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From Algebraic Counterflow Towards Vortex Line Density Evolution

Transient Heat Transport in Forced Flow He II

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Physics behind of He II – Landau theorem

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Mass conservation

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

Entropy balance

$$\partial_t(\rho s) + \nabla \cdot (\rho s \boldsymbol{u}_n) = \frac{\boldsymbol{F}_{ns} \cdot (\boldsymbol{u}_n - \boldsymbol{u}_s)}{T}$$

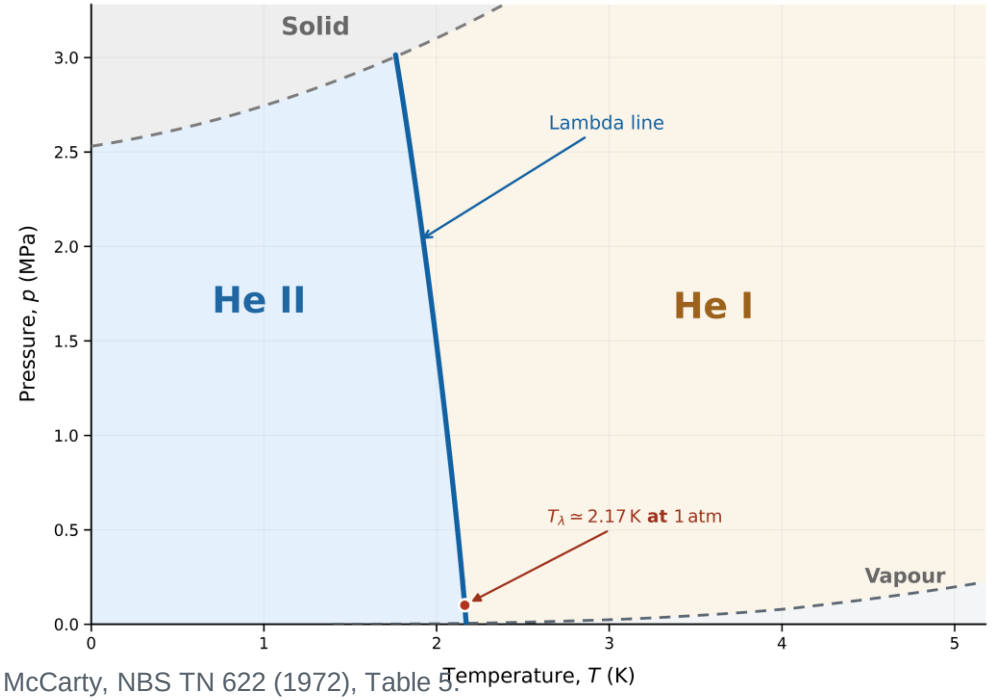
Superfluid momentum

$$\partial_t \boldsymbol{u}_s + (\boldsymbol{u}_s \cdot \nabla)\boldsymbol{u}_s + \frac{1}{\rho}\nabla p - s\nabla T = \frac{\boldsymbol{F}_{ns}}{\rho_s}$$

Total momentum

$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n + \rho_s \boldsymbol{u}_s \otimes \boldsymbol{u}_s) + \nabla p = \eta_n \nabla^2 \boldsymbol{u}_n$$

Helium phase diagram



McCarty, NBS TN 622 (1972), Table 5.

$\rho = \rho_n + \rho_s$
$\rho \boldsymbol{u} = \rho_n \boldsymbol{u}_n + \rho_s \boldsymbol{u}_s$

$\boldsymbol{F}_{ns}$: mutual-friction force density
$+\boldsymbol{F}_{ns}$ on uperfluid; $-\boldsymbol{F}_{ns}$ on normal

Φ_{mf}: mutual-friction heating
$\Phi_{\mathrm{mf}} = \boldsymbol{F}_{ns} \cdot (\boldsymbol{u}_n - \boldsymbol{u}_s) \geq 0$

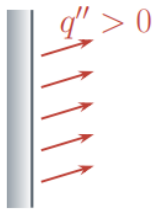
Landau two fluid description

- The **normal component** carries entropy and behaves as a viscous fluid.
- The **superfluid component** is inviscid in the ideal description and carries no entropy.
- Heat produces relative motion between both interpenetrating components.

$$\rho \boldsymbol{u} = \rho_n \boldsymbol{u}_n + \rho_s \boldsymbol{u}_s,$$

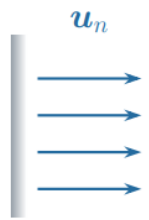
$$\boldsymbol{q}_{\mathrm{cf}} = \rho_s s T (\boldsymbol{u}_n - \boldsymbol{u}_s).$$

Heat input at the heated surface



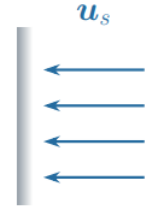
A wall heat flux creates a temperature gradient normal to the heated surface.

