

Massive QCD at Five Loops

Andreas Maier Peter Marquard York Schröder



INSTYTUT FIZYKI JĄDROWEJ
IM. HENRYKA NIEWODNICZAŃSKIEGO
POLSKIEJ AKADEMII NAUK

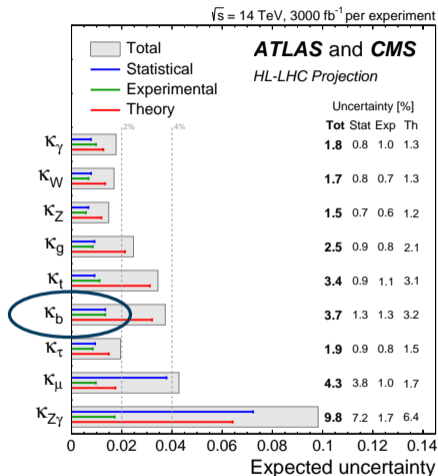
2 April 2026

Why Go to Five Loops?

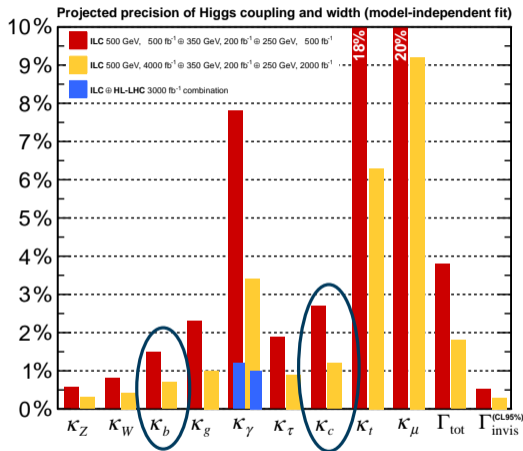
- **High-level answer:** Standard Model + general relativity are our best theories
 - ▶ Better calculational technology for maximum predictivity
 - ▶ Determine fundamental parameters as best as possible
- **Pragmatic answer:** Required for next generation of experiments
 - ▶ Higgs coupling measurements
 - ▶ Gravitational wave form modelling

Time scale for one additional loop order ~ 20 years \Rightarrow start **now**

Higgs Coupling Measurements



[arXiv:1902.00134]



[arXiv:1506.05992]

Coupling proportional to mass \Rightarrow need $\Delta m_b < 0.5\%$, $\Delta m_c < 1\%$



Decoupling

Quark masses

Higgs-gluon coupling

Quark condensate

Heavy Quark Masses at Four Loops

[Chetyrkin, Kühn, Maier, Maierhöfer, Marquard, Steinhauser, Sturm 2009 + 2017]

$$m_b(10 \text{ GeV}) = (3610 \pm 10(\text{exp}) \pm 12(\alpha_s) \pm 3(\mu)) \text{ MeV}$$

$$m_b(m_b) = (4163 \pm 16) \text{ MeV}$$

$$m_c(3 \text{ GeV}) = (993 \pm 7(\text{exp}) \pm 4(\alpha_s) \pm 2(\mu) \pm 1(\text{np})) \text{ MeV}$$

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[Dehnadi, Hoang, Mateu, Zebarjad 2013; Dehnadi, Hoang, Mateu 2015]

$$m_b(m_b) = (4176 \pm 4(\text{stat}) \pm 19(\text{sys}) \pm 7(\alpha_s) \pm 10(\mu)) \text{ MeV}$$

$$m_c(3 \text{ GeV}) = (994 \pm 6(\text{stat}) \pm 9(\text{sys}) \pm 10(\alpha_s) \pm 21(\mu) \pm 2(\text{np})) \text{ MeV}$$

Heavy Quark Masses at Four Loops

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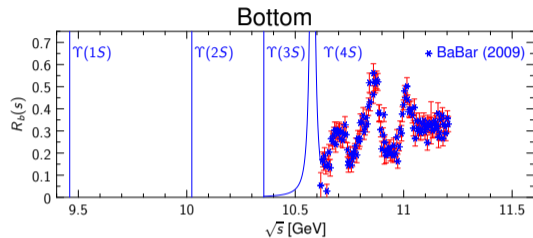
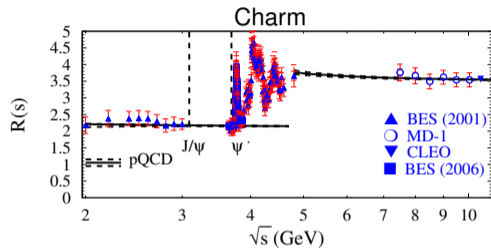
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Factor 3 to 10 disagreement in theory uncertainty

