Quantization of charge in loop electrodynamics and beyond

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NTWR Prime - redundant security based on NTRU Prime and LWR problems

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Classical electrodynamics

The action of electrodynamics sourced by the 4-current $j^\mu = (\rho, \vec{j})$ is:

$$S = -\frac{1}{4}\int_{\mathcal{M}} d^4x F^{\mu\nu}F_{\mu\nu} - \int_{\mathcal{M}} d^4x j^\mu A_\mu$$

where $\mathcal{M} = \Sigma \times \mathbb{R}$ (Σ denotes the spatial manifold),

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

is the field strength tensor, and A_μ is the four-potential.
Under the $U(1)$ gauge transformation:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \lambda,$$

with $\lambda \in \mathbb{R}$, the sourced action transforms as:

$$S[A'] = S[A] + \int_{\mathcal{M}} d^4x \lambda(\partial_\mu j^\mu) + \text{const.}$$

The action remains invariant under the gauge transformation when the continuity equation for the source term is satisfied:

$$\partial_\mu j^\mu = 0.$$

In order to perform the Legendre transform we find the canonical momenta:

$$\pi^0 := \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0,$$
$$\pi^a := \frac{\partial \mathcal{L}}{\partial \dot{A}_a} = -E^a.$$

Here, the electric field vector has been introduced as follows:

$$E^a = \partial^a A_0 - \dot{A}^a,$$

where $\dot{A}^a \equiv \partial_0 A^a$. As easily shown, the E^a field is invariant under gauge transformations:

$$E^a \to E^{a\prime} = \partial^a A_0 - \dot{A}^a.$$

The π_0 is a so-called primary constraint, from which follows that A_0 is a non-dynamical variable (Lagrange multiplier). Furthermore, the secondary constraint can be calculated:

$$0 = \dot{\pi}_0 = \{\pi_0, H\} = \partial_a E^a - \rho =: G(\vec{E}),$$

which is the Gauss constraint (Gauss law):

$$G(\vec{E}) = \partial_a E^a - \rho = 0.$$

the Poisson bracket is defined as follows:

$$\{M, N\} = \int_\Sigma d^3z \left[\frac{\delta M}{\delta \pi^0(z)} \frac{\delta N}{\delta A_0(z)} - \frac{\delta N}{\delta \pi^0(z)} \frac{\delta M}{\delta A_0(z)} \right] + \int_\Sigma d^3z \left[\frac{\delta M}{\delta E^a(z)} \frac{\delta N}{\delta A_a(z)} - \frac{\delta N}{\delta E^a(z)} \frac{\delta M}{\delta A_a(z)} \right].$$

Gauss constraint as a generator of gauge symmetry

One can consider the smeared Gauss constraint:

$$G[\lambda] := \int_\Sigma d^3x \lambda(x) G(\vec{E}),$$

with some test function $\lambda(x)$. This leads to the first class algebra:

$$\{G[\lambda_1], G[\lambda_2]\} = 0.$$

Therefore, the Gauss constraint is a generator of the underlying symmetry. Indeed, the gauge transformations of the canonical variables are now:

$$\begin{aligned}
\delta A_0 &= \{A_0, G[\lambda]\} = 0, \\
\delta A_a &= \{A_a, G[\lambda]\} = -\partial_a \lambda, \\
\delta \pi_0 &= \{\pi_0, G[\lambda]\} = 0, \\
\delta E^a &= \{E^a, G[\lambda]\} = 0.
\end{aligned}$$

Quantization

The usual quantization procedure, employed in QED, is performed by promoting the local field variables E^a and A_b to quantum operators. So that the canonical Poisson bracket:

$$\{E^a(x), A_b(y)\} = \delta^a_b \delta^{(3)}(x-y).$$

turns into the commutation relation:

$$\left[\hat{E}^a(x), \hat{A}_b(y)\right] = i\hat{\mathbb{I}}\delta^a_b \delta^{(3)}(x-y).$$

The excitations of the quantum field are described by the Fock space with photons being the quanta of the field.

In turn, the loop quantization employs non-local (and non-perturbative) field variables: holonomies (Wilson lines) and fluxes. The elementary excitations of the field (quanta of the field) are the field lines.