Normal component carries entropy away



The viscous normal component carries entropy and heat away from the heater.

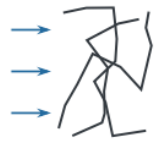
Superfluid component flows oppositely



The inviscid superfluid returns toward the heater to conserve the total mass flux.

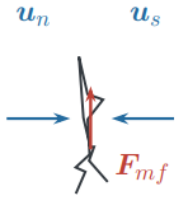
Relative counterflow and vortex tangle

$w = u_n - u_s$



The relative velocity produces a tangle of quantized vortices.

Mutual friction exchanges momentum



Mutual friction exchanges momentum and dissipates mechanical energy.

Forced flow modifies counterflow



Mean flow adds advection and changes thermal-pulse propagation.

Pietrowicz model

$$s\nabla T = \frac{1}{\rho}\nabla p - A_{\mathrm{GM}}\rho_n \left|\boldsymbol{u}_n - \boldsymbol{u}_s\right|^2 \left(\boldsymbol{u}_n - \boldsymbol{u}_s\right)$$

$$\boldsymbol{u}_n - \boldsymbol{u}_s = -\left(\frac{s}{A_{\mathrm{GM}}\rho_n}\right)^{1/3} \frac{\nabla T - \dfrac{1}{\rho s}\nabla p}{\left|\nabla T - \dfrac{1}{\rho s}\nabla p\right|^{2/3}}.$$

Retained in the reduced model

- thermomechanical driving by ∇T,
- contribution of the physical pressure gradient,
- nonlinear Gorter–Mellink resistance,
- counterflow heat transport in the energy equation.

Removed by the algebraic reduction

- transient normal and superfluid momentum,
- independent evolution of relative motion,
- an internal state carrying resistance history,
- first and second sound propagation.

Why the Pietrowicz reduction is useful?

$$\boldsymbol{q}_{\mathrm{cf}} = \rho_s s T (\boldsymbol{u}_n - \boldsymbol{u}_s),$$

$$\boldsymbol{q}_{\mathrm{cf}} = -k_{\mathrm{eff}} \left(\nabla T - \frac{1}{\rho s} \nabla p \right),$$

$$k_{\mathrm{eff}} \propto \left| \nabla T - \frac{1}{\rho s} \nabla p \right|^{-2/3}.$$

Main advantage

- Counterflow enters the energy equation through one nonlinear diffusion coefficient.
- Separate component momentum equations are not required.
- The formulation is inexpensive enough for engineering CFD calculations.

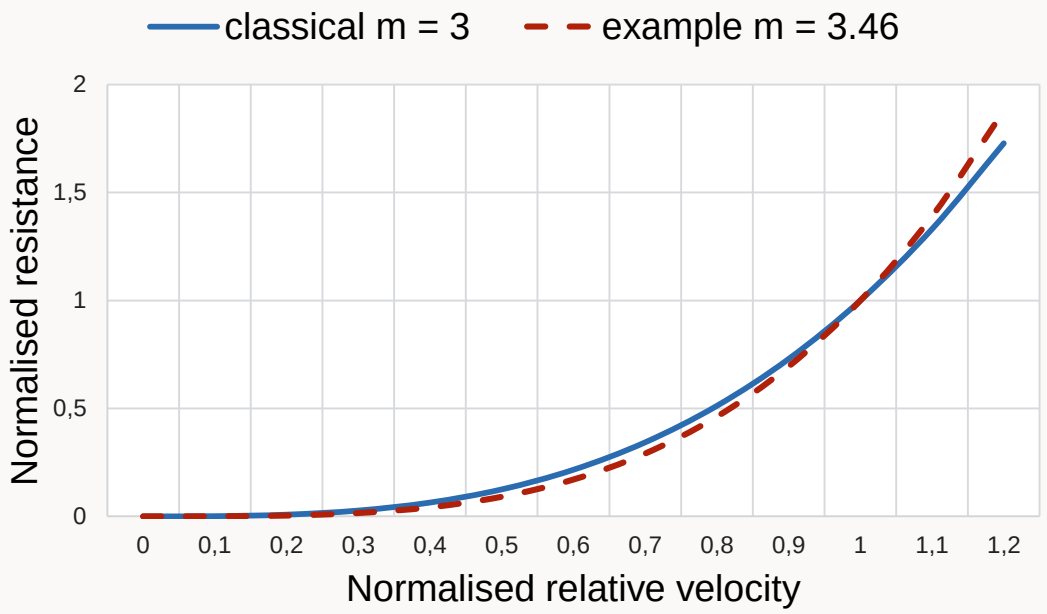
k_{eff} is a coefficient obtained by rewriting the algebraic counterflow law. It is not a thermophysical conductivity of He II.