Sum Rules

Consider $R_Q(s) = \frac{\sigma(e^+e^- \rightarrow Q\bar{Q})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$



- Cannot predict resonances in standard perturbation theory
- Idea: consider “smeared” cross section [Poggio, Quinn, Weinberg 1975]
- In practice: consider (inverse) moments [Shifman, Weinstein, Zakharov 1978]

$$\mathcal{M}_n = \int_{s_0}^{\infty} ds \frac{R_Q(s)}{s^{n+1}}$$

Sum Rules

Theory moments: use dispersion relation \rightarrow massive vacuum diagrams

$$\mathcal{M}_n = \int_{s_0}^{\infty} ds \frac{R_Q(s)}{s^{n+1}} = \frac{12\pi^2}{n!} \left(\frac{d}{dq^2} \right)^n \Pi_Q(q^2) \Big|_{q^2=0}$$

$$\Pi_Q(q^2) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots$$

[Chetyrkin, Kühn, Sturm 06; Boughezal, Czakon, Schutzmeier 06; Maier, Maierhöfer, Marquard, Smirnov 09]

Sum Rules

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Estimate of perturbative uncertainty


- [Chetyrkin, Kühn, Maier, Maierhöfer, Marquard, Steinhauser, Sturm 2009 + 2017]:
 - ▶ Charm: choose central $\mu = 3 \text{ GeV} \approx m_{J/\psi}$
 - ▶ Bottom: choose central $\mu = 10 \text{ GeV} \approx m_{\Upsilon}$
 - ▶ Vary by μ factor of 2
- [Dehnadi, Hoang, Mateu, Zebarjad 2013; Dehnadi, Hoang, Mateu 2015]:
 - ▶ Choose central $\mu = m_Q(m_Q)$.
 - ▶ Express moments in terms of $\alpha_s(\mu)$, $m(\mu_m)$, μ_m independent of μ .
 - ▶ Independently vary μ, μ_m by factor of 2

Calculating Five-Loop Moments

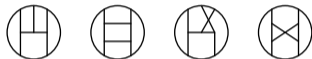
1 Generate diagrams

 ~ seconds

2 Identify scalar diagram families:

 ~ seconds

34 mass colourings of




3 Insert Feynman rules \rightarrow scalar Feynman integrals

 ~ days

4 Reduce to 143 basis integrals: **to be done**

 ~ months

5 Insert basis integrals

 ~ seconds

Need to calculate basis integrals



Decoupling

Quark masses

Higgs-gluon coupling

Quark condensate

The Quark Condensate

First tackle easier calculation:

$$\langle \bar{\psi}\psi \rangle = \text{[vacuum bubble]} + \text{[vacuum bubble with gluon loop]} + \dots$$

- Leading non-analytic contribution in
 - ▶ Operator Product Expansion ($m \sim \Lambda_{\text{QCD}}$)
 - ▶ Asymptotic small-mass expansion ($m \gg \Lambda_{\text{QCD}}$)

Extrapolation from heavy to light quarks via
Renormalisation Group Optimised Perturbation Theory

[Kneur, Neveu 2010-2020]

The Quark Condensate

First tackle easier calculation:

$$\langle \bar{\psi}\psi \rangle = \text{[Bubble Diagram]} + \text{[Bubble Diagram with vertical line]} + \dots$$

- Leading non-analytic contribution in
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Extrapolation from heavy to light quarks via
Renormalisation Group Optimised Perturbation Theory

[Kneur, Neveu 2010-2020]

- Direct relation to vacuum anomalous dimension:


[Spiridonov, Chetyrkin 1988]

$$\mu^2 \frac{d}{d\mu^2} m \langle \bar{\psi}\psi \rangle = -4m^4 \gamma_0$$


↔ independent check of five-loop result [Baikov, Chetyrkin 2018]

