Standard Model

Lecture 2
Broken symmetries

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Contents

Lecture 2

- Discrete symmetries
- Continuous symmetries
- Higgs mechanism

Broken symmetries

- Global symmetries

  → lead to conservation laws through Noether's theorem

  → if symmetry is non-Abelian - degeneracy in spectrum

    ⇒ states of theory must provide representation of symmetry

      e.g. hydrogen atom where states are in multiplets of *SO(3)* rotation group

Broken symmetry

- Explicit breaking → there are terms in the action not invariant under symmetry

  e.g. hydrogen atom in magnetic field → rotation symmetry is broken

  → may be 'small' symmetry ⇒ approximate symmetry

  e.g. isospin extension to SU(3), chiral symmetry

- **Spontaneous symmetry breaking**

  → theory is invariant under symmetry, but ground state is not

  → occurs in different contexts in SM

  → arises also in condensed matter physics *(magnets, superconductors)*

Discrete symmetries

Discrete symmetries

Spontaneous symmetry breaking appears in some simple classical mechanics systems

- **Assumption:** real, classical degree of freedom $\phi(t)$ with action

$$S = \int dt \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) \quad \text{with} \quad V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

→ in Newtonian mechanics → $\phi(t)$ - position of particle denoted as $x(t)$

→ potential has discrete $\mathbb{Z}_2$ symmetry: $\mathbb{Z}_2 : \phi \mapsto -\phi$

- Spontaneous symmetry breaking

→ if $\lambda > 0$ ⇒ sign of first term in potential

Symmetry unbroken if $m^2 > 0$ ⇒ potential has minimum at $\phi = 0$

one point that is invariant under the symmetry $\phi \mapsto -\phi$

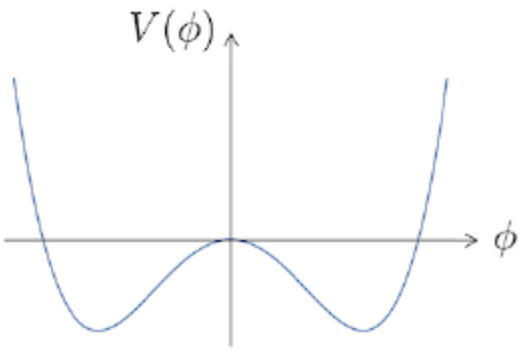
Symmetry spontaneously broken if $m^2 < 0$

→ ϕ^2 term → negative coefficient

→ point $\phi = 0$ is local maximum

→ **double well potential**

minimum: $\phi = \pm v \equiv \pm\sqrt{-\frac{m^2}{\lambda}}$



Discrete symmetries

- There is not unique ground state
- Neither ground state is invariant under $\mathbb{Z}_2$ symmetry

  → symmetry exchanges two ground states

Conclusions:

→ if discrete symmetry is spontaneously broken, theory has multiple ground states with a potential barrier between them

→ acting with symmetry transforms among ground states

- Sits in one of two ground states

  → small oscillations about minimum

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 + \text{constant}$$

→ near ground state $\phi = +v$: $\phi(t) = v + \sigma(t)$

→ final potential: $V(\sigma) = \frac{\lambda}{4}(2v\sigma + \sigma^2)^2 = \lambda\left(v^2\sigma^2 + v\sigma^3 + \frac{\sigma^4}{4}\right) + \text{constant}$

While full potential $V(\phi)$ has symmetry → if near minimum → $\mathbb{Z}_2$ symmetry is hidden/broken

→ generate multiple ground states

Quantum tunnelling

No spontaneous symmetry breaking in quantum mechanics

→ ground state is always invariant under $\mathbb{Z}_2$ symmetry

all energy eigenstates are invariant under $\mathbb{Z}_2$ symmetry

- **Assumption:** construct ground state localised near one of minima (wavefunction):

$$\psi_{\text{left}}(\phi) \approx \exp\left(-\frac{\sqrt{\lambda}v}{2}(\phi+v)^2\right) \quad \text{or} \quad \psi_{\text{right}}(\phi) \approx \exp\left(-\frac{\sqrt{\lambda}v}{2}(\phi-v)^2\right)$$

→ neither of these are eigenstates of the $\mathbb{Z}_2$ symmetry or Hamiltonian

Quantum tunnelling

If system placed in $\psi_{left}(\phi)$, wavefunction will leak through the barrier → quantum tunnelling

→ true ground state wavefunction takes approximate form:

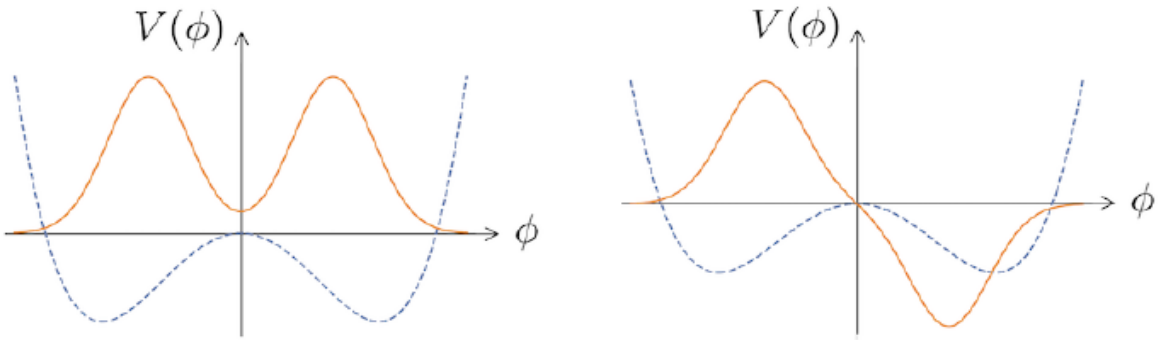
$$\psi_{\text{ground}}(\phi) \approx \psi_{\text{left}}(\phi) + \psi_{\text{right}}(\phi)$$

ground state has no zeros other than at $\phi \to \pm\infty$

→ first excited state: $\psi_{\text{excited}}(\phi) \approx \psi_{\text{left}}(\phi) - \psi_{\text{right}}(\phi)$

single node → crosses the axis once

Quantum tunnelling



Another way to see tunnelling (quantum field theory)

- Amplitude for particle to start in minimum $\phi = -v$, end in $\phi = +v$:

  $\rightarrow$ path integral + Wick rotating (to work with imaginary time $\tau = it$):

$$\langle +v|e^{-H\tau}|-v\rangle = \int \mathcal{D}\phi \; e^{-S_E[\phi]}$$

  $\rightarrow S_E[\phi]$ - Euclidean action (no minus sign wrt real):

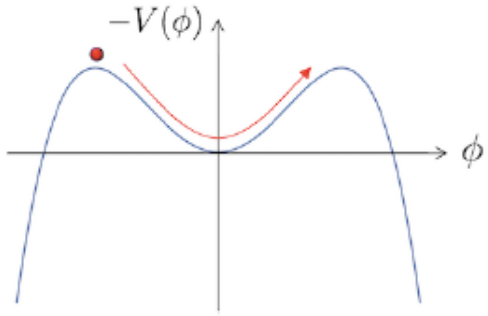
$$S_E[\phi] = \int d\tau \; \left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

- Calculate amplitude

  $\rightarrow$ path integral on paths starting in left-hand vacuum, ending in right-hand vacuum

  $\rightarrow$ Euclidean action flips sign of potential term

Quantum tunnelling



→ paths that start on left-hand peak

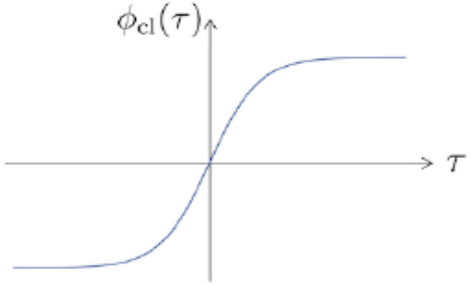
→ roll down to the minimum

→ rise again to end on right-hand peak

- Saddle point approximation

→ dominant contribution come from paths with classical equation of motion:

$\ddot{\phi} = \lambda\phi(\phi^2 - v^2)$ → solution: $\phi_{\rm cl}(\tau) = v\tanh\left(\sqrt{\frac{\lambda v^2}{2}}\tau\right)$



interpolates from -*v* to +*v* over time period $\Delta\tau \sim 1/\sqrt{\lambda v^2} \sim 1/|m|$

→ Euclidean action on this solution: $S_{\rm cl} = \int_{-\infty}^{+\infty} d\tau \left(\frac{1}{2}\dot{\phi}_{\rm cl}^2 + \frac{\lambda}{4}(\phi_{\rm cl}^2 - v^2)^2\right) = \frac{2}{3}\sqrt{2\lambda}v^3$

→ first guess at amplitude to tunnel from one minimum to the other:

$$\lim_{\tau\to\infty}\langle +v|e^{-H\tau}|-v\rangle = Ke^{-S_{\rm cl}}$$

→ *K* is overall constant accounting for assumptions and model simplifications

Discrete symmetry breaking in QFT

- No spontaneous symmetry breaking in quantum mechanics

  → energy difference between ground state and first excited state:

$$E_{\text{excited}} - E_{\text{ground}} \approx \sqrt{\lambda v^2} e^{-S_{\text{cl}}}$$

  → splitting of two states exponentially suppressed!

Discrete Symmetry Breaking in Quantum Field Theory (QFT)

- Double well in field theory → $\phi(x)$ function of spacetime

  → action: $S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi) \right)$ with $V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$

  → $\mathbb{Z}_2$ symmetry: $\mathbb{Z}_2 : \phi \mapsto -\phi$ when $m < 0$ ⇒ double well potential with 2 minima:

$$\phi = \pm v = \pm\sqrt{-m^2/\lambda}$$

  Is this symmetry spontaneously broken?

