to single dynamical quark of $m >> \Lambda_{QCD}$

  Energy between quark-anti & quark

  → large distance $r \gg \Lambda_{\mathrm{QCD}}^{-1}$ $\Rightarrow$ confining potential: $V(r) = \sigma r$

  → stretching particles over $\sigma r > 2m$ $\Rightarrow$ create quark-anti-quark pair from vacuum

  $\Rightarrow$ quark pair breaks the string → 2 meson-like states

  Regime of confining force is limited

  → string tension scales as: $\sigma \sim \Lambda_{\mathrm{QCD}}^2$

  → confining force only in: $\frac{1}{\Lambda_{\mathrm{QCD}}} \ll r \ll \frac{m}{\Lambda_{\mathrm{QCD}}^2}$

  Real world QCD - no such window

  → lightest quark has mass $m << \Lambda_{QCD}$

  → string breaks through pair creation before getting to confining regime: $r \gg \Lambda_{\mathrm{QCD}}^{-1}$

Chiral symmetry breaking

Chiral symmetry breaking

- 'Actual' symmetry group of QCD action: $U(1)^{N_f}$

  → rotates phase of each Dirac quark field

Assume that (3) quarks are massless

- $\Lambda_{QCD} \approx 200$ MeV

  → charm, bottom and top quarks much heavier than QCD scale

  ⇒ decouple from low-energy dynamics *(not considered)*

  → m_{up} , $m_{down} << \Lambda_{QCD}$ - approximation to be massless

  → $m_{strange} < \Lambda_{QCD}$ - pretend to be massless

- $G = SU(N_c)$ Yang-Mills coupled to N_f flavours of massless fundamental fermions (quarks)

  → additional symmetry

  ⇒ each 4-component Dirac spinor decomposes to two 2-component Weyl spinors

$$q = \begin{pmatrix} q_L \\ q_R \end{pmatrix}$$

  → Weyl spinors q_L and q_R

  ⇒ colour index (1,...,N_c), flavour index (1,...,N_f) & 2-component spinor index

Chiral symmetry breaking

- QCD action in terms of Weyl fermions:

$$S = \int d^4x \left(-\frac{1}{2} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu} + i \sum_{i=1}^{N_f} \bar{q}_{L\,i} \bar{\sigma}^\mu \mathcal{D}_\mu q_{L\,i} + \bar{q}_{R\,i} \sigma^\mu \mathcal{D}_\mu q_{R\,i} \right)$$

→ classical Lagrangian has global symmetry:

$$G_F = U(N_f)_L \times U(N_f)_R$$

$$U(N_f)_L : q_{L\,i} \mapsto L_{ij} q_{L\,j} \quad \text{and} \quad U(N_f)_R : q_{R\,i} \mapsto R_{ij} q_{R\,j} \qquad L, R \in U(N_f)$$

G_F - chiral symmetry

→ acts differently on left-handed and right-handed Weyl spinors

→ only with massless fermions

⇒ addind mass term **breaks chiral symmetry** to its diagonal subgroup

- Axial symmetry $U(1)_A$ *(Abelian)*

→ left-handed & right-handed fermions transform with opposite phase:

$$q_{L\,i} \to e^{i\beta} q_{L\,i}$$
$$q_{R\,i} \to e^{-i\beta} q_{R\,i}$$

→ not a symmetry of full quantum theory ⇒ anomaly

Global symmetry group of quantum theory:

$$G_F = U(1)_V \times SU(N_f)_L \times SU(N_f)_R$$

→ $U(1)_V$ - baryon number related, both q_L and q_R transform in same way

Chiral Lagrangian

Chiral symmetry breaking $\Rightarrow$ existence of Goldstone bosons

- Theory with spontaneously broken continuous symmetry

  $\rightarrow$ manifold of ground states: $\mathcal{M}_0 = [SU(N_f)_L \times SU(N_f)_R] / SU(N_f)_V$

  $SU(N_f)_V$ is the diagonal subgroup of $SU(N_f)_L \times SU(N_f)_R$

  $\rightarrow$ points in $\mathcal{M}_0$ parameterised as: $\langle \bar{q}_{Li} q_{Rj} \rangle = -\sigma U_{ij} \quad U = L^\dagger R \in SU(N_f)$

  $\rightarrow$ $N_f^2 - 1$ Goldstone bosons excited in space and time ($U = U(x)$):

$$U(x) = \exp\left(\frac{2i}{f_\pi}\pi(x)\right) \quad \text{with} \quad \pi(x) = \pi^a(x)\, T^a$$

  $\pi(x)$ valued in Lie algebra $su(N_f)$, matrices T^a_{ij} - generators of $su(N_f)$

- Component fields $\pi^a(x)$, $a = 1,\ldots,N_f^2 - 1$ $\Rightarrow$ **pions**

  $\rightarrow$ only accurate for $N_f = 2$

  $\rightarrow$ expand field π in generators of $SU(2)$: $\pi = \frac{1}{2}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0 \end{pmatrix}$