The relative velocity follows the current temperature and pressure gradients without an independent evolution equation.

The main limitation of the model

$$\boldsymbol{w} \equiv \boldsymbol{u}_n - \boldsymbol{u}_s, \qquad \underbrace{\boldsymbol{F}_{ns} = A_m(T,p)\rho_n |\boldsymbol{w}|^{m-1}\boldsymbol{w}}_{\text{mutual-friction closure}},$$

$$\underbrace{s\nabla T = \frac{1}{\rho}\nabla p - \boldsymbol{F}_{ns}}_{\text{reduced superfluid momentum balance}} \quad \Longrightarrow \quad \boldsymbol{w} = -\left[\frac{s}{A_m(T,p)\rho_n}\right]^{1/m} \frac{\nabla T - \frac{1}{\rho s}\nabla p}{\left|\nabla T - \frac{1}{\rho s}\nabla p\right|^{(m-1)/m}}.$$

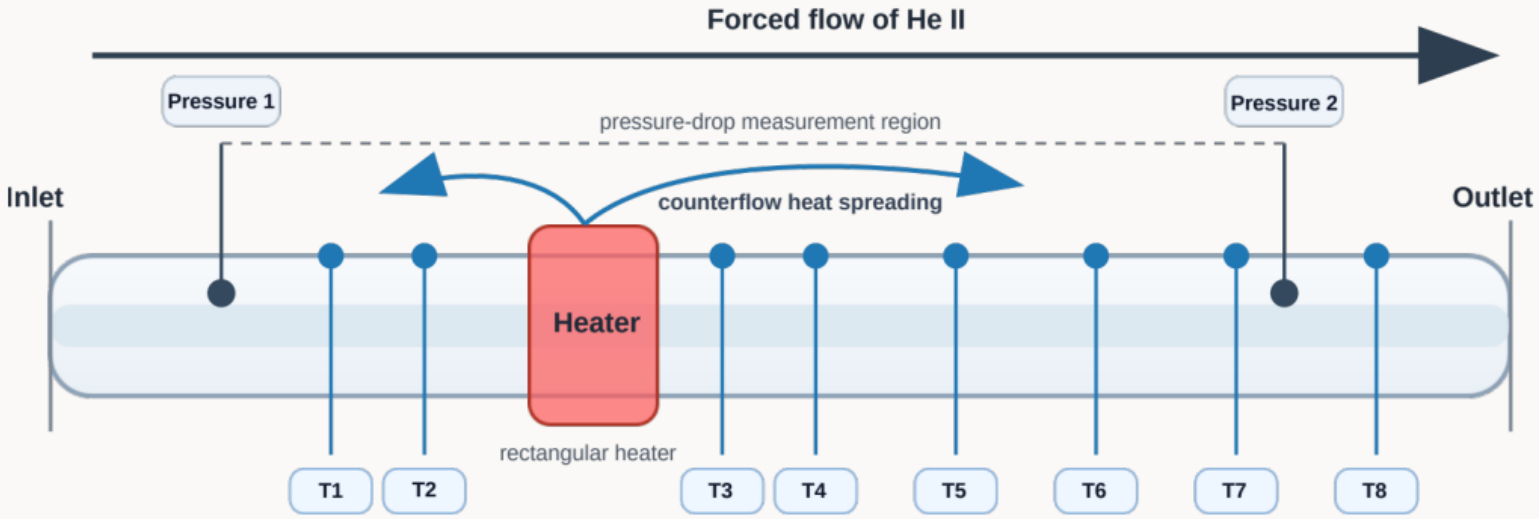


The exponent m requires calibration

- The classical formulation assumes $m = 3$, but this value is not universal.
- Kuehn and Ahlers obtained $m \approx 3.46$ near the lambda transition. Sato et al. reported a temperature-dependent exponent.
- A_m sets the resistance level. The exponent m changes the nonlinear response, especially when the thermal and pressure contributions partly cancel.

Both A_m and m require calibration for the operating range. Values fitted under different conditions cannot be transferred directly.

Fuzier and Van Sciver experiment



