Partonic entropy from DGLAP evolution

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Local tradition

- Pioneering works on parton entropy:
  K. Kutak, Phys. Lett. B 705, 217 (2011)
  M. Hentschinski, K. Kutak, and R. Straka, Eur. Phys. J. C 82, 1147 (2022)
  M. Hentschinski, D. E. Kharzeev, K. Kutak, and Z. Tu, Phys. Rev. Lett. 131, 241901 (2023)
  M. Hentschinski, D. E. Kharzeev, K. Kutak, and Z. Tu, Rept. Prog. Phys. 87, 120501 (2024)
  P. Caputa and K. Kutak, Phys. Rev. D 110, 085011 (2024)
  K. Kutak and M. Praszalowicz, Eur. Phys. J. C 85, 1215 (2025)
  K. Kutak and S. Lökös, Phys. Rev. D 112, 096017 (2025)

- Works on entropy in 2-D QCD:
  Y. Liu, M. A. Nowak, and I. Zahed, Phys. Rev. D 105, 114027 (2022)
  Y. Liu, M. A. Nowak, and I. Zahed, Phys. Rev. D 105, 114028 (2022)
  Y. Liu, M. A. Nowak, and I. Zahed, Phys. Rev. D 107, 054010 (2023)

Classical entropy

Thermodynamic entropy

- In thermodynamics, energy E is a function of state $A = (T, V, N)$:

$$\oint dE|_{rev} = 0 \quad \rightarrow \quad E(B) = \int_A^B dE|_{rev} + E(A)$$

- Claussius discovered entropy S as a function of state by proving that for any cyclic reversible process:

$$\oint \left.\frac{DQ}{T}\right|_{rev} = 0 \quad \rightarrow \quad S(B) = \int_A^B \left.\frac{DQ}{T}\right|_{rev} + S(A)$$

- The very existence of entropy relies on the existence of statistically irreversible processes (second law):
  - Heat cannot spontaneously flow from a colder body to a hotter body (Classius)
  - Heat cannot be converted into work without additional changes (Kelvin)
- Second law of thermodynamics in terms of entropy: *Entropy of a thermally isolated system never decreases*

$$\Delta S = S(fin) - S(in) \geqslant 0$$

Boltzmann insight

- Atomic hypotesis - microscopic reality behind macroscopic observations
- Entropy of a macrostate A is determined by the number of corresponding microstates $\Gamma(A)$:

$$S(A) = k_B \ln \Gamma(A)$$

- Second law: $S(A)$ is maximized in equilibrium.
- Planck's black body formula from application of the Boltzmann formula.

Shannon entropy

(C. E. Shannon, *A Mathematical Theory of Communication*, 1948)

- Probability p_i is our prejudice about random states $i \in \{1, 2, \ldots, N\}$

$$p_i \geqslant 0\,, \qquad\qquad \sum_{i=1}^{N} p_i = 1$$

- Shannon entropy of a probability distribution p_i:

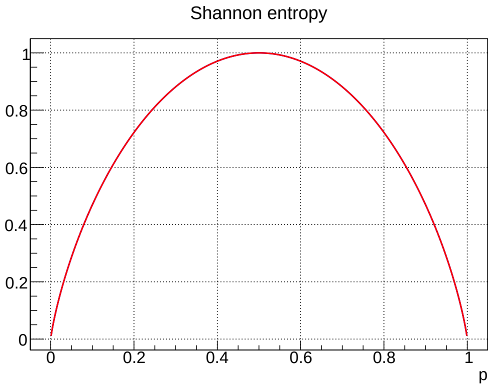
$$S(p) = -\sum_{i=1}^{N} p_i \log p_i \;\geqslant\; 0$$

- $S(p)$ is a measure of uncertainty in the probability distribution p_i:

$$\begin{array}{lcl} p_i^0 & = & 1/N \qquad\qquad S = \log N \qquad \text{max. uncert.} \\ p_i^m & = & \delta_{ij} \qquad\qquad\;\; S = 0 \qquad\quad\;\; \text{min. uncert.} \end{array}$$

- Maximal uncertainty = minimum information (and vice versa)
- $S = \log N$ is the Boltzmann formula for equally probable states.

Example for $N = 2$ states



$$S(p) = -p \log_2(p) - (1-p) \log_2(1-p)$$

- $p = 1/2$ maximal uncertainty, minimum information
- $p = 0, 1$ minimal uncertainty, maximum information
- Wigner: Information $I = -S < 0$

Missing information

- Relative entropy - “distance” of the distribution p_i from q_i

$$S(p\|q) = \sum_i p_i \log\left(\frac{p_i}{q_i}\right) \geqslant 0$$

- Missing information in the probability distribution p:

$$I(p) = S(p^m\|p^0) - S(p\|p^0) \qquad = \begin{cases} 0 & \text{for } p_i = p_i^m = \delta_{ij} \\ \log N & \text{for } p_i = p_i^0 = 1/N \end{cases}$$

- Computing:

$$I(p) = \log N - \textstyle\sum_i p_i \log(p_i N) = -\textstyle\sum_i p_i \log p_i = S(p)$$

- Shannon entropy equals missing information.