The loop quantization, for the case of $SU(2)$ gauge symmetry is at the foundation of loop quantum gravity (Rovelli, Ashtekar, Smolin,...).

The loop quantization of $U(1)$ gauge fields is mostly considered in the literature as a toy model to $SU(2)$ loop quantization.

- A. Ashtekar and C. Rovelli, "A Loop representation for the quantum Maxwell field," Class. Quant. Grav. **9** (1992), 1121-1150.
- A. Ashtekar and J. Lewandowski, "Relation between polymer and Fock excitations," Class. Quant. Grav. **18** (2001), L117-L128.
- M. Varadarajan, "Fock representations from U(1) holonomy algebras," Phys. Rev. D **61** (2000), 104001.
- A. Corichi and K. V. Krasnov, "Ambiguities in loop quantization: Area versus electric charge," Mod. Phys. Lett. A **13** (1998), 1339-1346.

Holonomy

We consider a path $\gamma : [0, 1] \to \Sigma$, parametrised by the affine parameter $\xi \in [0, 1]$. Here $\Sigma = \mathbb{R}^3$. We will denote $\gamma(0) \equiv s$ and $\gamma(1) \equiv t$, corresponding to the *source* and *target* of parallel transport via the holonomy, respectively.

Holonomy of the connection A_a satisfies the following differential equation:

$$\frac{d}{d\xi} h_\gamma[A, \xi] = ie\dot{x}^a A_a h_\gamma[A, \xi].$$

such that $h_\gamma[A, 0] = 1$. Here, constant e has been introduced for dimensional reasons.

For the considered $U(1)$ case the solution is:

$$h_\gamma[A] = \exp\left(ie \int_\gamma A \right),$$

The flux of the electric field E_a can be defined as:

$$F_S[E] = \int_S \epsilon_{abc} E^a dx^b \wedge dy^c,$$

where $S \in \Sigma$ is a 2-surface. The flux is gauge-invariant by the invariance of the electric field.

Based on the definition of the flux and the Stokes theorem, the Gauss law for the case with charge leads to:

$$F_S[E] = q$$

for a closed surface S, enclosing the total charge q.

The algebra of newly defined variables can be computed via the Poisson bracket definition:

$$
\begin{aligned}
\{F_S[E], h_\gamma[A]\} &= \int_\Sigma d^3z \left[\frac{\delta F_S[E]}{\delta E^a(z)}\frac{\delta h_\gamma[A]}{\delta A_a(z)} - \frac{\delta h_\gamma[A]}{\delta E^a(z)}\frac{\delta F_S[E]}{\delta A_a(z)}\right] \\
&= \int_\Sigma d^3z \frac{\delta F_S[E]}{\delta E^a(z)}\frac{\delta h_\gamma[A]}{\delta A_a(z)} \\
&= \int_S \varepsilon_{abc} dx^b \wedge dx^c \{E^a, h_\gamma[A]\} \\
&= \int_S \varepsilon_{abc} dx^b \wedge dx^c \{E^a, \mathcal{P}\exp(ie\int_e A_d dy^d)\} \\
&= ieh_\gamma[A] \underbrace{\int_S d\sigma_a \int_\gamma \delta^{(3)}(x-y) dy^a}_{\iota(\gamma,S)},
\end{aligned}
$$

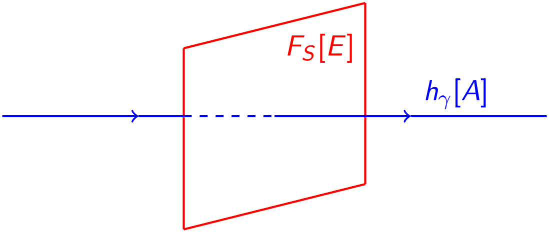
where $\iota(\gamma, S) = 0, \pm 1$.

Holonomy-flux algebra

So that the holonomy-flux algebra is:

$$\{F_S[E], h_\gamma[A]\} = ieh_\gamma[A]\iota(\gamma, S).$$

where $\iota(\gamma, S) = 0, \pm 1$.



