Massive QCD at Five Loops

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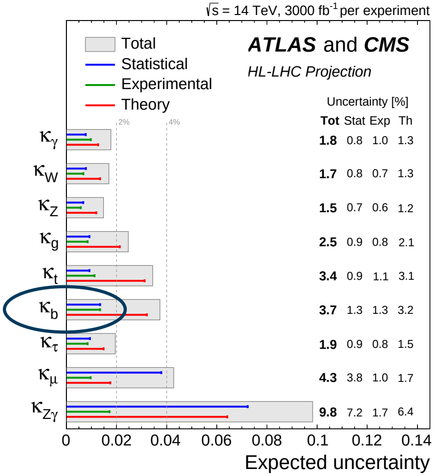
Why Go to Five Loops?

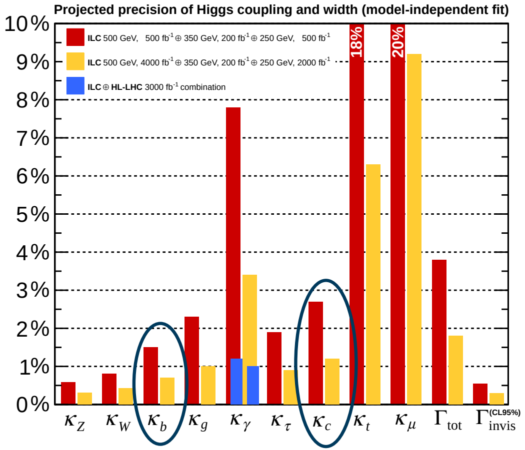
- High-level answer: Standard Model + general relativity are our best theories
  - Better calculational technology for maximum predictivity
  - Determine fundamental parameters as best as possible
- Pragmatic answer: Required for next generation of experiments
  - Higgs coupling measurements
  - Gravitational wave form modelling

Time scale for one additional loop order $\sim$ 20 years $\Rightarrow$ start now

Higgs Coupling Measurements



[arXiv:1902.00134]



[arXiv:1506.05992]

Coupling proportional to mass $\Rightarrow$ need $\Delta m_b < 0.5\%$, $\Delta m_c < 1\%$


Quark masses
Higgs-gluon coupling
Decoupling
Quark condensate

Heavy Quark Masses at Four Loops

[Chetyrkin, Kühn, Maier, Maierhöfer, Marquard, Steinhauser, Sturm 2009 + 2017]

$$m_b(10\,\text{GeV}) = \big(3610 \pm 10(\text{exp}) \pm 12(\alpha_s) \pm 3(\mu)\big)\,\text{MeV}$$
$$m_b(m_b) = (4163 \pm 16)\,\text{MeV}$$
$$m_c(3\,\text{GeV}) = \big(993 \pm 7(\text{exp}) \pm 4(\alpha_s) \pm 2(\mu) \pm 1(\text{np})\big)\,\text{MeV}$$

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[Dehnadi, Hoang, Mateu, Zebarjad 2013; Dehnadi, Hoang, Mateu 2015]

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m_b(m_b) &= \big(4176 \pm 4(\text{stat}) \pm 19(\text{sys}) \pm 7(\alpha_s) \pm 10(\mu)\big)\,\text{MeV} \\
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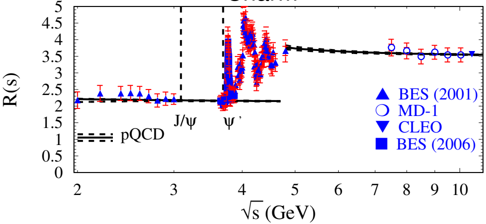
Factor 3 to 10 disagreement in theory uncertainty

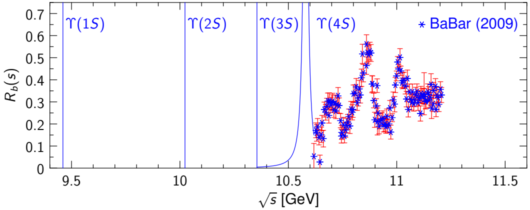
Sum Rules

Consider $R_Q(s) = \frac{\sigma(e^+e^- \to Q\bar{Q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$

Charm



Bottom



- Cannot predict resonances in standard perturbation theory
- Idea: consider “smeared” cross section [Poggio, Quinn, Weinberg 1975]
- In practice: consider (inverse) moments [Shifman, Weinstein, Zakharov 1978]