  → QFT is extension of quantum mechanics ⇒ no spontaneous symmetry breaking?

- QFT has two ground states with vacuum expectation value: $\langle \phi \rangle = \pm v$

  → amplitude to go from one ground state to another: $\langle +v | e^{-H\tau} | -v \rangle = \int \mathcal{D}\phi \; e^{-S_E[\phi]}$

Discrete symmetry breaking in QFT

- Euclidean action: $S_E[\phi] = \int d\tau d^3x \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi)\right)$

  → saddle point approximation (amplitude dominated by classical solutions):

  $$\partial^2\phi = \lambda\phi(\phi^2 - v^2)$$

  $\ddot{\phi}$ term replaced by Laplacian on spacetime $\partial^2 = \partial_\tau^2 + \nabla^2$

  → solution same as before:

  $$\phi_{\rm cl}(\tau) = v\tanh\left(\sqrt{\frac{\lambda v^2}{2}}\tau\right)$$

- Classical action of this solution:

  $$S = \int d\tau d^3x \left(\frac{1}{2}\partial_\mu\phi_{\rm cl}\,\partial^\mu\phi_{\rm cl} + V(\phi_{\rm cl})\right) = \mathcal{V}S_{\rm cl}$$

  $S_{\rm cl}$ - quantum mechanical action $\mathcal{V}$ - volume of space

  → tunnelling amplitude & energy splitting of ground states proportional to:

  $$e^{-\mathcal{V}S_{\rm cl}} \to 0 \quad \text{as} \quad \mathcal{V}\to\infty$$

  → in QFT ground state in one minimum → $\phi(x)$ same for all x

  to tunnel to other minimum ⇒ shift value of field at every point in space

  → infinite action → not occurs

  Discrete symmetries can be broken in QFT

Domain walls

- Presence of a spontaneously broken symmetry

  → existence of **domain wall** (field configuration interpolating from one vacuum to the other)

- Replace imaginary time with one of spatial coordinates $\boldsymbol{x} = (x, y, z)$ in tunnelling solution:

$$\phi(z) = v \tanh\left(\sqrt{\frac{\lambda v^2}{2}} z\right)$$

  → interpolates from vacuum $\phi = -v$ at $z \to -\infty$ to vacuum $\phi = +v$ at $z \to +\infty$

  → describes excitation of field localised around $z = 0$, extended in x and y-directions

  ⇒ domain wall

- Domain wall has finite energy density: $\mathcal{E} = \frac{2}{3}\sqrt{2\lambda} v^3$

  → infinite energy ⇒ stretches to infinity in the (x, y)-plane

Exception: domain walls in $d = 1+1$ dimensions → kind of particle in theory

$d = 3 + 1$ dimensions

→ domain wall forms sphere of radius R

→ one vacuum $\phi = -v$ inside and other vacuum $\phi = +v$ outside

→ has finite energy: $E = 4\pi R^2 \mathcal{E}$

Domain walls

- Static configuration no longer solve equation of motion

  → find the classical solution ⇒ solve full time-dependent partial differential equation

In quantum theory

- build a Fock space of states above either of two ground states by exciting the field:

$$\phi(x) = \pm v + \sigma(x)$$

  → creates particles of mass: $M = \sqrt{2\lambda v^2}$

- Hilbert space of theory decomposes: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

  → tensor sum ⇒ pick either state from $\mathcal{H}_+$ or state from $\mathcal{H}_-$

$$|\psi\rangle \in \mathcal{H}_+ \qquad \langle\psi|\phi(0,\mathbf{x})|\psi\rangle = +v \quad \text{for} \quad |\mathbf{x}| \to \infty$$

  → necessarily approach vacuum $|+v|$ when far away

- Many excitations on top of each other

  → possible to find region of one vacuum inside another

  → states are spherical domain wall

Continuous symmetries

Continuous symmetries

Continuous symmetry in quantum field theory

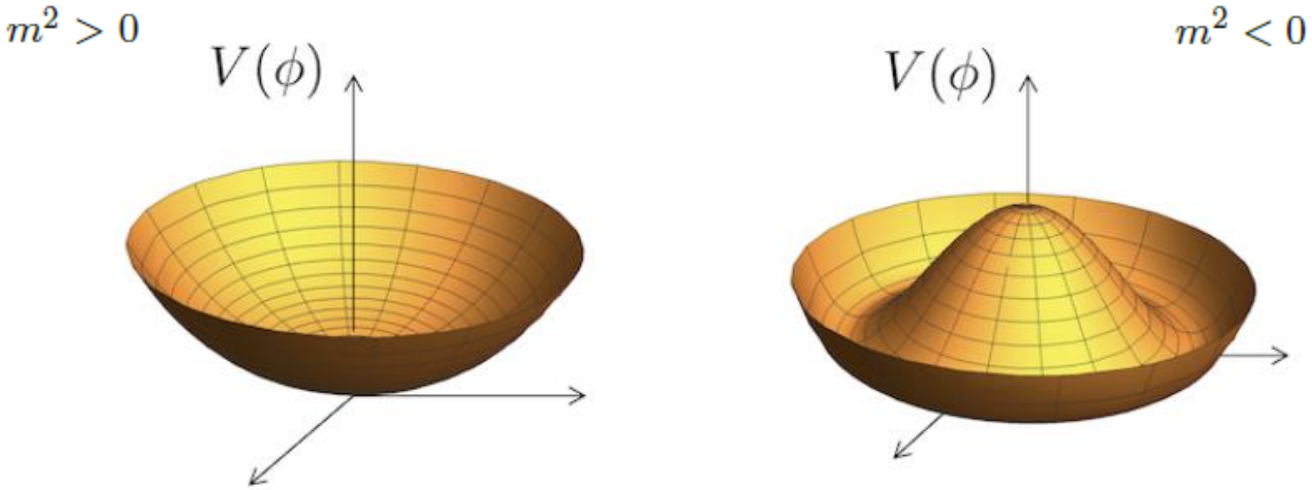
- Consider complex scalar field $\phi(x)$ in d = 3 + 1 dimensions with action:

$$S = \int d^4x \left(\partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi, \phi^\dagger) \right) \quad \text{with} \quad V(\phi, \phi^\dagger) = m^2 |\phi|^2 + \frac{1}{2} \lambda |\phi|^4$$

→ action is constructed to preserve global $U(1)$ symmetry (rotates phase):

$$\phi(x) \to e^{i\alpha} \phi(x)$$

- Physics depends on sign of m^2 term in potential



Continuous symmetries

- $m^2 > 0$

  → expand around vacuum $\phi = 0$

  → interacting particles of mass m with *U(1)* symmetry

- $m^2 < 0$

  → **'Mexican hat potential'**

  → not isolated minima

  → infinite number of ground states: $|\phi|^2 = -\frac{m^2}{\lambda}$

- Define *vacuum manifold* $\mathcal{M}_0$ - space of field configurations which have minimum energy

  → double well potential $\Rightarrow$ vacuum manifold are just two points

  → NOW: vacuum manifold is set of solutions - circle

$$\mathcal{M}_0 = \mathrm{S}^1$$

- Complex field in polar coordinates

$$\phi(x) = r(x)e^{i\theta(x)} \qquad r(x) \geq 0 \quad \theta(x) = \theta(x) + 2\pi$$

Continuous symmetries

- Substitute polar decomposition into original action (polar ansatz):

$$S = \int d^4x \left(\partial_\mu r \partial^\mu r + r^2 \partial_\mu \theta \partial^\mu \theta - \frac{\lambda}{2}(r^2 - v^2)^2 \right) \qquad v^2 = -m^2/\lambda$$

→ ground state of system: $r(x) = +v$

→ expand about vacuum $\Rightarrow$ $r(x) = v + \sigma(x)$:

$$S = \int d^4x \left(\partial_\mu \sigma \partial^\mu \sigma + (v+\sigma)^2 \partial_\mu \theta \partial^\mu \theta - \frac{\lambda}{2}\sigma^2(\sigma + 2v)^2 \right)$$

Physics:

→ $\sigma(x)$ excitations have mass $M^2 = 2\lambda v^2$

→ radial oscillations of the field

- Ground state

→ angular scalar field $\theta(x)$ parameterises vacuum manifold $\mathcal{M}_0 = \mathrm{S}^1$

→ each point on $\mathcal{M}_0$ corresponds to a different ground state of theory

→ specify ground state $\Rightarrow$ choose one such point

(physics same in each)

Continuous symmetries

Whatever choice of ground state $\Rightarrow$ spontaneously break *U(1)* symmetry

- Symmetry acts by moving from one point on $\mathcal{M}_0$ to another:

$$\theta(x) \rightarrow \theta(x) + \alpha$$

- Dynamics of field $\theta(x)$ that parameterises $\mathcal{M}_0$

$\rightarrow$ *U(1)* invariance of the potential $\Rightarrow$ no potential term for θ in action

$\rightarrow$ θ field governed by Lagrangian (if no coupling to σ):

$$\mathcal{L} = v^2 \partial_\mu \theta \, \partial^\mu \theta$$ Lagrangian for a massless scalar field

$\rightarrow$ existence of massless scalar field $\Rightarrow$ direct consequence of spontaneous breaking of *U(1)* global symmetry *(photon)*

Goldstone bosons

If continuous global symmetry is spontaneously broken, there are massless scalar fields - *Goldstone bosons*

$\rightarrow$ can't have potential terms (only derivative terms)

$\rightarrow$ can interact among themselves and with other fields

O(N) sigma model

- **Generalisation:** collection of N real scalar fields $\phi^a(x)$, with $a = 1, \ldots, N$

  → action: $S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - V(\phi) \right)$ with $V(\phi) = \frac{1}{2} m^2 \phi^a \phi^a + \frac{1}{4} \lambda (\phi^a \phi^a)^2$