Straight stainless steel tube, $L = 0.864$ m

Heater at $x = 0.303$ to 0.313 m. Pressure taps at $x = 0.20$ and 0.80 m.

Tube diameter	10 mm
Bath temperature	1.7 K
Heater pulse	20 ms
Heat flux	9.9 W cm^{-2}
Flow velocity	$2, 8, 16$ m/s
Measurement	eight Cernox sensors

The heat pulse probes the transient response of counterflow already subjected to forced motion and pressure loss.

Experimental layout redrawn for the manuscript from Fuzier and Van Sciver, Cryogenics (2008).

What shapes the temperature signal?

$$\underbrace{\frac{\partial T}{\partial t}}_{\text{local response}} + \underbrace{U_0 \frac{\partial T}{\partial x}}_{\text{enthalpy advection}} = \underbrace{-\frac{1}{\rho c_p} \nabla \cdot \boldsymbol{q}_{\mathrm{cf}}}_{\text{counterflow}} + \underbrace{\frac{Q_h}{\rho c_p}}_{\text{wall pulse}} + \underbrace{\mu_{\mathrm{JT}} U_0 \frac{\partial p}{\partial x}}_{\text{pressure loss}}$$

Reading the measured trace

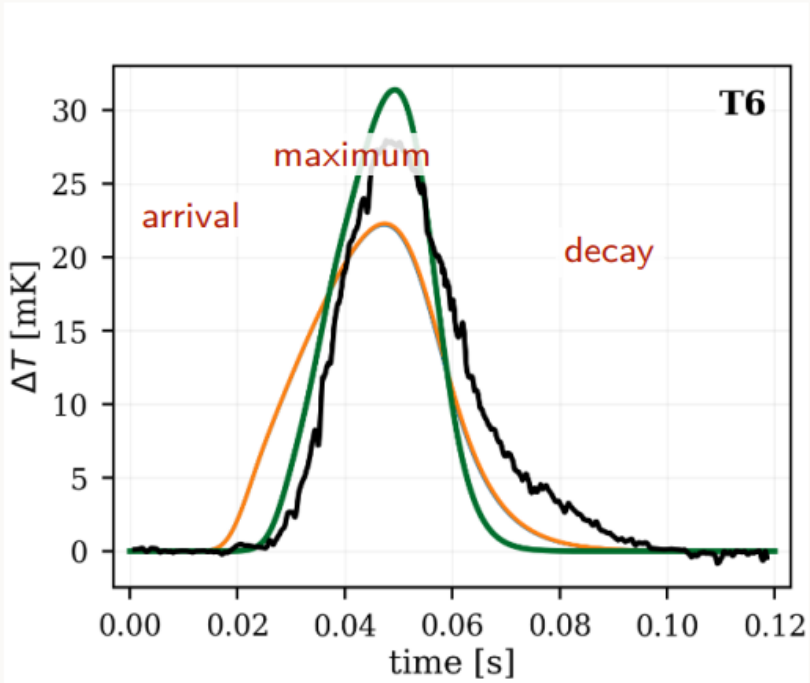
Arrival time — $U_0\,\partial T/\partial x$ sets the delay between the heater and the sensor.

Rise and maximum — Q_h adds energy while $-\nabla \cdot \boldsymbol{q}_{\mathrm{cf}}$ redistributes it.

Width and decay — After $Q_h = 0$, advection and counterflow determine the falling part.