Calculating the Quark Condensate

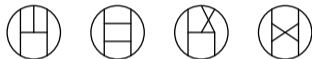
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Need to calculate basis integrals

Basis integrals

Sector decomposition [Hepp 1966; Speer 1968; Binoth, Heinrich 2000]

Compute basis integrals with FIESTA:

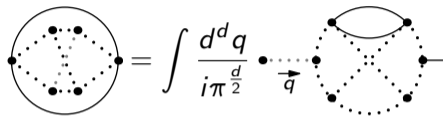
[Smirnov et al. 2008-2021]

$$\langle \bar{\psi} \psi \rangle \Big|_{\left(\frac{\alpha_s}{\pi}\right)^4} = \frac{m^3}{16\pi^2} \left[\frac{(-3.5 \pm 3.0) \times 10^{-8}}{\epsilon^{11}} + \frac{(-2.1 \pm 7.6) \times 10^{-6}}{\epsilon^{10}} \right. \\ \left. + \frac{(0.2 \pm 2.1) \times 10^{-4}}{\epsilon^9} + \frac{(-0.5 \pm 7.5) \times 10^{-2}}{\epsilon^8} + \dots \right]$$

Need to calculate basis integrals *with high precision*

Basis integrals

Direct integration c.f. [Faisst, Chetyrkin, Kühn 2004]



The diagram shows an equality between two Feynman diagrams. On the left is a circle with a dashed internal loop and two external lines. This is equal to an integral over $d^d q$ with a denominator $i\pi^{d/2}$. The integral is followed by a diagram where the dashed loop is replaced by a solid loop with a self-energy insertion, and an arrow labeled \vec{q} indicates the momentum flow.

$$\text{Diagram 1} = \int \frac{d^d q}{i\pi^{d/2}} \text{Diagram 2}$$

Basis integrals

Direct integration c.f. [Faisst, Chetyrkin, Kühn 2004]

$$\text{Diagram} = \int \frac{d^d q}{i\pi^{\frac{d}{2}}} \text{Diagram}$$

With $z = -q^2$:

$$\text{Diagram} = \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \text{Diagram}$$

Basis integrals

Direct integration c.f. [Faisst, Chetyrkin, Kühn 2004]

$$\text{Diagram} = \int \frac{d^d q}{i\pi^{\frac{d}{2}}} \text{Diagram}'$$

With $z = -q^2$:

$$\text{Diagram} = \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \text{Diagram}'$$

q does not need to be a line momentum:

$$\text{Diagram} = \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \text{Diagram}''$$

Expand in ϵ , integrate numerically

Basis integrals

Infrared & ultraviolet divergences

Problem: integrand is infrared & ultraviolet divergent

Basis integrals

Infrared & ultraviolet divergences

Problem: integrand is infrared & ultraviolet divergent

Initial idea:

- Choose q as massive line momentum
 - ▶ All heavy fermion lines are closed
 - ⇒ No massless cuts in integrand
 - ⇒ No infrared divergences
- Take propagator with momentum q to higher power
 - ⇒ No ultraviolet divergence

Basis integrals

Infrared & ultraviolet divergences

Problem: integrand is infrared & ultraviolet divergent

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Drawbacks:

- Needs five-loop basis change
- Needs closed massive lines
- Needs unknown ingredients → next slides

Alternative: infrared and ultraviolet subtraction

Basis integrals

Infrared subtraction

Subtract infrared divergences:

$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \left[\text{Diagram} - \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\text{IR}}} \Delta_{k,l}^{\text{IR}} \left(\text{Diagram} \right)^{k+l\epsilon} \right]$$

The diagram on the left is a complex loop diagram with several vertices and internal lines, some of which are dashed. A momentum vector \vec{q} is shown entering from the left. The diagram on the right is a simpler diagram consisting of two vertices connected by a dashed line, with a momentum vector \vec{q} entering from the left.

Basis integrals

Infrared subtraction

Subtract infrared divergences:

$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \left[\text{Diagram 1} - \sum_{l=0}^{L-1} \sum_{k=2}^{k_{lR}} \Delta_{k,l}^{\text{IR}} \left(\text{Diagram 2} \right)^{k+l\epsilon} \right]$$

=

$$\text{Diagram 3} - \underbrace{\text{Diagram 4}}_{=0}^{k+l\epsilon}$$

The diagrams represent Feynman diagrams with external momenta \vec{q} . Diagram 1 is a complex multi-loop diagram with a bubble. Diagram 2 is a simpler multi-loop diagram. Diagram 3 is a diagram with a bubble. Diagram 4 is a single loop diagram.