  π^0 - neutral pion, $\pi^\pm$ - charged pions

- Pion decay constant f_π

  $\rightarrow$ expect: $f_\pi \sim \Lambda_{\mathrm{QCD}}$ measured: $f_\pi \approx 130\ \mathrm{MeV}$

Chiral Lagrangian - low-energy effective action

Theory that governs Goldstone bosons *U*

- **Assume:** Theory invariant under full global chiral symmetry $G_F = U(1)_V \times SU(N_f)_L \times SU(N_f)_R$

$$U(x) \to L^\dagger U(x) R$$

→ kinetic term: $\mathrm{tr}\,(\partial^\mu U^\dagger \partial_\mu U)$

Two-derivative Lagrangian describing dynamics of pions - **chiral Lagrangian**

$$\mathcal{L}_{\text{pion}} = \frac{f_\pi^2}{4}\,\mathrm{tr}\,(\partial^\mu U^\dagger\,\partial_\mu U)$$

→ non-linear sigma model *(by Gell-Mann and Lévy in 1960)*

→ *U* valued in $SU(N_f)$ ⇒ cannot set $U = 0$

Sigma-model describes degeneracy of ground states $U \neq 0$

→ ensures chiral Lagrangian spontaneously breaks $SU(N_f)_L \times SU(N_f)_R$

→ *U* - Goldstone boson associated to symmetry breaking

Chiral Lagrangian - pion scattering

Chiral Lagrangian contains infinite number of interaction terms

→ packed by demand of symmetry

- Chiral Lagrangian in terms of pion fields *U(x)*: $\mathcal{L}_{\text{pion}} = \text{tr}\,(\partial_\mu \pi)^2 - \frac{2}{3f_\pi^2}\text{tr}\,\left(\pi^2(\partial_\mu\pi)^2 - (\pi\partial_\mu\pi)^2\right) + \ldots$

  → if $\text{tr}\,T^a T^b = \frac{1}{2}\delta^{ab}$ for *su(N$_f$)* generators ⇒ kinetic term: $\text{tr}\,(\partial_\mu\pi)^2 = \frac{1}{2}\,\partial^\mu\pi^a\partial_\mu\pi^a$

- *N$_f$* = 2 & *su(2)* generators ~ Pauli matrices: $T^a = \frac{1}{2}\sigma^a$

  → quartic interaction terms: $\mathcal{L}_{\text{int}} = -\frac{1}{6f_\pi^2}\left(\pi^a\pi^a\partial\pi^b\partial\pi^b - \pi^a\partial\pi^a\pi^b\partial\pi^b\right)$

Tree-level $\pi\pi \to \pi\pi$ scattering amplitude:

$$i\mathcal{A}^{abcd} = \frac{i}{6f_\pi^2}\Big[\ \delta^{ab}\delta^{cd}\Big(4(p_a\cdot p_b + p_c\cdot p_d) + 2(p_a\cdot p_c + p_a\cdot p_d + p_b\cdot p_c + p_b\cdot p_d)\Big)\ + (b\leftrightarrow c) + (b\leftrightarrow d)\Big]$$

incoming momenta p_a, p_b, outgoing momenta - p_c, p_d $\quad p_a + p_b = p_c + p_d$

→ using Mandelstam variables (all particles massless):

$$s = (p_a+p_b)^2 = 2p_a\cdot p_b = 2p_c\cdot p_d \quad t = (p_a-p_c)^2 = -2p_a\cdot p_c = -2p_b\cdot p_d \quad u = (p_a-p_d)^2 = -2p_a\cdot p_d = -2p_b\cdot p_c$$

→ using relation *s* + *t* + *u* = 0: $i\mathcal{A}^{abcd} = \frac{i}{f_\pi^2}\Big[\delta^{ab}\delta^{cd}s + \delta^{ac}\delta^{bd}t + \delta^{ad}\delta^{bc}u\Big]$

Higher loop corrections + non-linear sigma model non-renormalisable

⇒ theory valid only at low-energies

Chiral Lagrangian - currents

- Operators - currents associated to chiral symmetry: $SU(N_f)_L \times SU(N_f)_R$

  → flavour currents for $SU(N_f)_L$ and $SU(N_f)_R$

$$J_L^{a\,\mu} = \bar{q}_{L\,i}\bar{\sigma}^\mu T^a_{ij} q_{L\,j} \quad \text{and} \quad J_R^{a\,\mu} = \bar{q}_{R\,i}\sigma^\mu T^a_{ij} q_{R\,j}$$

  T^a_{ij} - generators of $su(N_f)$

  → in terms of the vector and axial combinations $J_V^{a\,\mu} = J_L^{a\,\mu} + J_R^{a\,\mu} \quad J_A^{a\,\mu} = J_L^{a\,\mu} - J_R^{a\,\mu}$

$$J_V^{a\,\mu} = \bar{q}_i T^a_{ij} \gamma^\mu q_j \quad \text{and} \quad J_A^{a\,\mu} = \bar{q}_i T^a_{ij} \gamma^\mu \gamma^5 q_j$$