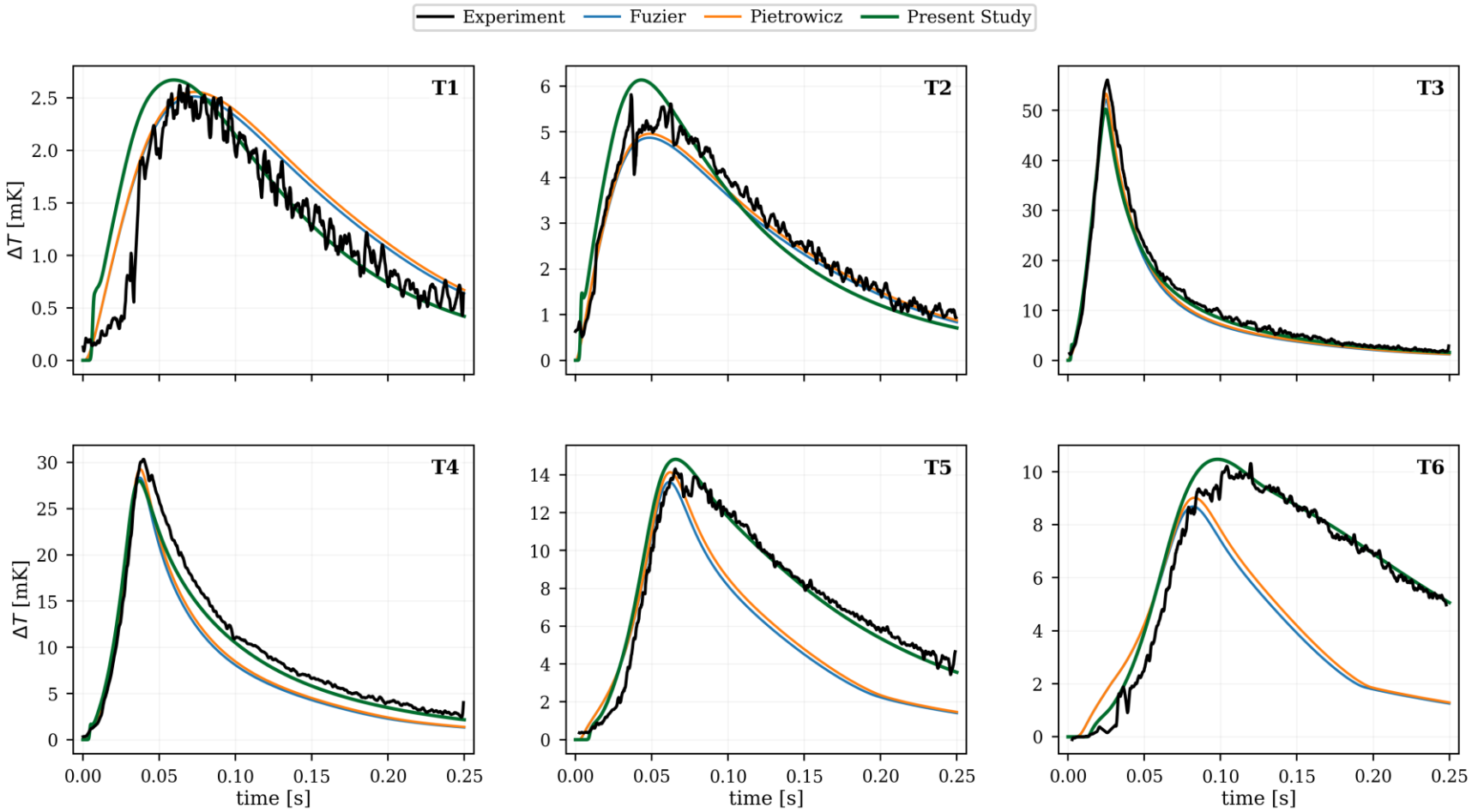
Baseline shift — $\mu_{\mathrm{JT}} U_0\,\partial p/\partial x$ accounts for the response to pressure loss.