Translate infrared to ultraviolet divergences

Basis integrals

Ultraviolet subtraction

$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \left[\begin{array}{l} \text{Diagram 1} - \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\text{IR}}} \Delta_{k,l}^{\text{IR}} \left(\text{Diagram 2} \right)^{k+l\epsilon} \\ - \sum_{l=0}^{L-1} \sum_{k=k_{\text{UV}}}^2 \Delta_{k,l}^{\text{UV}} \left(\text{Diagram 3} \right)^{l\epsilon} \left(\text{Diagram 4} \right)^k \end{array} \right]$$


$$= \text{Diagram 5} - \sum_{l=0}^{L-1} \sum_{k=1}^2 \Delta_{k,l}^{\text{UV}} \frac{\Gamma((l+1)\epsilon + k - 2)\Gamma(2 - (l+1)\epsilon)}{\Gamma(2-\epsilon)\Gamma(k)}$$

The diagrams are:

- Diagram 1: A circle with 6 vertices. A dashed line with momentum \vec{q} connects two vertices. A solid line with a loop connects two other vertices.
- Diagram 2: A circle with 6 vertices and a dashed line with momentum \vec{q} connecting two vertices.
- Diagram 3: A circle with 6 vertices and a dashed line with momentum \vec{q} connecting two vertices.
- Diagram 4: A circle with 2 vertices and a dashed line with momentum \vec{q} connecting them.
- Diagram 5: A circle with 6 vertices and a dashed line with momentum \vec{q} connecting two vertices.

Differential Equations

Generalised power series ansatz

$P_0 =$  from differential equations for propagator basis integrals:

$$z \frac{\partial}{\partial z} P_i = Q_{ij} P_j$$

[Kotikov 1991, Remiddi 1997,...]

Infrared and ultraviolet counterterms from series ansätze:

$$P_i = \sum_{l=0}^{L-1} \sum_{k=k_{\text{IR}}}^{N-1} c_{ikl} z^{k-l\epsilon} + \mathcal{O}(z^N), \quad P_i = \sum_{l=0}^{L-1} \sum_{k=k_{\text{UV}}}^{N-1} d_{ikl} \frac{1}{z^{k-l\epsilon}} + \mathcal{O}\left(\frac{1}{z^N}\right)$$

Boundary conditions $c_{ik_{\text{IR}}l}$, $d_{ik_{\text{UV}}l}$ from asymptotic expansion:

known massless propagators and massive vacuum diagrams with ≤ 4 loops

[Laporta 2002; Czakon 2004, Schröder, Vuorinen 2005; Lee, Terekhov 2010; Baikov, Chetyrkin 2010; Lee, Smirnov, Smirnov 2011]

Differential Equations

Padé approximation

Idea: rational function approximation from

$$P_i = \sum_{l=0}^{L-1} \sum_{k=k_{IR}}^{N-1} c_{ikl} z^{k-l\epsilon} + \mathcal{O}(z^N), \quad P_i = \sum_{l=0}^{L-1} \sum_{k=k_{UV}}^{N-1} d_{ikl} \frac{1}{z^{k-l\epsilon}} + \mathcal{O}\left(\frac{1}{z^N}\right)$$