Quantization of the holonomy-flux algebra

To consider a quantum theory, we can canonically quantize the holonomy flux algebra by promoting holonomies and fluxes to operators, and the Poisson bracket to a commutator:

$$[\hat{F}_S[E], \hat{h}_\gamma[A]] = -e\hat{h}_\gamma[A]\iota(\gamma, S).$$

Here, we choose holonomies as the configuration variables and consider functions diagonalizing the holonomy operator, so that:

$$\hat{h}_\gamma[A]\Psi = h_\gamma[A]\Psi.$$

Such a choice is motivated by the fact that holonomies, due to their transformation properties, are suitable for constructing gauge-invariant states. This property generalizes beyond the $U(1)$ case.

Kinematical states

Space of holonomies is infinitely dimensional, just like the space of functionals over them, so to regularize this, we only take cylindrical functions, that by definition depend on a finite amount N of holonomies:

$$\Psi_\Gamma \equiv \Psi(h_{\gamma_1}, h_{\gamma_2}, ..., h_{\gamma_N}),$$

where Γ is consists of all the edges γ_i.

On space of such functions, which is now finite-dimensional, we can define the Ashtekar-Lewadowski measure $d\mu_{AL}$, which is defined as:

$$d\mu_{AL} := \prod_{\gamma \in \Gamma} dg_\gamma,$$

where dg_γ is the normalized Haar measure.

This lets us define the L^2 kinematical Hilbert space of functions over a graph Γ with N edges:

$$\Psi_\Gamma \in \mathcal{H}_{\text{kin}} = L^2(U(1)^N, d\mu_{AL}).$$

The full kinematical Hilbert space is a direct sum of all possible graphs.

According to Peter-Weyl theorem, for L^2 spaces with Haar measure, such as $U(1)$, we can write any function in terms of matrix elements of its irreducible representations.

Recall that the elements in $U(1)$ are $e^{i\varphi}$. The irreducible representations V_n of the group $U(1)$ are complex one-dimensional spaces labeled by $n \in \mathbb{Z}$. The space V_n can be written as:

$$V_n = \{\alpha e^{in\varphi} : \alpha \in \mathbb{C}\}$$

where the basis vector is $\psi_n(\varphi) = e^{in\varphi}$.

Following the Peter-Weyl theorem, $\Psi \in L^2(U(1), \frac{\varphi}{2\pi})$ can be decomposed as:

$$\Psi(\varphi) = \sum_n f_n \psi_n(\varphi)$$

in the sense of a series with L^2 norm.

Here, the basis functions satisfy the complementarity relation:

$$\int_{U(1)} \frac{d\varphi}{2\pi} \psi_n(\varphi) \psi_m^*(\varphi) = \delta_{nm}.$$

Consequently, the n-th irreducible representation of a holonomy is:

$$(h_\gamma[A])^n = \exp\left(ine \int_\gamma A \right).$$

Since $U(1)$ is abelian, it follows from Schur's lemma that all irreducible representations of functions Ψ_Γ will be of the form:

$$\Psi_\Gamma[A] = (h_{e_1}[A])^{n_1}(h_{e_2}[A])^{n_2}...$$

where $n_1, n_2, ... \in \mathbb{Z}$ correspond to irreducible representations of $U(1)$.

A general element of $L^2(U(1)^N, d\mu_{AL})$ can, therefore, be written as:

$$\Psi_\Gamma[A] = \sum_{n_1,n_2,...,n_N} f_{n_1,n_2,...,n_N}(h_{e_1}[A])^{n_1}(h_{e_2}[A])^{n_2}...(h_{e_N}[A])^{n_N},$$

with some coefficients $f_{n_1,n_2,...,n_N}$.

The inner product can be defined as:

$$\langle \Phi_\Gamma | \Psi_{\Gamma'} \rangle := \int \Pi_{\gamma\in\Gamma\cap\Gamma'} dg_\gamma \overline{\Phi_\Gamma[A]}\Psi_{\Gamma'}[A].$$

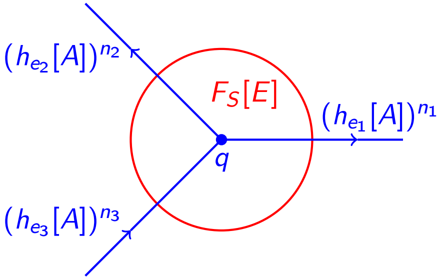
These states are still kinematical and do not satisfy the Gauss constraint.