  → $\phi^a(x)$ rotates under *O(N)* symmetry

  → for $N = 2$ coincide with action *(slide 12)* for complex scalar field
  real and imaginary parts: ϕ^1 and ϕ^2

- Spontaneous symmetry breaking

  → $m^2 < 0$ - Mexican hat potential with higher dimensional head

  → minima: $\phi^a \phi^a = v^2 := -\frac{m^2}{\lambda}$ ⇒ (N-1)-dimensional sphere

  → vacuum manifold: $\mathcal{M}_0 = \mathrm{S}^{N-1}$ ⇒ vaccum of theory: one point on $\mathcal{M}_0$

- 'South pole' → vacuum $\phi^a = (0, 0, \ldots, 0, v)$

  → fluctuations: $\phi^a(x) = \left(\pi^1(x), \ldots, \pi^{N-1}(x), v + \sigma(x) \right)$

  → action: $S = \int d^4x \left(\frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{2} \partial_\mu \sigma \, \partial^\mu \sigma - V(\pi^a, \sigma) \right)$ $V(\pi^a, \sigma) = \lambda v^2 \sigma^2 + \lambda v \sigma \left(\sigma^2 + \pi^a \pi^a \right) + \frac{1}{4} \lambda (\pi^a \pi^a + \sigma^2)^2$

  → σ field with quadratic term ⇒ massive particle

  → π^a give $N - 1$ massless particles ⇒ Goldstone bosons from spontaneous symmetry breaking

O(N) sigma model

- π^a fields are massless, but still appear in potential (in higher order terms)

  → $\pi^a(x)$ fields - linear displacements away from vacuum

- Fields follow polar ansatz *(slide 14)*

  → at low energies ⇒ only dynamics of Goldstone bosons

  → restrict to $\mathcal{M}_0$ ⇒ minima condition obeyed everywhere: $(\pi^a)^2(x) + (\phi^N)^2(x) = v^2$

  eliminates $\phi_N(x)$ in action → potential term vanishes *(only kinetic terms)*

$$S = \int d^4x \, \frac{1}{2} \left(\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})(\vec{\pi} \cdot \partial^\mu \vec{\pi})}{v^2 - \vec{\pi} \cdot \vec{\pi}} \right)$$

  → only kinetic interactions between Goldstone modes

- Restrict to $N = 3$ ⇒ polar coordinates on $\mathbb{R}^3$: $\pi^1 = v \sin\theta \cos\varphi \,, \quad \pi^2 = v \sin\theta \sin\varphi \,, \quad \phi^3 = v \cos\theta$

  → $\theta(x)$ and $\phi(x)$ - fields parameterising vacuum manifold $\mathcal{M}_0 = \mathrm{S}^2$

  → action for Goldstone bosons: $S = \int d^4x \, \frac{1}{2} g_{ab}(\pi) \partial_\mu \pi^a \, \partial^\mu \pi^b$ g_{ab} - round metric on $\mathcal{M}_0$

Non-linear sigma models: actions with fields being coordinates on some manifold

→ non-renormalisable ⇒ not valid at arbitrarily high energy scales

Goldstone theorem in Classical Field Theory

- **Assumption:** Theory with ϕ scalar fields transforming in representation of global symmetry Lie group *G (continuous symmetry)*

  → potential with minima defining vacuum manifold: $\mathcal{M}_0 = \{\phi_0 \mid V(\phi_0) = V_{\min}\}$

  $\phi_0 = 0 \;\rightarrow\; \mathcal{M}_0$ - single point $\Rightarrow$ symmetry realised only on excited states

- ϕ_0 not unique

  → all points in $\mathcal{M}_0$ can be reached by symmetry transformation:

$$\phi_0' = g\phi_0 \qquad \phi_0, \phi_0' \in \mathcal{M}_0 \quad g \in G$$

- Stability group *H*

  → some elements of *G* move around $\mathcal{M}_0$, other not change ϕ_0

  H defined as elements of *G* which don't change ϕ_0

$$H = \{h \in G \mid h\phi_0 = \phi_0\} \qquad \phi_0 \in \mathcal{M}_0$$

- *G = O(N)* model

  → for any point in $\mathcal{M}_0 = \mathrm{S}^{N-1}$ stability group is *H = O(N−1)*

  for any element in $\mathcal{M}_0$ there is always *O(N−1)* subgroup

Goldstone theorem in Classical Field Theory

- *G = O(N)* model

  G is spontaneously broken to the group *H:* $G \to H$

  → field ϕ - order parameter for symmetry *G*

  its value in ground state (zero or non-zero) – test if symmetry *G* is broken

  Vacuum manifold - coset space: $\mathcal{M}_0 \cong G/H$

Goldstone's theorem

If global continuous symmetry *G* is spontaneously broken to *H*, number of massless Goldstone bosons is:

$$\dim(G/H) = \dim G - \dim H$$

→ Goldstone bosons - modes that fluctuate along vacuum manifold $\mathcal{M}_0$

→ proved by Goldstone, Weinberg, Salam (1962)