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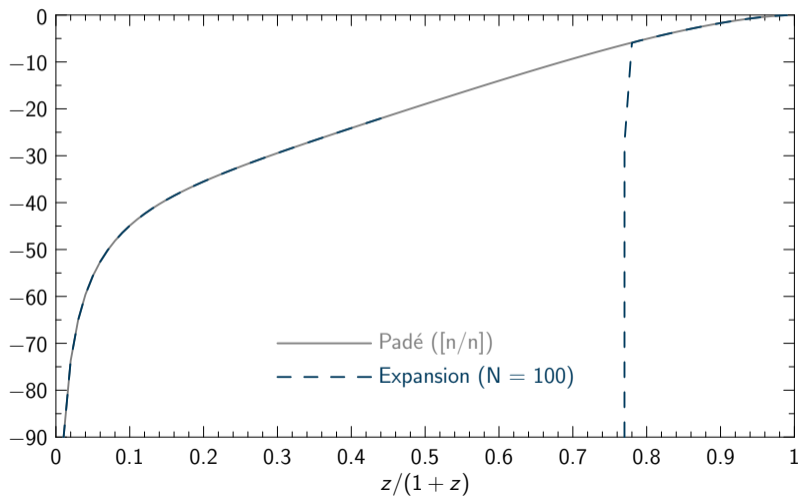
- Subtract logarithms from expansion of $z^{-\epsilon}$
- Combine into expansion around single point
- Conformal mapping $z \rightarrow \frac{4\omega}{(1-\omega)^2}$

$$P_i \approx \frac{a_0 + a_1\omega + \cdots + a_n\omega^n}{1 + b_1\omega + \cdots + b_m\omega^m} z(\omega)^{1-N} + P_i^{\text{subtr}}(\omega)$$

Differential Equations

Padé approximation

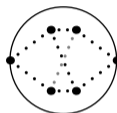
Integrand at order ϵ^0 :



Differential Equations

Padé approximation

Result for $N = 100$:


$$\begin{aligned} &= - 1.0369277551433699263313654864570341680570809195019 \epsilon^{-2} \\ &\quad - 2.2715175939186296129359966601358890918938395071698 \epsilon^{-1} \\ &\quad - (51.677219826125100976097934332151000544783722 \pm 10^{-42}) \\ &\quad - (74.113630654937474804957005927608652069299 \pm 10^{-39}) \epsilon \\ &\quad - (1534.5823448304860203496262262737878903221 \pm 10^{-37}) \epsilon^2 \\ &\quad - (997.924115204237781450466544814814445621 \pm 10^{-37}) \epsilon^3 \\ &\quad - (69346.269757752899137898405329811054610282 \pm 10^{-36}) \epsilon^4 \\ &\quad + \mathcal{O}(\epsilon^5) \end{aligned}$$

Result for the Quark Condensate

For n_f massless quark flavours:

$$\langle \bar{q}q \rangle \Big|_{\mu=m_q} = -\frac{3}{16\pi^2} m_q^3 \left[4 + 13.3333 \frac{\alpha_s}{\pi} + (132.723 - 9.27398 n_f) \left(\frac{\alpha_s}{\pi} \right)^2 \right. \\ \left. + (1320.23 - 134.592 n_f + 1.05832 n_f^2) \left(\frac{\alpha_s}{\pi} \right)^3 \right. \\ \left. + (17721.5 - 3448.7 n_f + 168.599 n_f^2 - 1.88974 n_f^3) \left(\frac{\alpha_s}{\pi} \right)^4 \right]$$

- Poor convergence, but in Operator Product Expansion always multiplied by Wilson coefficient
- ✓ Confirm five-loop anomalous dimension [Baikov, Chetykin 2018]



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Decoupling at five loops

Motivation

Top quarks are problematic for QCD processes with $E < m_t$:

- Diagrams with massive internal lines \Rightarrow hard to calculate
- Large logarithms $\ln\left(\frac{E}{m_t}\right)$ spoil perturbative convergence

Decoupling at five loops

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Solution: effective 5-flavour theory with coupling

$$\alpha_s^{(5)} = \alpha_s^{(6)} \times \left[\frac{\left(\text{triangle diagram} \right)^2}{\left(\text{circle diagram} \right)^2 \cdot \left(\text{circle diagram} \right)} \right]_{\substack{q=0 \\ m_t \neq 0 \\ m_Q = 0}}$$


Relation known to four loops [Schröder, Steinhauser 2005]

Calculating Decoupling Coefficients

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 ~ seconds

✓ Identify scalar diagram families:

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34 mass colourings of    


✓ Insert Feynman rules → scalar Feynman integrals

 ~ days

✓ Reduce to 143 basis integrals

 ~ months

~~✗~~ Insert basis integrals

 ~ seconds

Conclusions

- First five-loop QCD result: quark condensate
- More to come:
 - ▶ Decoupling of heavy quarks
 - ▶ Charm and bottom mass determinations
 - ▶ ...