- “Entropy is a human concept” - state of lack of knowledge of the observer

Statistical mechanics as inference problem

(E. T. Jaynes, *"Information Theory and Statistical Mechanics I"*, PR 106 (1957) 620)

- Physical system in a microstate i with energy E_i and probability p_i
- What can we say about p_i if we know only mean value of energy

$$\overline{E} = \sum_i p_i E_i$$

- Maximum entropy principle - maximize constrained Shannon entropy:

$$\delta \left\{ \underbrace{-\sum_i p_i \ln p_i}_{S(p)} - \alpha \left(\sum_i p_i - 1 \right) - \beta \left(\sum_i p_i E_i - \overline{E} \right) \right\} = 0$$

- Variation $\delta p_i, \delta\alpha, \delta\beta$ are independent:

$$\ln p_i = -1 - \alpha - \beta E_i \qquad \sum_i p_i = 1 \qquad \sum_i p_i E_i = \overline{E}$$

- Boltzmann distribution:

$$p_i(\beta) = \frac{1}{Z(\beta)} \mathrm{e}^{-\beta E_i} \qquad Z(\beta) = \sum_i \mathrm{e}^{-\beta E_i}$$

Statistical mechanics relations

- Mean energy for Boltzmann distribution:

$$\overline{E} = \sum_i p_i E_i = -\frac{\partial \ln Z}{\partial \beta} \qquad \rightarrow \qquad \beta = \beta(\overline{E}) \equiv 1/T.$$

- Shannon entropy for Boltzmann distribution is thermodynamic entropy (Gibbs):

$$S = -\sum_i p_i \ln p_i = \ln Z + \beta \overline{E}$$

- First law of thermodynamics from $d\overline{E}$ and dS:

$$d\overline{E} = \sum_i (dp_i\, E_i + p_i\, dE_i) \qquad\qquad dS = -\sum_i dp_i \ln p_i = \beta \sum_i dp_i\, E_i$$

and

$$d\overline{E} = TdS + \underbrace{\left(\sum_i p_i \frac{\partial E_i}{\partial V}\right)}_{-P} dV = TdS - PdV$$

Quantum entropy

von Neumann entropy

- Density operators - describe mixed states in QM:

$$\rho = \frac{1}{2}\,|z+\rangle\langle z+| + \frac{1}{2}\,|x+\rangle\langle x+| = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}_x \rightarrow \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}_d$$

- von Neumann entropy of the state ρ:

$$S_{\mathrm{vN}}(\rho) = -\mathrm{Tr}\,(\rho \log \rho) \quad = -\sum_i p_i \log p_i = S_{\mathrm{Sh}}(p)$$

- For pure states, $\rho = \rho^2$ is a projection operator:

$$\rho = |x+\rangle\langle x+| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_x \quad \rightarrow \quad S_{\mathrm{vN}} = 0$$

- von Neumann entropy of pure states always equals zero

- Unitary evolution conserves von Neumann entropy

$$\rho_{fin} = U^\dagger \rho_{in}\, U \quad \rightarrow \quad S_{vN}(\rho_{fin}) = S_{vN}(\rho_{in})$$

Entanglement entropy

- Two spin $\frac{1}{2}$ system: $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ in pure singlet state:

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left\{ |+\rangle_A |-\rangle_B - |-\rangle_A |+\rangle_B \right\} \qquad \rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$$

- Reduced state ρ_A and its von Neumann entropy S_A:

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}(\rho_{AB}) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad S_A = -\mathrm{Tr}(\rho_A \log \rho_A) = \log 2$$

- S_A is entanglement entropy of the state $|\psi_{AB}\rangle$. The same for system B:

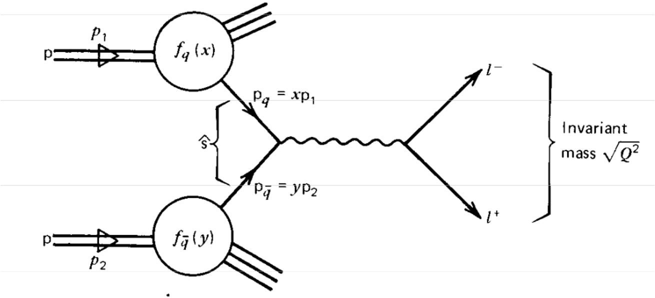
$$S_A = S_B$$

- If $S_A \neq 0$, the state $|\psi_{AB}\rangle$ is entangled - cannot be written as $|\psi_{AB}\rangle = |\psi_A\rangle|\psi_B\rangle$
- Maximal entanglement: - $S_A = \log N$ for $p_i = \frac{1}{N}$ is maximal.
- Entanglement entropy is produced after averaging over unobserved system.