In *G = O(N)* model

→ sphere as coset: $\mathrm{S}^{N-1} = O(N)/O(N-1)$

Goldstone theorem in QFT

- Classical field theory

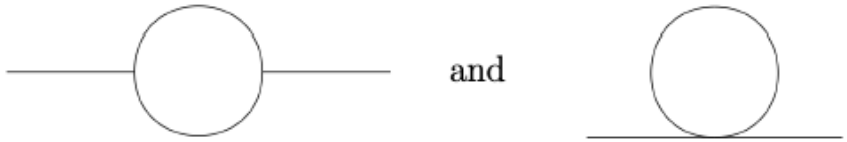
  → no difficulty in theory for a massless scalar ⇒ set $m^2 = 0$ in potential

- Quantum field theories

  → massless scalars *(and 'light' scalars as well)* are difficult to obtain

  physical mass is not just m^2 in Lagrangian

  → mass of scalar has contributions from cloud of other fields



  → external legs - scalars

  → particle running in loop - anything that scalar interacts with *(including itself)*

- Diagrams contribute to mass renormalisation of the scalar

  → quantum corrections push the mass of scalar particle to UV-cut off (Λ_{UV})

  → Lagrangian with $m^2 = 0$ will not describe quantum scalar particle with physical mass = 0

  after renormalisation it will describe scalar with physical mass $m^2 \sim \Lambda^2_{UV}$

Goldstone theorem in QFT

If QFT describing massless scalar

→ tune mass in Lagrangian to $m^2 \sim -\Lambda^2_{UV}$ *(bare mass)*

→ coefficient precisely cancels contributions from quantum corrections ⇒ **fine tuning**

- Quantum field theories tend not to have massless or light scalar fields

→ mass pushed up to some cut-off scale

not true for fermions - only logarithmic correction to the mass

fermions have extra chiral symmetry when massless ⇒ protects mass from being renormalised

Spontaneous symmetry breaking

Mechanism to deliver massless scalars, whose masses are protected from renormalization corrections.

Hierarchy problem

→ Higgs boson not particularly light ($m_H \approx 126$ GeV)

→ why mass of Higgs boson not pushed up to higher scales?

⇒ hierarchy problem

Broken symmetries on Hilbert space

How Goldstone theorem manifests in quantum field theory

- Physics seen in how symmetries act on Fock space of particles

By Noether theorem

Any continuous symmetry G has associated set of currents $J^A{}_\mu$, with $A = 1, \ldots, \dim G$

→ conserved charges: $Q^A = \int d^3x \; J^A_0$

'Inverse Noether theorem'

Given conserved charge, you can always reconstruct associated symmetry

- Vacuum expectation value

→ classical theory: ϕ is order parameter for symmetry G

→ quantum theory: order parameter - *vacuum expectation value* (*vev*) of ϕ

$\langle\phi\rangle = \langle\Omega|\phi|\Omega\rangle$ $|\Omega\rangle$ - vaccum of full interacting theory

→ if $\langle\phi\rangle \neq 0 \;\Rightarrow \phi$ condenses: $\langle\Omega|[Q^A, \phi^a]|\Omega\rangle = (T^A)^a{}_b\langle\phi^b\rangle \neq 0$

This is true if: $Q^A|\Omega\rangle \neq 0 \quad \text{for some } A$

Symmetry spontaneously broken in QFT: symmetry generators do not annihilate the vacuum

Broken symmetries on Hilbert space

- For any symmetry generator (broken or unbroken): $[Q^A, H] = 0$

  → if symmetry is broken ⇒ vacuum is degenerate

  in QFT every different choice of $\langle\phi\rangle$ corresponds to different vacuum of theory

  excitations above vacuum no longer in multiplets of full symmetry group *G*

  → if $\langle\phi\rangle = 0$ vacuum annihilated by symmetry generators: $Q^A|\Omega\rangle = 0$

  symmetry is unbroken

- Goldstone bosons

  → classical theory - correspond to fluctuations along directions of broken symmetry

  → states for each broken symmetry generator:

$$|\pi^A(\mathbf{p})\rangle \sim \int d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}} J_0^A(x)|\Omega\rangle$$

  states with 3-momentum **p**

  → limit of vanishing momentum: $\lim_{\mathbf{p}\to 0} |\pi^A(\mathbf{p})\rangle \sim Q^A|\Omega\rangle$

- Generators spontaneously broken

  → state $Q^A|\Omega\rangle \neq 0$ has same energy as original vacuum $|\Omega\rangle$ because $[Q^A,H] = 0$

  → Goldstone boson $|\pi^A(\mathbf{p})\rangle$ has energy $E \to 0$ as $\mathbf{p} \to 0$

  ⇒ Goldstone boson is massless

Coleman-Mermin-Wagner theorem

We assumed that spontaneous symmetry breaking occurs in quantum theory

→ e.g. if $\langle\phi\rangle \neq 0$ ground state must shift under symmetry: $Q|\Omega\rangle \neq 0$

how we know this happens?