Imposing quantum Gauss constraint

The Gauss constraint is implemented at the nodes of the graph (using the Dirac method) to guarantee gauge-invariance. At a given node s it can be given by:

$$\hat{G}^{(s)} |\Psi\rangle = \left(\sum_i \hat{F}_i[E] - q\hat{\mathbb{I}} \right) |\Psi\rangle = 0,$$

where the sum goes over all surfaces encapsulating node s, and q is the charge at the node. It states that the fluxes entering the node and escaping it do not change the charge at the node.



Action of the flux operator in the configuration (holonomy) representation is:

$$\hat{F}_S[E] = i \int_S dn_a(x) \frac{\delta}{\delta A_a(x)}.$$

Based on this, we can determine the action of the flux operator on the holonomies:

$$\hat{F}_S[E]\hat{h}_\gamma^{(n)}[A] = -en\hat{h}_\gamma[A]\iota(\gamma, S).$$

Let us consider the action of the flux operator on two links $N = 2$:

$$\begin{aligned}
\hat{F}_S(h_{e_1}^{n_1} h_{e_2}^{n_2}) &= \left(\hat{F}_S h_{e_1}^{n_1}\right) h_{e_2}^{n_2} + h_{e_1}^{n_1} \left(\hat{F}_S h_{e_2}^{n_2}\right) \\
&= -en_1\iota(e_1, S)h_{e_1}^{n_1} h_{e_2}^{n_2} + h_{e_1}^{n_1}(-en_2\iota(e_2, S))h_{e_2}^{n_2} \\
&= -e\left[n_1\iota(e_1, S) + n_2\iota(e_2, S)\right] h_{e_1}^{n_1} h_{e_2}^{n_2}.
\end{aligned}$$

Generalizing this result to an arbitrary number of edges N yields:

$$\hat{F}_S \Psi = -e \left(\sum_i^N n_i \iota(\gamma_i, S) \right) \Psi.$$

So that, satisfying the Gauss constraint at the node leads to:

$$\hat{G}^{(s)} \Psi = \left[-e \left(\sum_i^N n_i \iota(\gamma_i, S) \right) - q \right] \Psi = 0,$$

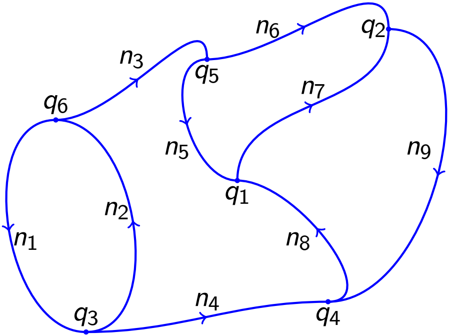
where q is the electric charge at the node. This tells us that any charge q has to be an integer multiplicity of some elementary charge e:

$$-q/e = \sum_i n_i \iota(\gamma_i, S), \; n_i \in \mathbb{Z}, \; \iota(\gamma_i, S) = \pm 1$$

$$\frac{q}{e} \in \mathbb{Z}.$$

$U(1)$ gauge-invariant networks

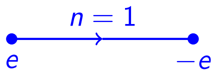
Satisfying the Gauss constraint at the nodes of the graph leads to gauge-invariant $U(1)$ networks spanning the physical Hilbert space $\mathcal{H}_{\text{kphys}} = L^2(U(1)^N/U(1)^V, d\mu_{AL}) \simeq L^2(U(1)^{b_1(\Gamma)}, d\mu_{AL})$. Here, $b_1(\Gamma) = N - V + 1$ is the number of independent loops in the graph (Betti number).



Here, $b_1(\Gamma) = 9 - 6 + 1 = 4$ loops.

Further consequences

- In a hydrogen atom, the charges are linked by a single quantum of flux:



- The same for particle-$\overline{\text{particle}}$ pairs.
- Each link gives $n_i + (-n_i) = 0$ contribution to two nodes. Consequently:

$$\sum_{v \in V} \sum_{e_{v,i}} n_{v,i} = -\frac{1}{e} \sum_{v \in V} q_v = 0.$$

Therefore, the network is electrically neutral:

$$\sum_{v \in V} q_v = 0.$$

- States with individual charges are not allowed.