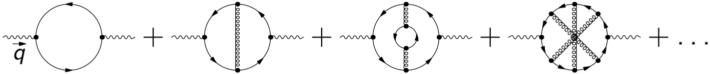
$$\mathcal{M}_n = \int_{s_0}^{\infty} ds \, \frac{R_Q(s)}{s^{n+1}}$$

Sum Rules

Theory moments: use dispersion relation → massive vacuum diagrams

$$\mathcal{M}_n = \int_{s_0}^{\infty} ds\, \frac{R_Q(s)}{s^{n+1}} = \frac{12\pi^2}{n!} \left(\frac{d}{dq^2} \right)^n \Pi_Q(q^2) \Big|_{q^2=0}$$

$$\Pi_Q(q^2) =$$



[Chetyrkin, Kühn, Sturm 06; Boughezal, Czakon, Schutzmeier 06; Maier, Maierhöfer, Marquard, Smirnov 09]

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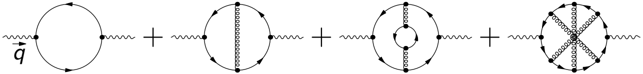


$+ \ldots$

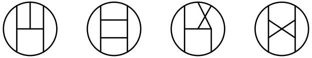
[Chetyrkin, Kühn, Sturm 06; Boughezal, Czakon, Schutzmeier 06; Maier, Maierhöfer, Marquard, Smirnov 09]

Estimate of perturbative uncertainty

- [Chetyrkin, Kühn, Maier, Maierhöfer, Marquard, Steinhauser, Sturm 2009 + 2017]:
  - Charm: choose central $\mu = 3\,\mathrm{GeV} \approx m_{J/\psi}$
  - Bottom: choose central $\mu = 10\,\mathrm{GeV} \approx m_{\Upsilon}$
  - Vary by μ factor of 2
- [Dehnadi, Hoang, Mateu, Zebarjad 2013; Dehnadi, Hoang, Mateu 2015]:
  - Choose central $\mu = m_Q(m_Q)$.
  - Express moments in terms of $\alpha_s(\mu)$, $m(\mu_m)$, μ_m independent of μ.
  - Independently vary μ, μ_m by factor of 2

Calculating Five-Loop Moments

1. Generate diagrams ~ seconds
2. Identify scalar diagram families: ~ seconds

   34 mass colourings of 

3. Insert Feynman rules $\rightarrow$ scalar Feynman integrals ~ days
4. Reduce to 143 basis integrals: to be done ~ months
5. Insert basis integrals ~ seconds

Need to calculate basis integrals


Decoupling
Quark masses
Higgs-gluon coupling
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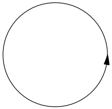
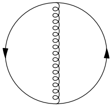
The Quark Condensate

First tackle easier calculation:

$\langle\bar{\psi}\psi\rangle =$  $+$  $+ \; \ldots$

- Leading non-analytic contribution in
  - Operator Product Expansion ($m \sim \Lambda_{\mathrm{QCD}}$)
  - Asymptotic small-mass expansion ($m \gg \Lambda_{\mathrm{QCD}}$)

  Extrapolation from heavy to light quarks via
  Renormalisation Group Optimised Perturbation Theory [Kneur, Neveu 2010-2020]

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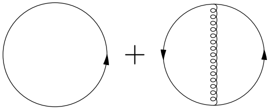
$\langle\bar{\psi}\psi\rangle =$  $+ \dots$

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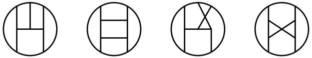
  Extrapolation from heavy to light quarks via
  Renormalisation Group Optimised Perturbation Theory [Kneur, Neveu 2010-2020]
- Direct relation to vacuum anomalous dimension: [Spiridonov, Chetyrkin 1988]

$$\mu^2 \frac{d}{d\mu^2} m\langle\bar{\psi}\psi\rangle = -4m^4\gamma_0$$

$\hookrightarrow$ independent check of five-loop result [Baikov, Chetyrkin 2018]

Calculating the Quark Condensate

1. Generate diagrams — ∼ seconds
2. Identify scalar diagram families: — ∼ seconds

   34 mass colourings of 

3. Insert Feynman rules → scalar Feynman integrals — ∼ minutes
4. Reduce to 142 basis integrals — ∼ days
5. Insert basis integrals — ∼ seconds

Need to calculate basis integrals

Basis integrals

Sector decomposition [Hepp 1966; Speer 1968; Binoth, Heinrich 2000]