- **Assumption:** $SU(N_f)_L$ with infinitesimal transformation: $L = e^{i\alpha^a T^a} \approx 1 + i\alpha^a T^a$

  → Under $SU(N_f)_L$: $U \to L^\dagger U \quad \Rightarrow$ infinitesimally: $\delta_L U = -i\alpha^a T^a U$

  → Lagrangian no longer invariant $\Rightarrow$ transforms: $\delta\mathcal{L} = \partial_\mu \alpha^a J^a_{L\,\mu}$

Current: $J^a_{L\,\mu} = \frac{i f_\pi^2}{4} \mathrm{tr}\left(U^\dagger T^a \partial_\mu U - (\partial_\mu U^\dagger) T^a U\right)$

  → expand in pion fields *(leading order - LO)*: $J^a_{L\,\mu} \approx -\frac{f_\pi}{2}\partial_\mu \pi^a$

- Similar for $SU(N_f)_R \quad \Rightarrow$ *'+' instead of '−'*

Goldstones associated to broken symmetry generators for axial current $J^a_{A_\mu}$ *(pseudovector)*

$\Rightarrow$ pions - pseudoscalars *(odd under parity)*

Chiral Lagrangian - mass term

- Adding masses

$$S = \int d^4x \left(-\frac{1}{2} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu} + \sum_{i=1}^{N_f} \left(i \bar{q}_i \not{D} q_i - m_i \bar{q}_i q_i \right) \right)$$

→ if masses large wrt Λ_{QCD} ⇒ quarks play no role in low-energy physics (*c,b,t*)

For up, down, strange quarks assume: $\langle \bar{q}_{L\,i} q_{R\,j} \rangle \approx -\sigma\, U_{ij} \qquad \sigma \sim \Lambda^3_{\mathrm{QCD}}$

→ masses in chiral Lagrangian ⇒ $N_f \times N_f$ mass matrix: $M = \mathrm{diag}(m_1, \ldots, m_{N_f})$

→ low-energy effective theory ⇒ renormalised masses *(no bare quark masses)*

LO chiral Lagrangian: $\mathcal{L}_{\mathrm{pion}} = \frac{f_\pi^2}{4} \mathrm{tr}\, (\partial^\mu U^\dagger\, \partial_\mu U) + \frac{\sigma}{2} \mathrm{tr}\, \left(M U + U^\dagger M^\dagger \right)$

→ lifts vacuum manifold of theory ⇒ unique ground state

→ expand $U = e^{2i\pi/f_\pi}$: $\mathcal{L}_2 = \mathrm{tr}\, (\partial \pi)^2 - \frac{\sigma}{f_\pi^2} \mathrm{tr} \Big((M + M^\dagger) \pi^2 \Big) + \ldots$

⇒ mass term for pions (***pseudo-Goldstone bosons***)

For $N_f = 2$: $M = \mathrm{diag}(m_d, m_u)$

⇒ quadratic terms: $\mathcal{L}_2 = \frac{1}{2} \partial_\mu \pi^0 \partial^\mu \pi^0 + \partial_\mu \pi^+ \partial^\mu \pi^- - \frac{\sigma}{2 f_\pi^2} (m_d + m_u) \left((\pi^0)^2 + 2 \pi^+ \pi^- \right)$

→ 3 pions with equal mass: $m_\pi^2 = \frac{\sigma}{f_\pi^2} (m_u + m_d)$

Gell-Mann-Oakes-Renner relation: pion mass scales linearly with quark masses

Phases of massless QCD

Assumption: Gauge theory with $G = SU(N_c)$ Yang-Mills coupled to N_f flavours of massless quarks

What is low-energy physics of theory?

- $N_f = 0$: pure Yang-Mills $\Rightarrow$ confining phase + mass gap
- $N_f = 1$: no chiral symmetry group $\Rightarrow$ no chiral symmetry breaking + mass gap
  $\Rightarrow$ quarks bound in mesons and baryons
- $2 \leq N_f \leq N^*$: theory confines + chiral symmetry breaking $\Rightarrow$ freely interacting Goldstones
  $\max N^* \approx 4N_c$ $\Rightarrow$ for $N_f = 2$ or 3 $\rightarrow$ chiral symmetry breaking
- $N_f \geq 11/2\ N_c$: beta function positive $\rightarrow$ $b_0 = \frac{11}{3}N_c - \frac{2}{3}N_f$

  $\rightarrow$ theory weakly coupled in infra-red (IR) low-energy

  $\rightarrow$ massless gluons weakly interacting with massless quarks

  $\rightarrow$ in far IR $\Rightarrow$ physics is 'free'

  $\rightarrow$ strongly coupled in UV

Phases of massless QCD

- $N^{**} < N_f < 11/2\, N_c$

  → need two-loop beta function:

$$\beta(g) = -\frac{b_0}{(4\pi)^2} g^3 - \frac{b_1}{(4\pi)^4} g^5 + \dots \qquad b_0 = \frac{11}{3} N_c - \frac{2}{3} N_f \qquad b_1 = \frac{34 N_c^2}{3} - \frac{N_f (N_c^2 - 1)}{N_c} - \frac{10 N_f N_c}{3}$$