Example: sensor T6 at $8\ \mathrm{m/s}$



Experiment in black, Pietrowicz model in oragne, Opalski model in gr

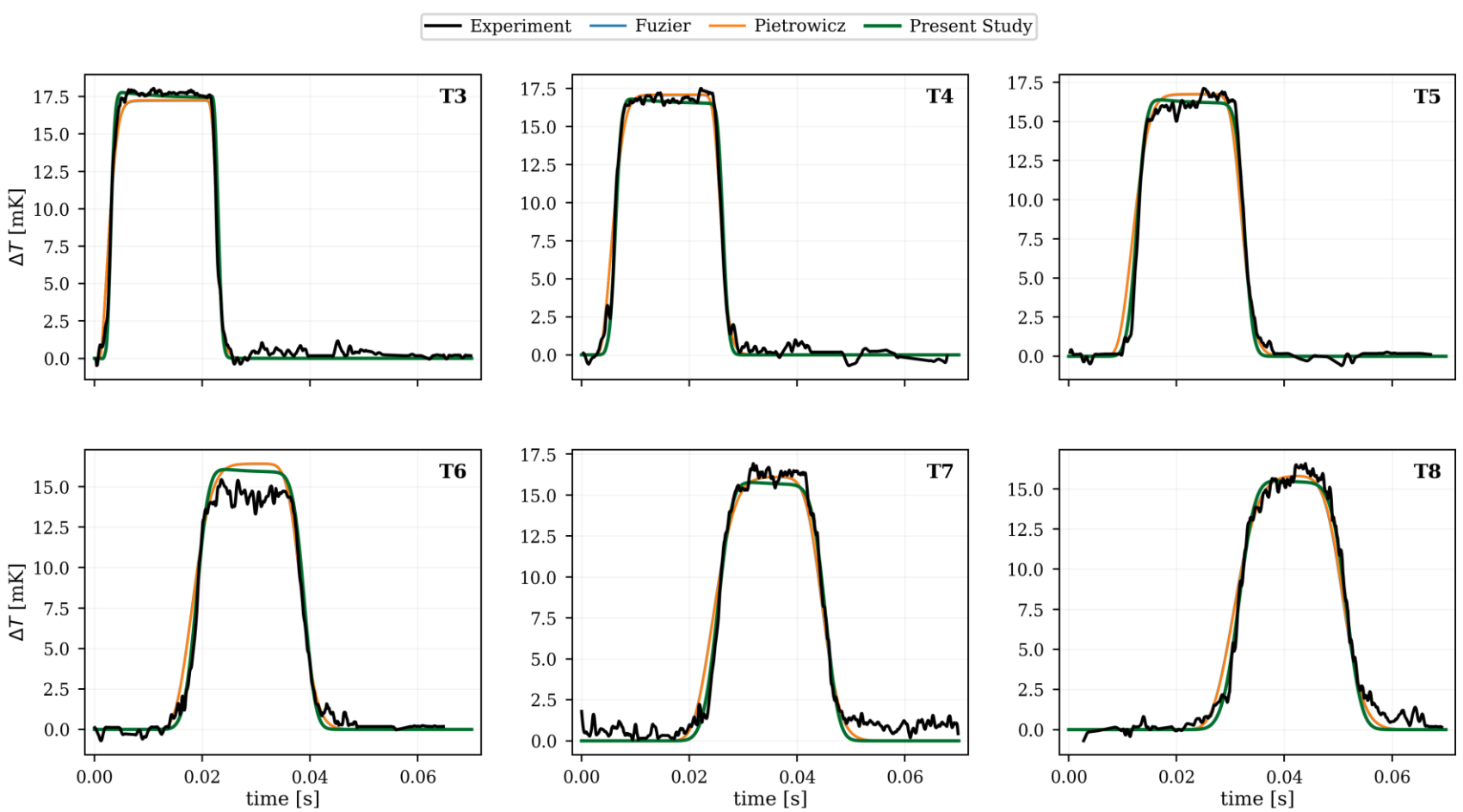
Model comparison at 2 m/s



- The algebraic formulations capture the main peak and its propagation through the tube.
- Their largest error appears in the downstream decay at T5 and T6.
- The dynamic formulation improves the late response, but does not change the basic picture.