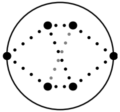
Compute basis integrals with `FIESTA`: [Smirnov et al. 2008-2021]

$$\langle\bar{\psi}\psi\rangle\Big|_{\left(\frac{\alpha_s}{\pi}\right)^4} = \frac{m^3}{16\pi^2}\left[\frac{(-3.5\pm3.0)\times10^{-8}}{\epsilon^{11}} + \frac{(-2.1\pm7.6)\times10^{-6}}{\epsilon^{10}} + \frac{(0.2\pm2.1)\times10^{-4}}{\epsilon^{9}} + \frac{(-0.5\pm7.5)\times10^{-2}}{\epsilon^{8}} + \dots\right]$$

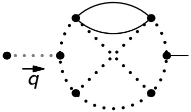
Need to calculate basis integrals *with high precision*

Basis integrals

Direct integration c.f. [Faisst, Chetyrkin, Kühn 2004]



$$= \int \frac{d^d q}{i\pi^{\frac{d}{2}}}$$



Basis integrals

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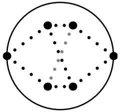
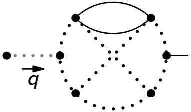
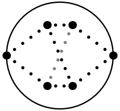
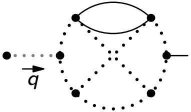
 $= \int \frac{d^d q}{i\pi^{\frac{d}{2}}}$ 

With $z = -q^2$:

 $= \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} \, z^{1-\epsilon}$ 

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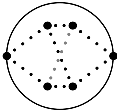
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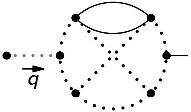

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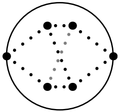
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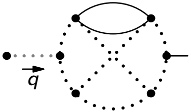
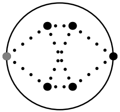


q does not need to be a line momentum: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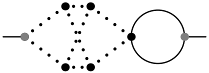

Expand in ϵ, integrate numerically

Basis integrals

Infrared & ultraviolet divergences

Problem: integrand is infrared & ultraviolet divergent

Basis integrals

Infrared & ultraviolet divergences

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Initial idea:

- Choose q as massive line momentum
  - All heavy fermion lines are closed

  $\Rightarrow$ No massless cuts in integrand

  $\Rightarrow$ No infrared divergences
- Take propagator with momentum q to higher power

  $\Rightarrow$ No ultraviolet divergence

Basis integrals

Infrared & ultraviolet divergences

Problem: integrand is infrared & ultraviolet divergent

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  - ▸ All heavy fermion lines are closed
  - $\Rightarrow$ No massless cuts in integrand
  - $\Rightarrow$ No infrared divergences
- Take propagator with momentum q to higher power

  $\Rightarrow$ No ultraviolet divergence

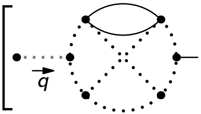
Drawbacks:

- Needs five-loop basis change
- Needs closed massive lines
- Needs unknown ingredients $\rightarrow$ next slides

Alternative: infrared and ultraviolet subtraction

Subtract infrared divergences:

$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)}\, z^{1-\epsilon} \left[\right.$$



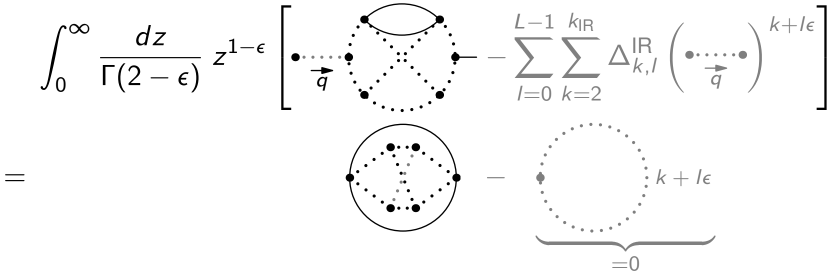
$$\left. - \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\mathrm{IR}}} \Delta^{\mathrm{IR}}_{k,l} \left(\right. \right.$$

$$\left. \left. \right)^{k+l\epsilon} \right]$$

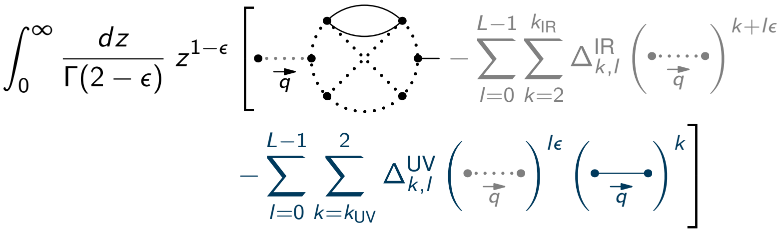
Basis integrals

Infrared subtraction

Subtract infrared divergences:

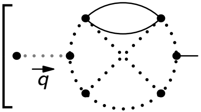


Translate infrared to ultraviolet divergences



$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon}$
$\vec{q}$
$- \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\mathrm{IR}}} \Delta^{\mathrm{IR}}_{k,l}$
$k+l\epsilon$
$- \sum_{l=0}^{L-1} \sum_{k=k_{\mathrm{UV}}}^{2} \Delta^{\mathrm{UV}}_{k,l}$
$l\epsilon$
k

$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)}\, z^{1-\epsilon} \Bigg[$$



$$- \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\mathrm{IR}}} \Delta^{\mathrm{IR}}_{k,l} \Bigg($$

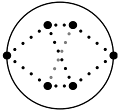
$$\Bigg)^{k+l\epsilon}$$

$$- \sum_{l=0}^{L-1} \sum_{k=k_{\mathrm{UV}}}^{2} \Delta^{\mathrm{UV}}_{k,l} \Bigg($$

$$\Bigg)^{l\epsilon} \Bigg($$

$$\Bigg)^{k} \Bigg]$$

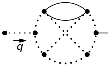
$$=$$



$$- \sum_{l=0}^{L-1} \sum_{k=1}^{2} \Delta^{\mathrm{UV}}_{k,l} \frac{\Gamma((l+1)\epsilon + k - 2)\Gamma(2-(l+1)\epsilon)}{\Gamma(2-\epsilon)\Gamma(k)}$$

Differential Equations

Generalised power series ansatz

$P_0 =$ 

from differential equations for propagator basis integrals:

$$z \frac{\partial}{\partial_z} P_i = Q_{ij} P_j$$

[Kotikov 1991, Remiddi 1997,...]

Infrared and ultraviolet counterterms from series ansätze:

$$P_i = \sum_{l=0}^{L-1} \sum_{k=k_{\mathrm{IR}}}^{N-1} c_{ikl} z^{k-l\epsilon} + \mathcal{O}(z^N), \qquad P_i = \sum_{l=0}^{L-1} \sum_{k=k_{\mathrm{UV}}}^{N-1} d_{ikl} \frac{1}{z^{k-l\epsilon}} + \mathcal{O}\left(\frac{1}{z^N}\right)$$

Boundary conditions $c_{ik_{\mathrm{IR}}l}$, $d_{ik_{\mathrm{UV}}l}$ from asymptotic expansion:
known massless propagators and massive vacuum diagrams with ≤ 4 loops

[Laporta 2002; Czakon 2004, Schröder, Vuorinen 2005; Lee, Terekhov 2010; Baikov, Chetyrkin 2010; Lee, Smirnov, Smirnov 2011]

Differential Equations

Padé approximation

Idea: rational function approximation from

$$P_i = \sum_{l=0}^{L-1} \sum_{k=k_{\text{IR}}}^{N-1} c_{ikl} z^{k-l\epsilon} + \mathcal{O}(z^N), \qquad P_i = \sum_{l=0}^{L-1} \sum_{k=k_{\text{UV}}}^{N-1} d_{ikl} \frac{1}{z^{k-l\epsilon}} + \mathcal{O}\left(\frac{1}{z^N}\right)$$

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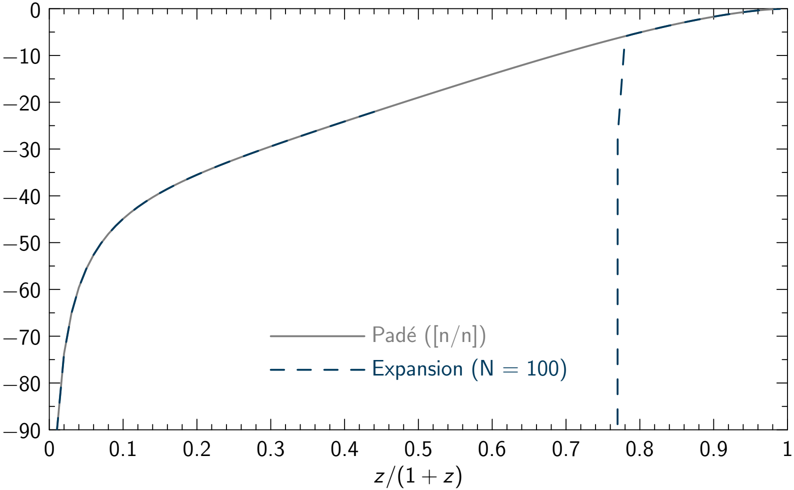
- Subtract logarithms from expansion of $z^{-\epsilon}$
- Combine into expansion around single point
- Conformal mapping $z \to \frac{4\omega}{(1-\omega)^2}$