  → $b_0 > 0$ and $b_1 < 0$ ⇒ find zero of beta function: $g_\star^2 = -(4\pi)^2 \frac{b_0}{b_1}$

  ⇒ massless modes, no longer free in IR - *Banks-Zaks fixed point*

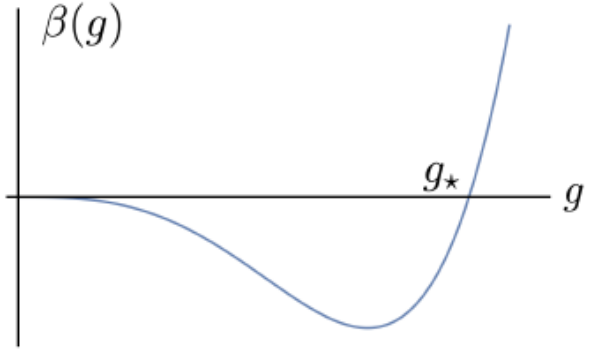
  → g_* small ⇒ perturbation theory

  N_f decreases → g_* increases

  → conformal field theory only for $N^{**} < N_f < 11/2\, N_c$

  ⇒ **conformal window**

  → don't know the value of N^{**}

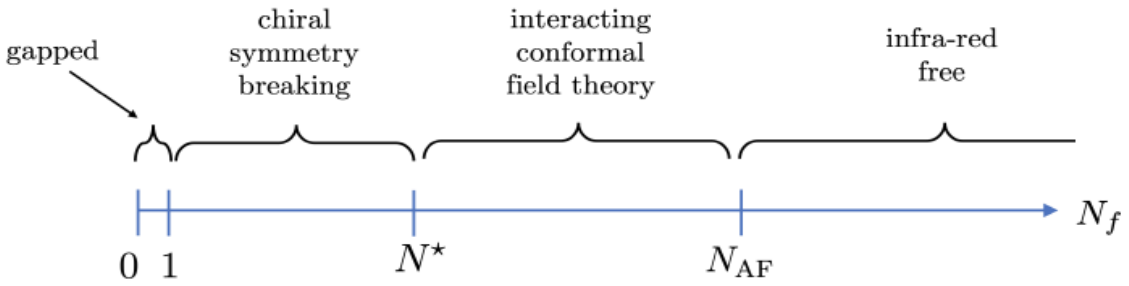


- $N^* < N_f \leq N^{**}$

  → no such regime?

  → $N^{**} = N^*$

  → motivated by numerics



Upper edge of chiral symmetry breaking phase = lower edge of conformal window

Hadrons

Quark model

Confinement - quarks bound to colour singlets

quark-anti-quark pairs → mesons, collection of 3 quarks → baryons

⇒ **hadrons**

- **Assume** quarks are massless

→ global symmetry with symmetry breaking:

$$U(1)_V \times SU(N_f)_L \times SU(N_f)_R \to U(1)_V \times SU(N_f)_V$$

→ keep *SU(N_f)* symmetry ⇒ no need quarks to be massless *(but same masses)*

Different masses

Quark	Charge	Mass (in MeV)
d = down	-1/3	5
u = up	+2/3	2
s = strange	-1/3	93
c = charm	+2/3	1270
b = bottom	-1/3	4200
t = top	+2/3	170,000

$m_{\text{up}} \approx m_{\text{down}}$ - *$SU(2)_V$* symmetry

Quark model

- Approximate $SU(3)_V$ symmetry *(u,d,s)*

  → Gell-Mann and Ne'eman in 1961 (**eightfold way** - dim *SU(3)* = 8)

  → ***SU(3)$_V$* not gauge group *SU(3)* of QCD !**

  ⇒ $SU(3)_V$ - global symmetry rotating different flavours of light quarks

QCD symmetries used to assign quantum numbers to particles, e.g.:

- Rotations

  → angular momentum

  → spin of particle *J*

- Parity *P*

- Charge conjugation *C*

e.g. particle label - J^{PC}

- Isospin I^G *(proton & neutron as 2 different nucleon states - Heisenberg, 1932)*

Mesons

- Simple model - meson as bound state of quark & anti-quark
  or linear combinations of them

→ quarks are fermions ⇒ mesons are bosons *(integer spin)*

Mesons with spin 0 or 1 with *u, d, s*

- (*d, u, s*) transform in **3** of $SU(3)_V$

→ little group ⇒ quark & anti-quark transform: $3 \otimes \bar{3} = 1 \oplus 8$

→ mesons in 2 representations of $SU(3)_V$: singlet **1** & adjoint octet **8**

Pseudoscalar mesons

- Lowest mass mesons with spin 0: $J^{PC} = 0^{-+}$

Meson	Quark Content	Mass (in MeV)	Lifetime (in s)
pion π^+	$u\bar{d}$	140	10^{-8}
pion π^0	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$	135	10^{-16}
eta η	$\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$	548	10^{-19}
eta Prime η'	$\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$	958	10^{-21}
kaon K^+	$u\bar{s}$	494	10^{-8}
kaon K^0	$d\bar{s}$	498	$10^{-8} - 10^{-11}$

→ π^0, η and η' are their own anti-particles ⇒ real scalar fields *(9 mesons)*

Pseudoscalar mesons

Masses of mesons not equal to sum of masses of their constituent quarks!