Compactified phase spaces

- The holonomies introduce $U(1)$ compactification of the connection (A) part of the phase space. However, the flux part of the phase space remains unconstrained. The phase space is locally:

$$(F[E], h[A]) \in \mathbb{R} \times U(1).$$

- Can we make the theory locally $U(1) \times U(1)$, so the quantum theory is described by the finite-dimensional Hilbert space?
- We investigate compact generalizations of QFTs within the program of Nonlinear Field Space Theory (NFST):

1. J. Mielczarek and T. Trześniewski, "The Nonlinear Field Space Theory," Phys. Lett. B **759** (2016), 424-429 Phys. Rev. D **96** (2017) no.4, 043522
2. D. Artigas, J. Mielczarek and C. Rovelli, "A minisuperspace model of compact phase space gravity," Phys. Rev. D **100** (2019) no.4, 043533
3. J. Mielczarek and T. Trześniewski, "Nonlinear Field Space Cosmology," Phys. Rev. D **96** (2017) no.4, 043522

$U(1) \times U(1)$ phase space electrodynamics

Here, we generalize the flux to:

$$F_S[E] \to \mathcal{F}_S[E] = \exp\left(i\lambda F_S[E]\right),$$

with some parameter $\lambda \in \mathbb{R}$ so holonomy-flux algebra:

$$\{F_S[E], h_e[A]\} = ieh_e[A]\iota(e, S)$$

turns into compactified holonomy-flux (torus) algebra:

$$\{\mathcal{F}_S[E], h_e[A]\} = -\lambda e\iota(e, S)h_e[A]\mathcal{F}_S[E].$$

In quantum theory, the algebra writes as:

$$[\hat{\mathcal{F}}_S, \hat{h}_e] = (e^{-i\lambda e\iota(e,S)} - 1)\hat{h}_e\hat{\mathcal{F}}_s.$$

Holonomies form a complete basis for the kinematical Hilbert space $\mathcal{H}_{kin}$. Then, given a state

$$|n\rangle := (h_e)^n,$$

we have the following representation of the $\hat{\mathcal{F}}_S$ and $\hat{h}_e$ operators:

$$\hat{\mathcal{F}}_s |n\rangle = e^{i\lambda \hat{F}_s} |n\rangle = e^{-i\lambda e \iota(e,S) n} |n\rangle$$
$$\hat{h}_e |n\rangle = \hat{h}_e (h_e)^n = (h_e)^{n+1} = |n+1\rangle .$$

Then, the action of fluxes to be consistent with the toroidal phase space, we require:

$$\lambda e = \frac{2\pi}{N},$$

where $N \in \mathbb{N}_+$, and it depends on the size of the torus.

Gauss constraint and quantization of charge

Since the Gauss constraint is a function of the flux, it requires suitable generalization to the exponentiated variables. For this purpose, we apply the often-used technique of building self-adjoint operator from exponentiations of a self-adjoint operator $\hat{\mathcal{A}}$, with the control parameter λ:

$$\hat{\mathcal{A}} \to \frac{1}{2i\lambda}\left(e^{i\lambda\hat{\mathcal{A}}} - e^{-i\lambda\hat{\mathcal{A}}}\right) = \hat{\mathcal{A}} + \mathcal{O}(\lambda),$$

so that the original case is recovered in the $\lambda \to 0$.

However, ambiguities remain so that eventually there are two possible expression for the total charge of n elementary charges q_e:

- $q = nq_e$ - standard expression
- $q = q_e \frac{\sin \frac{2\pi}{N} n}{\sin \frac{2\pi}{N}}$ - deformed expression

Summary and outlook

- Loop Electrodynamics (LED) provides an alternative quantization of electrodynamics, employing non-local and non-perturbative variables.
- In LED, quantization and conservation of electric charge is a simple consequence of $U(1)$ gauge invariance (via the Gauss constraint).
- No Dirac monopole is needed to explain the quantization of electric charge.
- As currently understood, the flux quantization is not captured in the Fock Hilbert space, since it covers only a tiny fraction of the LED Hilbert space (spanned by the $U(1)$ networks).
- The framework has been extended to the case of compact phase space ($U(1) \times U(1)$), providing a finite-dimensional Hilbert space description of electrodynamics.
- Dynamics and further consequences of LED are still poorly understood.