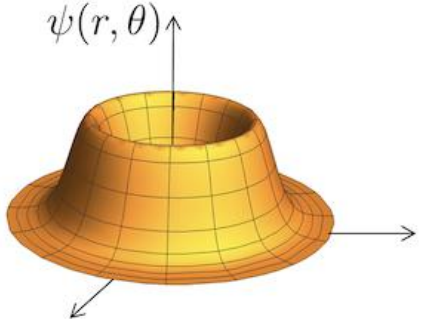
- Mexican hat potential in quantum mechanics → quantum particle moving in potential

→ peaked in the trough at $r = v$

→ fully delocalised in angular θ direction

→ wavefunction spreads around circle parameterised by θ

⇒ no spontaneous symmetry breaking



Why is quantum field theory different from quantum mechanics?

Coleman-Mermin-Wagner theorem

Continuous symmetry cannot be broken in quantum theories in d=0+1 (quantum mechanics) or d=1+1 dimensions

→ more difficult to spontaneously break continuous symmetries than discrete symmetries

→ for continuous symmetry to be spontaneously broken we must be in d=2+1 or higher

Coleman-Mermin-Wagner theorem

- **Assumption:** *d*-dimensional Euclidean space

  → assume massless scalar field ϕ with no potential *(choice of vacuum)*

  → choose $\phi = 0$ as ground state

  → excite scalar field introducing delta function source at origin: $\nabla^2\phi = \delta(\mathbf{x})$

  equation for the Green's function of the d-dimensional Laplacian

Solution: $$\phi(\mathbf{x}) \sim \begin{cases} |x| & \text{for } d = 1 \\ \log|\mathbf{x}| & \text{for } d = 2 \\ 1/|\mathbf{x}|^{d-2} & \text{for } d \geq 3 \end{cases}$$

- $d = 1$ and $d = 2$

  → exciting scalar field at origin it can no longer take $\phi = 0$ asymptotically

  → any disturbance at origin is felt at $|\mathrm{x}| \to \infty$

- $d \geq 3$

  → field excited near origin

  → back down to $\phi \to 0$ as $|\mathrm{x}| \to \infty$

Coleman-Mermin-Wagner theorem

- Quantum theory (free massless scalar + Euclidean spacetime)

→ 2-point function: $\langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle$

Solution: $$\langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle \sim \begin{cases} |x-y| & \text{for } d=1 \\ \log|\mathbf{x}-\mathbf{y}| & \text{for } d=2 \\ 1/|\mathbf{x}-\mathbf{y}|^{d-2} & \text{for } d\geq 3 \end{cases}$$

- $d = 1$ and $d = 2$

→ infra-red divergence

→ no normalisable ground state

- $d \geq 3$

→ each value of $\langle\phi\rangle$ specifies different ground state

→ massless ϕ field - Goldstone boson for shift symmetry $\phi \to \phi$ + constant

Existence of spontaneous symmetry breaking is due to infinite volume of space

→ quantum field theory on compact spatial manifold

⇒ long-time behaviour as in quantum mechanics

⇒ no spontaneous symmetry breaking

Higgs mechanism

Higgs mechanism

Goldstone theorem: continuous symmetry spontaneously broken ⇒ massless boson

What happens if symmetry is gauged?

→ there is no massless Goldstone boson

→ **gauge boson gets mass!**

Abelian Higgs model

- Complex scalar ϕ with Mexican hat potential

→ couple scalar to *U(1)* gauge field ⇒ action:

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi - \frac{\lambda}{2} (|\phi|^2 - v^2)^2 \right)$$

→ covariant derivative: $\mathcal{D}_\mu \phi = \partial_\mu \phi - ieA_\mu \phi$

→ ground state: $|\phi|^2 = v^2$

U(1) around symmetric S^1 is gauge symmetry

$$\phi \to e^{ie\alpha(x)} \phi \quad \text{and} \quad A_\mu \to A_\mu + \partial_\mu \alpha$$

→ gauge theory: only single ground state

→ no manifold of ground states

Abelian Higgs model

- **Sigma model:** classical vacuum $\phi = v$

→ fluctuations:

$$\phi(x) = e^{i\theta(x)}\big(v + \sigma(x)\big)$$

→ covariant derivarive :

$$\mathcal{D}_\mu\phi = e^{i\theta}\big(\partial_\mu\sigma + i(v+\sigma)(\partial_\mu\theta - eA_\mu)\big)$$

→ expanded action:

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\sigma\partial^\mu\sigma + (v+\sigma)^2(\partial_\mu\theta - eA_\mu)(\partial^\mu\theta - eA^\mu) - V(\sigma)\right) \qquad V(\sigma) = \frac{\lambda}{2}\sigma^2(\sigma+2v)^2$$