→ chiral Lagrangian ⇒ Gell-Mann-Oakes-Renner relation

- 9 meson states decomposed into **1** ⊕ **8** multiplets:

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes (\bar{u}, \bar{d}, \bar{s}) = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix} = \mu_0 \mathbb{1} + \sum_{a=1}^{8} \mu_a \lambda_a$$

λ_a - Gell-Mann matrices (generators of $SU(3)_V$)

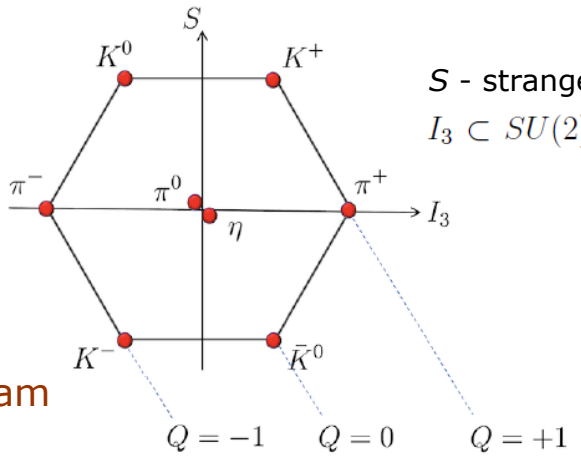
Mesons in 8 ⇒ would-be Goldstone bosons

→ μ_a rearrange into 2 real & 3 complex fields ⇒ pions, kaons, eta

$$\pi^0 = \mu_3 \ , \quad \pi^{\pm} = \frac{1}{\sqrt{2}}(\mu_1 \mp i\mu_2) \qquad K^0 = \frac{1}{\sqrt{2}}(\mu_6 - i\mu_7) \ , \quad K^{\pm} = \frac{1}{\sqrt{2}}(\mu_4 \mp i\mu_5) \ , \quad \eta = \mu_8$$

→ expand field π to $N_f = 3$ quarks:

$$\pi = \frac{1}{2}\sum_{a=1}^{8} \mu_a \lambda^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}$$



S - strangeness

$I_3 \subset SU(2)_V \subset SU(3)_V$

SU(3) weight diagram

Pseudoscalar mesons

- Mesons in chiral massive Lagrangian

→ mass matrix: $M = \mathrm{diag}(m_u, m_d, m_s)$

$$\mathcal{L}_{\mathrm{mass}} = \frac{-\sigma}{f_\pi^2}\left[\frac{1}{2}(m_u+m_d)\left((\pi^0)^2+2\pi^+\pi^-\right)+(m_u+m_s)K^-K^+ \right.$$
$$\left. +(m_d+m_s)\bar{K}^0K^0+\frac{1}{2}\left(\frac{m_u}{3}+\frac{m_d}{3}+\frac{4m_s}{3}\right)\eta^2+\frac{1}{\sqrt{3}}(m_u-m_d)\pi^0\eta\right]$$

→ mixing between π^0 and η ⇒ disappears when $m_u = m_d$ *(isospin restored)*

→ ratios ⇒ eliminate scale σ / f^2_π : $\dfrac{m_{K^+}^2 - m_{K^0}^2}{m_\pi^2} = \dfrac{m_u - m_d}{m_u + m_d}$

Gell-Mann-Okubo relation: if $m_u \approx m_d$: $4m_K^2 \approx 3m_\eta^2 + m_\pi^2$

Scalar meson → singlet in $3 \otimes \bar{3} = 1 \oplus 8$

→ field μ_0 - η' $\quad \eta' = \dfrac{1}{\sqrt{3}}(u\bar{u}+d\bar{d}+s\bar{s})$

→ heaviest scalar meson ⇒ not pseudo-Goldstone boson

→ if quark masses = 0 ⇒ pions, kaons & eta massless, η' massive

$U(1)_A$ - symmetry of classical action, not symmetry of quantum theory *(anomaly)*

Pseudovector mesons

Pattern of **1** ⊕ **8** repeated in excited meson states

→ spins of quarks aligned

→ quarks with additional relative orbital angular momentum *L*

→ parity $P = (-1)^{L+1}$

- Quarks spins aligned, $L = 0$

→ 9 pseudovector mesons $J^{PC} = 1^{--}$

→ lightest spin 1 mesons:
$\rho^{\pm}$ and ρ^{0} *(excited pions)*

→ heaviest - singlet **1**
ϕ meson

Meson	Quark Content	Mass (in MeV)	Lifetime (in s)
rho ρ^+	$u\bar{d}$	770	10^{-24}
rho ρ^0	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$	770	10^{-24}
omega ω	$\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$	780	10^{-22}
phi ϕ	$s\bar{s}$	1020	10^{-22}
kaon $K^{+\star}$	$u\bar{s}$	890	10^{-24}
kaon $K^{0\star}$	$d\bar{s}$	890	10^{-24}