Partonic entropy

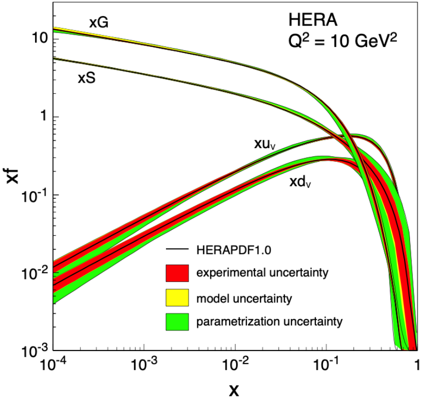
Parton distributions functions (PDFs)

- PDFs are used both in DIS and hadronic scattering, e.g. Drell-Yan process



$$\sigma_{DY} = \sum_f \int_0^1 dx \int_0^1 dy \left[q_f(x, Q^2)\,\bar{q}_f(y, Q^2) + (x \leftrightarrow y)\right] \hat{\sigma}(q\bar{q} \to l^+ l^-)$$

PDFs



- Snapshot at the resolution scale Q^2 for parton's momentum fraction

$$x = \frac{p_{\parallel}}{P_{\parallel}}$$

- Gluons and sea-quarks ($q\bar{q}$ pairs) dominate at small x.

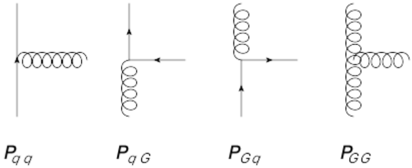
Altarelli-Parisi (DGLAP) evolution equations

(Dokshitzer, Gribov, Lipatov, Altarelli, Parisi, 1972-77)

- DGLAP evolution equations with "evolution time" $t = \log(Q^2/Q_0^2)$:

$$\frac{\partial q_f(x,t)}{\partial t} = P_{qq} \otimes q_f + P_{qG} \otimes G$$
$$\frac{\partial \bar{q}_f(x,t)}{\partial t} = P_{qq} \otimes \bar{q}_f + P_{qG} \otimes G$$
$$\frac{\partial G(x,t)}{\partial t} = P_{Gq} \otimes \sum_f (q_f + \bar{q}_f) + P_{GG} \otimes G$$

- Splitting functions $P_{ij}(\alpha_s)$:



Partonic sum rules

- DGLAP equations obey valence quark number sum rule

$$\int_0^1 dx\,[q_f(x,t) - \bar{q}_f(x,t)] = N_f$$

  and momentum sum rule

$$\int_0^1 dx\left[\sum_f (xq_f(x,t) + x\bar{q}_f(x,t)) + xG(x,t)\right] = 1$$

- Individual PDFs (not multiplied by x) are not normalized.
- How to define entropy of partons, following Shannon formula?

Partonic entropy

- Sum of all PDFs multiplied by x as the probability distribution:

$$P(x,t) = \sum_f \Big[x q_f(x,t) + x\bar{q}_f(x,t) \Big] + xG(x,t) \geqslant 0$$

- Momentum sum rule (MOM) as the normalization condition:

$$\int_0^1 dx\, P(x,t) = 1$$

- Partonic entropy in analogy to Shannon up to minus sign:

$$S(t) = \int_0^1 dx\, P(x,t) \ln P(x,t) \geqslant 0$$

- Positivity from relation: $x \ln x \geqslant (x-1)$ for $x \geqslant 0$:

$$\int_0^1 dx\, P(x,t) \ln P(x,t) \geqslant \int_0^1 dx\, [P(x,t) - 1] = 1 - 1 = 0$$

Unlimited growth of partonic entropy

- Compute

$$\frac{dS}{dt} = \int_0^1 dx\, \frac{\partial P(x,t)}{\partial t} \ln P(x,t)$$

- Use DGLAP equations for the partial derivative computation:

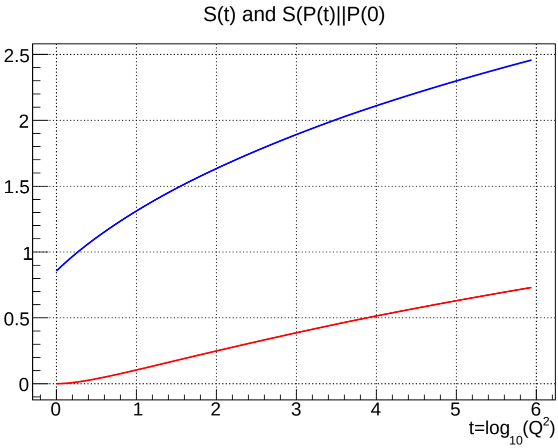
$$\frac{dS}{dt} = \sum_i \int_0^1 dx\, x\, D_i(x,t) \int_0^1 dy \left(\sum_j y\, P_{ji}(y,t) \right) \ln\left(\frac{P(yz,t)}{P(z,t)} \right)$$

  Since $P(yz,t) > P(z,t)$ we find:

$$\frac{dS}{dt} > 0$$

- Partonic entropy S is monotonically rising function of $t = \ln(Q^2/Q_0^2)$

Illustration



Relative entropy

- Relative entropy of probability distributions $p(x)$ and $q(x)$ on $[0,1]$:

$$S(p\|q) = \int_0^1 dx\, p(x) \ln\left(\frac{p(x)}{q(x)}\right) \geqslant 0$$

"Distance" of p from q.

- Partonic entropy is relative entropy. "Distance" of $P(x)$ from $\mathbb{1}(x) = 1$

$$S(t) = \int_0^1 dx\, P(x,t) \ln P(x,t) = S(P(t)\|\mathbb{1})$$

- "Distance" of $P(x,t)$ from initial $P(x,0)$ - dynamical entropy

$$S(P(t)\|P(0)) = \int_0^1 dx\, P(x,t) \ln\left(\frac{P(x,t)}{P(x,0)}\right)$$

$S(t)$ growth interpretation

- Sea-quark and gluon distributions rise rapidly as $x \to 0$ and $t \to \infty$:

$$P(x,t) = A(t)\, x^{\alpha(t)-1}(1-x)^{\beta(t)-1}\,, \qquad A(t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}$$

  where $\alpha(t) > 0$ and $\beta(t) > 1$

- Partonic entropy grows indefinitely for $\alpha(t) \to 0^+$:

$$S(t) = \frac{1}{\alpha} + \ln\alpha + \text{regular terms}$$

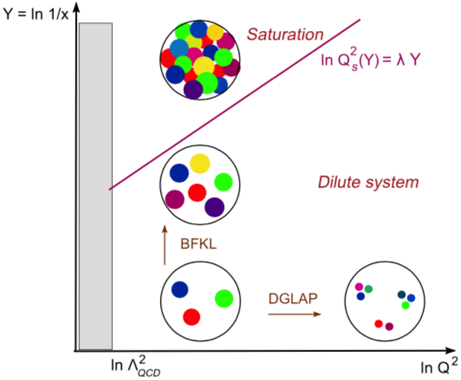
- Contribution to $S(t)$ from $x < x_{\rm min} \ll 1$ gives

$$S_<(t) = \int_0^{x_{\rm min}} dx\, P(x,t) \ln P(x,t) \,\simeq\, \frac{1}{\alpha} + \ln\alpha$$

- Unlimited growth of $S(t)$ is driven by small-x partons.

Parton saturation

- Saturation of $P(x)$ for $x \leqslant x_{\rm min}$ tames partonic entropy:

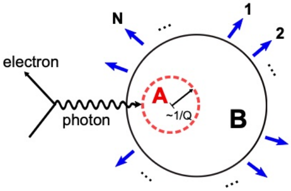


$$P(x,t) = (A/\alpha)\, x_{\rm min}^{\alpha-1} \quad \rightarrow \quad S_<(t) = \ln(1/x_{\rm min})$$

- $x_{\rm min} = x_{\rm min}(t)$ is the critical line → saturation scale $Q_s(x)$

Entanglement entropy at small x

D. E. Kharzeev and E. M. Levin, Phys. Rev. D 95, 114008 (2017)
M. Hentschinski, K. Kutak, and R. Straka, Eur. Phys. J. C 82, 1147 (2022)
K. Kutak and S. Lökös, Phys. Rev. D 112, 096017 (2025)



- In DIS we have only access to reduced density operator $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$
- From the equality of entanglement entropies

$$S_A = -\mathrm{Tr}\,(\rho_A \log \rho_A) = -\mathrm{Tr}\,(\rho_B \log \rho_B) = S_B$$

the equality: "partonic etropy" ≃ "hadronic entropy"

$$\log(xG(x)) \simeq -\sum_N P_N(y) \log P_N(y)$$

Summary

- Entropy lies at the very bottom of our understanding of the world.
- Shannon entropy - describes our perception of uncertainty and information.
- von Neumann entropy - offers a new perspective through quantum information theory and entantglement.
- Application of Shannon entropy to QCD partons confirms importance of parton saturation at small x