Mass spectrum of theory

→ quadratic mass term for scalar σ : $m_\sigma^2 = 2\lambda v^2$

same mass as for global symmetry

→ in electroweak (*EW*) theory: analogous particle is Higgs boson

Abelian Higgs model

Higgs mechanism

- Scalar field $\theta(x)$ $\rightarrow$ Goldstone boson (in absence of gauge field)
- **Gauge field** $\rightarrow$ field appears only in kinetic terms

$$\partial_\mu \theta - eA_\mu$$

$\rightarrow$ allows to eliminate field $\theta(x)$ $\Rightarrow$ new gauge field:

$$A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \theta$$

$\rightarrow$ A'_μ does not change under gauge transformation *(unitary gauge)*

shift $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$ now compensated by $\theta \rightarrow \theta + e\alpha$

$\rightarrow$ field $\theta(x)$ no longer in action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \sigma \partial^\mu \sigma + e^2 (v+\sigma)^2 A'_\mu A'^\mu - V(\sigma) \right)$$

Generated mass term for gauge field: $e^2 v^2 A'_\mu A'^\mu \qquad m_\gamma^2 = 2e^2 v^2$

$\rightarrow$ term forbidden by gauge invariance

$\rightarrow$ mass term arises naturally when spontaneously break gauge symmetry

$\Rightarrow$ **Higgs mechanism**

Abelian Higgs model

- Particle degrees of freedom

  → massive spin 1 particles: 3 degrees of freedom

  $(2l + 1)$-dimensional representation of little group for $l = 1$

  → massless spin 1 particles: 2 degrees of freedom *(2 polarisation states)*

- Would-be Goldstone mode

  → photon absorbing would-be Goldstone mode $\theta(x)$

  $\Rightarrow$ extra degree of freedom - longitudinal mode of the photon

- Mass of Higgs boson and photon

  → different parameteric dependence on coupling constants

  → decouple Higgs boson: take $m_\sigma \to \infty$ leave finite mass m_γ

Couple photon to Goldstone mode θ, ignore radial Higgs mode σ

→ Higgs mechanism in non-Abelian theories $\Rightarrow$ σ model non-renormalisable

effective low energy theory

Non-Abelian Higgs mechanism

Higgs mechanism extended to non-Abelian theories

Novel → gauge group *G* not broken completely

→ surviving massless gauge bosons

- **Assumption:** *SO*(3) symmetry to a gauge symmetry

→ 3-vector of real scalars: ϕ^a with $a = 1, 2, 3$

→ covariant derivative:

$$\mathcal{D}_\mu \phi^a = \partial_\mu \phi^a + g\epsilon^{abc} A^b_\mu \phi^c$$

ϵ^{abc} - generators of *SO(3)*

$$T^a_{bc} = -i\epsilon^{abc}$$

→ action in terms of components of gauge field A^a_μ

$$S = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \frac{1}{2} \mathcal{D}_\mu \phi^a \mathcal{D}^\mu \phi^a - \frac{\lambda}{2} (\phi^a \phi^a - v^2)^2 \right)$$

→ field strength tensor: $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu$

Non-Abelian Higgs mechanism

- Ground state:

$$\phi \cdot \phi = v^2$$

→ choice of vacuum: $\phi = (0, 0, v)$

This breaks gauge symmetry:

$$G = SO(3) \rightarrow H = U(1)$$

→ expect massless photon corresponding to *H = U(1)*

→ other 2 gauge bosons become massive due to Higgs mechanism

- For chosen vacuum look at fluctuations

→ parametrisation: $\phi^a(x) = e^{i(\xi^1(x)T^1 + \xi^2(x)T^2)} \begin{pmatrix} 0 \\ 0 \\ v + \sigma(x) \end{pmatrix}$

T^1, T^2 - $SO(3)$ generators

$\xi^1(x)$ $\xi^2(x)$ - Goldstone bosons *(for global G = SO(3))*

Non-Abelian Higgs mechanism

- **Assumption:** Now *SO*(3) is gauged

  → 2 would-be Goldstone bosons $\xi^1(x)$, $\xi^2(x)$

  ⇒ can both be removed by *SO*(3) gauge transformation on scalar:

$$\phi \to e^{i\alpha^a T^a}\phi$$

  → allows to eliminate fields $\xi^1(x)$, $\xi^2(x)$

  ⇒ new gauge fields:

$$A^1_\mu \text{ and } A^2_\mu$$

- Action for gauge fields + remaining fluctuating scalar $\sigma(x)$

$$S = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}g^2(v+\sigma)^2(A^1_\mu A^{1\,\mu} + A^2_\mu A^{2\,\mu}) - V(\sigma) \right)$$

$$V(\sigma) = \frac{\lambda}{2}\sigma^2(\sigma+2v)^2$$

Non-Abelian Higgs mechanism

- Two massive gauge bosons, A^1_μ and A^2_μ

  → gauge bosons mass: $m_\gamma^2 = g^2 v^2$

- Gauge boson A^3_μ remains massless

  → photon associated to unbroken symmetry group $H = U(1)$

- Massive Higgs field σ

  → mass term: $m_\sigma^2 = 4\lambda v^2$

Theory of purely Goldstone modes is interacting sigma-model

→ non-renormalisable

Non-Abelian Higgs model

→ in contrast to Abelian, Goldstone boson eaten here is not free before gauging