Excited versions of η and η' ⇒ ω and ϕ

→ particle mixing $\eta : \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$ and $\eta' : \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ but $\omega : \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$ and $\phi : s\bar{s}$

→ quark content as linear combination

⇒ group theory dominates for pseudoscalars *(would-be Goldstone bosons)*

⇒ mass difference of strange quark dominates for pseudovectors

Mesons lifetimes

Lifetime of mesons

→ many lifetimes dictated by weak force

→ wide range of lifetimes: 10^{-8} *s* ($\pi^{\pm}$, $K^{\pm}$) - 10^{-24} *s* (ρ)

⇒ different ways of particles decay

- Neutral and charged pions

→ different lifetimes

π^0 $\tau = 10^{-16}$ *s (electromagnetic force)*: $\pi^0 \to \gamma + \gamma$

$\pi^{\pm}$ $\tau = 10^{-8}$ *s (weak force),* e.g: $\pi^+ \to \mu^+ + \nu_\mu \quad \text{and} \quad \pi^- \to \mu^- + \bar{\nu}_\mu$

Each force with characteristic time scale *(determines hadron lifetime)*

- Strong decay: ~10^{-22} - 10^{-24} *s*
- Electromagnetic decay: ~10^{-16} - 10^{-21} *s*
- Weak decay: ~10^{-7} - 10^{-13} *s*

Resonance - particle decaying via strong force

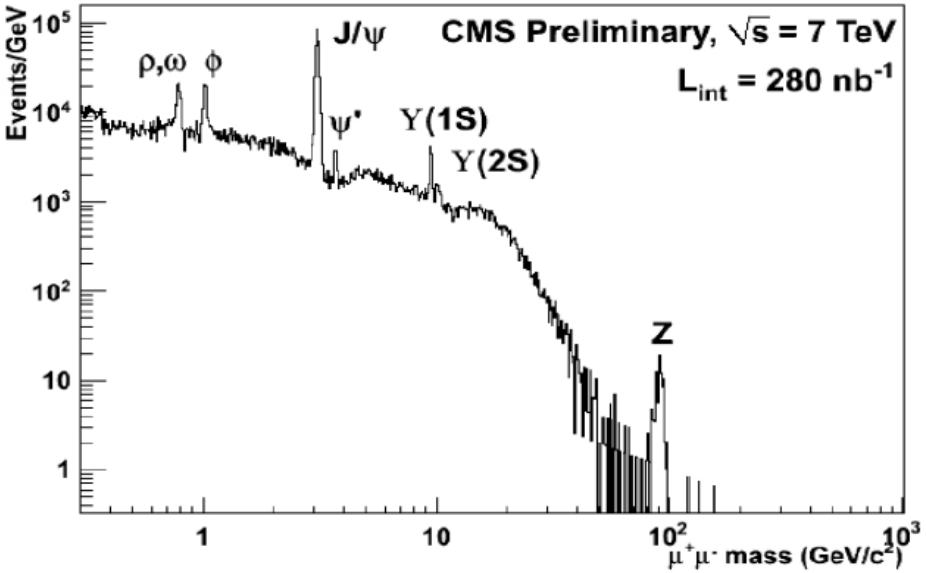
⇒ e.g. ρ meson

Mesons lifetimes

- Resonance lifetimes in terms of width *(energy scale)*

  → conversion factor: $100\ \mathrm{MeV} \approx 10^{-23}\ \mathrm{s}^{-1}$

  ⇒ energy scale of strong force: $\Lambda_{QCD} \sim 100$ MeV → $T_{QCD} \sim 10^{-24}$ *s*



→ bumps in number of back-to-back $\mu^+\mu^-$ (from CMS)

→ ρ, ω, ϕ

→ *J/ψ (charmed meson)*, Υ *(bottom meson)*, *Z*-boson *(gauge boson of weak force)*

Baryons

3 quarks can form gauge singlet $\Rightarrow$ anti-symmetrising over colour indices a = 1, 2,3

$$\mathcal{B} = \epsilon^{abc} q_a q_b q_c$$

- For baryons of light quarks *(d, u, s)*

  $\rightarrow$ representations of $SU(3)_V$ flavour symmetry (*not SU(3) gauge symmetry*)

  $\rightarrow$ little group $\Rightarrow$ 2 quarks: $\quad 3 \otimes 3 = \bar{3} \oplus 6$

Adding 3rd quark: $\quad 3 \otimes 3 \otimes 3 = (\bar{3} \otimes 3) \oplus (6 \otimes 3) = 1 \oplus 8 \oplus 8' \oplus 10$

Possible combinations (*quarks are fermions* $\rightarrow$ *Pauli exclusion principle*)