$$P_i \approx \frac{a_0 + a_1\omega + \cdots + a_n\omega^n}{1 + b_1\omega + \cdots + b_m\omega^m} z(\omega)^{1-N} + P_i^{\text{subtr}}(\omega)$$

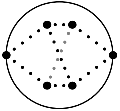
Integrand at order ϵ^0:



Differential Equations

Padé approximation

Result for $N = 100$:



$$
\begin{aligned}
= &- 1.036927755143369926331365486457034168057080919 5019\,\epsilon^{-2} \\
&- 2.2715175939186296129359966601358890918938395071698\,\epsilon^{-1} \\
&- (51.677219826125100976097934332151000544783722 \pm 10^{-42}) \\
&- (74.113630654937474804957005927608652069299 \pm 10^{-39})\,\epsilon \\
&- (1534.5823448304860203496262262737878903221 \pm 10^{-37})\,\epsilon^{2} \\
&- (997.92411520423778145046654481481444 5621 \pm 10^{-37})\,\epsilon^{3} \\
&- (69346.269757752899137898405329811054610282 \pm 10^{-36})\,\epsilon^{4} \\
&+ \mathcal{O}(\epsilon^{5})
\end{aligned}
$$

Result for the Quark Condensate

For n_l massless quark flavours:

$$\langle \bar{q}q \rangle \Big|_{\mu=m_q} = -\frac{3}{16\pi^2} m_q^3 \Bigg[4 + 13.3333 \frac{\alpha_s}{\pi} + (132.723 - 9.27398\, n_l) \left(\frac{\alpha_s}{\pi}\right)^2 + (1320.23 - 134.592\, n_l + 1.05832\, n_l^2) \left(\frac{\alpha_s}{\pi}\right)^3 + (17721.5 - 3448.7\, n_l + 168.599\, n_l^2 - 1.88974\, n_l^3) \left(\frac{\alpha_s}{\pi}\right)^4 \Bigg]$$

- Poor convergence, but in Operator Product Expansion always multiplied by Wilson coefficient
- ✓ Confirm five-loop anomalous dimension [Baikov, Chetykin 2018]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Top quarks are problematic for QCD processes with $E < m_t$:

- Diagrams with massive internal lines $\Rightarrow$ hard to calculate
- Large logarithms $\ln\left(\frac{E}{m_t}\right)$ spoil perturbative convergence

Decoupling at five loops

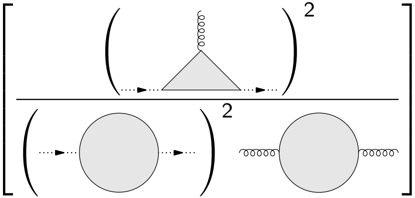
Motivation

Top quarks are problematic for QCD processes with $E < m_t$:

- Diagrams with massive internal lines $\Rightarrow$ hard to calculate
- Large logarithms $\ln\left(\frac{E}{m_t}\right)$ spoil perturbative convergence

Solution: effective 5-flavour theory with coupling

$$\alpha_s^{(5)} = \alpha_s^{(6)} \times$$

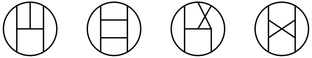


$q{=}0$
$m_t{\neq}0$
$m_Q{=}0$

Relation known to four loops [Schröder, Steinhauser 2005]

Calculating Decoupling Coefficients

- ✓ Generate diagrams — ∼ seconds
- ✓ Identify scalar diagram families: — ∼ seconds

  34 mass colourings of 
- ✓ Insert Feynman rules → scalar Feynman integrals — ∼ days
- ✓ Reduce to 143 basis integrals — ∼ months
- ✗ Insert basis integrals — ∼ seconds

Conclusions

- First five-loop QCD result: quark condensate
- More to come:
  - Decoupling of heavy quarks
  - Charm and bottom mass determinations
  - ...