- Singlet **1** fully anti-symmetrised in flavour indices

  $\rightarrow$ heavy & messy *(not discussed)*

- Decuplet **10** fully symmetrised in flavour indices

  $\rightarrow$ satisfy Pauli principle by symmetrising over spin degrees of freedom

  $\rightarrow$ decuplet of baryons with spin 3/2

Baryon	Quark Content	Mass (in MeV)	Lifetime (in s)
proton p	uud	938	stable
neutron n	udd	940	10^{3}
lambda Λ^0	uds	1115	10^{-10}
sigma Σ^+	uus	1189	10^{-10}
sigma Σ^0	uds	1193	10^{-19}
sigma Σ^-	dds	1197	10^{-10}
cascade Ξ^0	uss	1315	10^{-10}
cascade Ξ^-	dss	1321	10^{-10}

Baryons

- **8** and **8'** octets

  → **8** anti-symmetrised only in first 2 indices

  → **8'** symmetrised only in first 2 indices

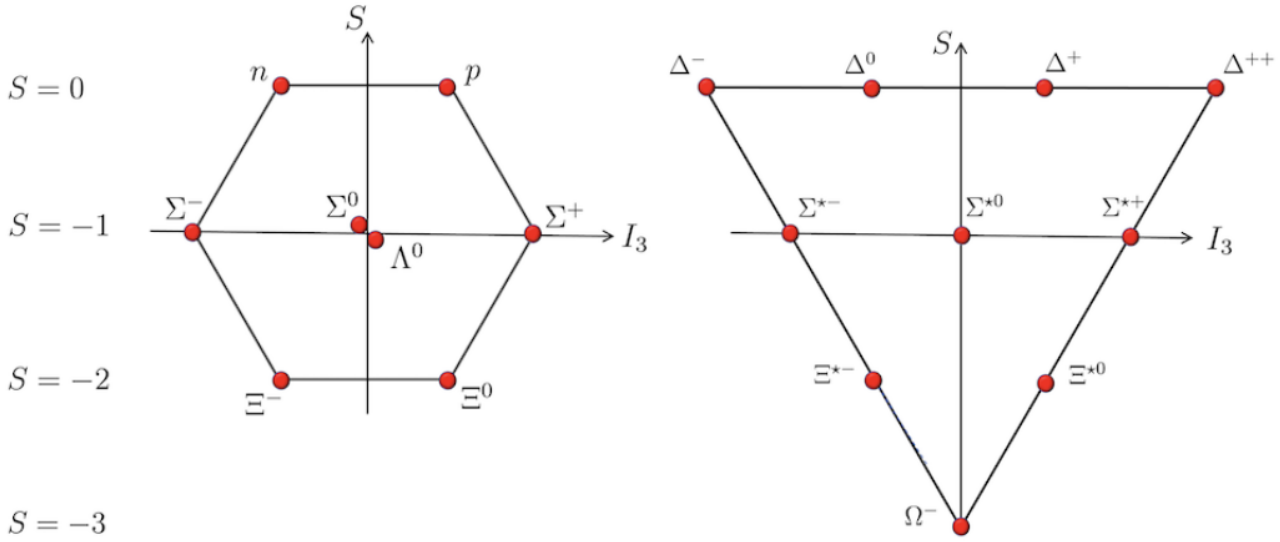
  Total quarks spin = ½:

  ⇒ linear combination gives fully anti-symmetrised wavefunction

  → octet contains proton & neutron *(nucleons)*

  → Σ baryons - single strange quark

  → Ξ baryons - 2 strange quarks



Baryons

Baryon masses

- If $m_u = m_d = 0$: proton & neutron masses unchanged wrt measured

  Mass of baryons in $SU(N_c)$ QCD scales as $N_c \Lambda_{QCD}$

  → Higgs responsible for mass of elementary particles *(leptons & quarks)*

  → mass in atoms ⇒ strongly interacting quantum fields

Strong force

- Binds quarks into baryons *(fermions)* and into mesons *(bosons)*
- Scalar particles mediate forces

  → **pions mediate Yukawa force:**

$$V(r) \sim -\frac{e^{-m_\pi r}}{r}$$

  → binds protons & neutrons in nucleus

  'mesonic force' or *'Yukawa force'*

  ⇒ residual secondary effect

Baryons

- Flavour decuplet **10**

→ spin 3/2 baryons

Δ (charges 0, ±1 and +2)

Σ^* (charges 0 and ±1),

Ξ^* (charges -1 and 0)

Ω^- (charge -1)

→ **3 outliers:**
(3 same quarks)

Δ^{++} - first particle with charge +2

Ω^- - mass predicted by Gell-Mann *(using simple quark model)*

Δ^- - no direct nucleon analogues (*as* Δ^{++})

Baryon	Quark Content	Mass (in MeV)	Lifetime (in s)
Δ^{++}	uuu	1232	10^{-24}
Δ^{+}	uud	1232	10^{-24}
Δ^{0}	udd	1232	10^{-24}
Δ^{-}	ddd	1232	10^{-24}
$\Sigma^{\star -}$	dds	1383	10^{-23}
$\Sigma^{\star 0}$	dus	1384	10^{-23}
$\Sigma^{\star +}$	uus	1387	10^{-23}
$\Xi^{\star -}$	dss	1535	10^{-23}
$\Xi^{\star 0}$	uss	1532	10^{-23}
Ω^{-}	sss	1672	10^{-11}

Much of results cannot be calculated from first principles *(starting from QCD Lagrangian)*

→ numerical simulations *(lattice QCD)*

→ ~5% accuracy

Heavy quarks

- No hadrons comprised of top quark

  → extreme high mass ⇒ decay lifetime ~10^{-25} *s* *(faster than T_{QCD} ~ 10^{-24} s)*

- **Charm & bottom**

  → masses determined more by bare quark mass (*not by Λ_{QCD}*)

charmonium → bound state of charm & anti-charm quark ($\tau = 10^{-21}$ *s*)

$J/\psi\ (\bar{c}c) \quad m \approx 3.1\ \mathrm{GeV}$, discovered in 1974

→ *D-mesons*: mesons with single charm quark, lightest: $D^0\ (c\bar{u}) \quad m \approx 1865\ \mathrm{MeV}$, $D^+\ (c\bar{d}) \quad m \approx 1869\ \mathrm{MeV}$

bottomonium → bound state of beauty & anti-beauty quark ($\tau = 10^{-20}$ *s*)

$\Upsilon\ (\bar{b}b) \quad m \approx 9.5\ \mathrm{GeV}$, discovered in 1976

→ *B-mesons*: mesons with single beauty quark, lightest: $B^+\ (u\bar{b})$ and $B^0\ (d\bar{b}) \quad m \approx 5280\ \mathrm{MeV}$

Baryons containing charm and bottom quarks

→ Σ^+_c (*uuc*), Σ^+_b (*uub*), ...

→ excited states

Theta term

Theta term

- QCD action: $S = \int d^4x \left(-\frac{1}{2} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu} + \sum_{i=1}^{N_f} \left(i \bar{q}_i \not{D} q_i - m_i \bar{q}_i q_i \right) \right)$

Are there any missed interaction terms?

Yang-Mills theta term $S_\theta = \frac{\theta g_s^2}{16\pi^2} \int d^4x \; \mathrm{Tr}\, G_{\mu\nu} {}^\star G^{\mu\nu} \qquad {}^\star G^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}$

→ *θ* angle - additional parameter of QCD

→ bound *θ* parameter to: $\theta < 10^{-10}$

⇒ *θ* appears to vanish

What if θ not vanish?

→ $G_{\mu\nu} {}^\star G^{\mu\nu} \sim \mathbf{E} \cdot \mathbf{B}$

$$P : \mathbf{E} \mapsto -\mathbf{E} \quad \text{and} \quad P : \mathbf{B} \mapsto +\mathbf{B}$$
$$C : \mathbf{E} \mapsto -\mathbf{E} \quad \text{and} \quad C : \mathbf{B} \mapsto -\mathbf{B}$$
$$T : \mathbf{E} \mapsto +\mathbf{E} \quad \text{and} \quad T : \mathbf{B} \mapsto -\mathbf{B}$$

θ* breaks *P*, *CP* and *T

→ consequence of CP violation → electric dipole moment

→ experimentally measured neutron dipole moment: $d_n < 10^{-26}\ \mathrm{e\,cm}$

Question why *θ* = 0? ⇒ ***strong CP problem***

Instantons

- **Assumption:** Theory weakly coupled

  → path integral dominated by saddle points

  ⇒ solutions to classical equations of motion

  → any θ dependence from field equations that wind at infinity ($n \neq 0$, $S_\theta = \theta n$, $n \in \mathbb{Z}$)

  ⇒ solve classical equations of motion: $\mathcal{D}_\mu F^{\mu\nu} = 0$

- Yang-Mills action: $S_{YM} = \frac{1}{2g^2} \int d^4x \ \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$

  → Euclidean space ⇒ action with '+' sign *(opposite to Minkowski space)*

  → Euclidean action: $S_{YM} = \frac{1}{4g^2} \int d^4x \ \mathrm{Tr}\,(F_{\mu\nu} \mp {}^\star F_{\mu\nu})^2 \pm \frac{1}{2g^2} \int d^4x \ \mathrm{Tr}\, F_{\mu\nu} {}^\star F^{\mu\nu} \geq \frac{8\pi^2}{g^2} |n|$

  → in sector with winding n ($n \neq 0$) ⇒ Yang-Mills action < $8\pi^2 n / g^2$

  Minimised if bound saturated: $F_{\mu\nu} = \pm {}^\star F_{\mu\nu}$ *((anti)-self-dual Yang-Mills equations)*

  solutions necessarily minimise the action

Solutions to (anti)-self-dual Yang-Mills equations have finite action

⇒ solutions correspond to point-like objects in Euclidean spacetime

⇒ occur for 'instant of time' ⇒ **instantons** *(analogous to the classical tunnelling)*