All probe RMSE
Pietrowicz: $2.097\ \mathrm{mK}$
Present model: $1.147\ \mathrm{mK}$

Model comparison at 16 m/s

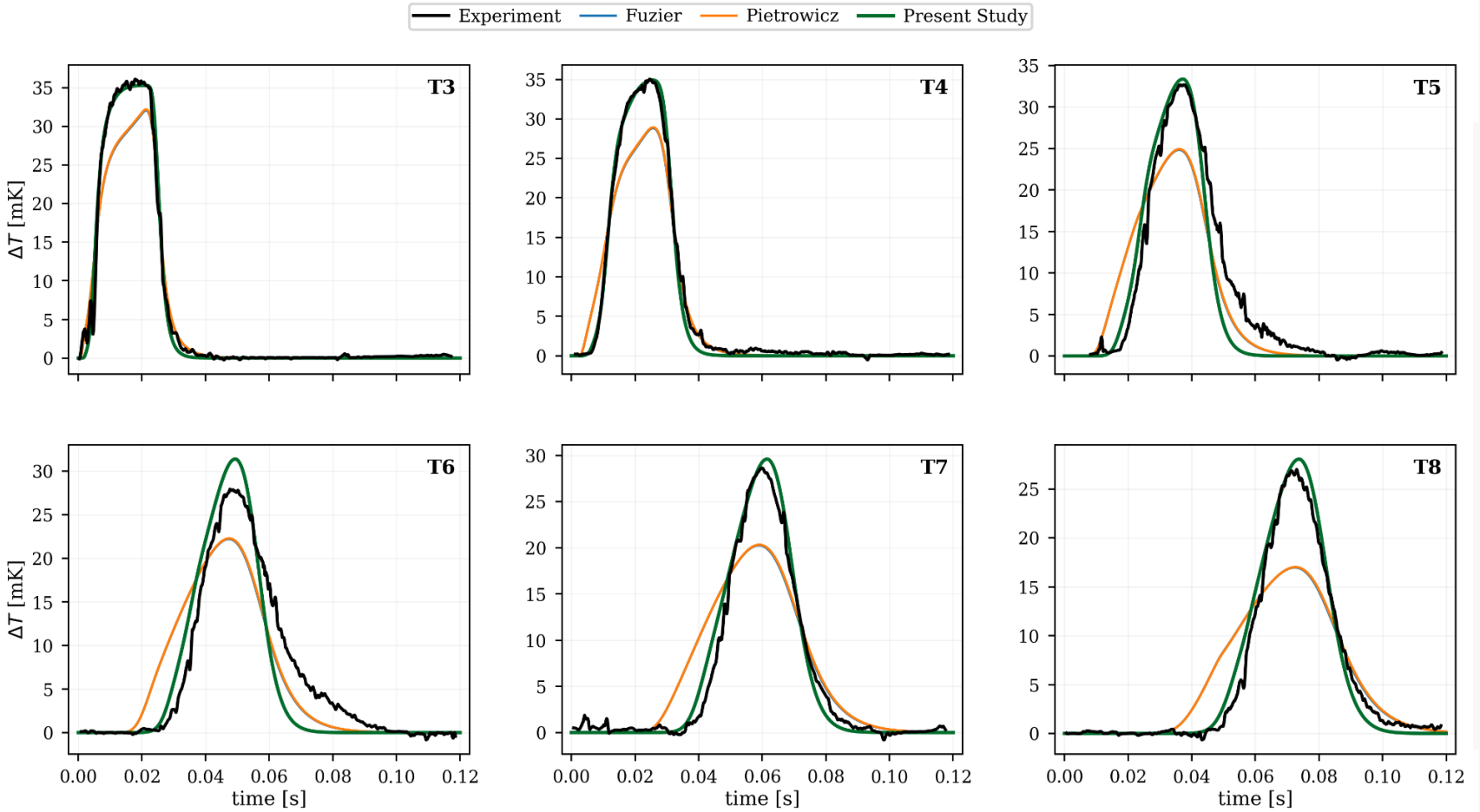


- All formulations reproduce the nearly rectangular temperature plateau.
- The Pietrowicz model already gives the correct pulse duration and approximate level.
- The dynamic formulation gives a smaller correction than in the difficult 8 m/s case.

All probe RMSE
Pietrowicz: $0.790\ \mathrm{mK}$
Present model: $0.576\ \mathrm{mK}$

Current manuscript analysis. Experimental curves digitised from Fuzier and Van Sciver, Cryogenics (2008).

Model comparison at 8 m/s

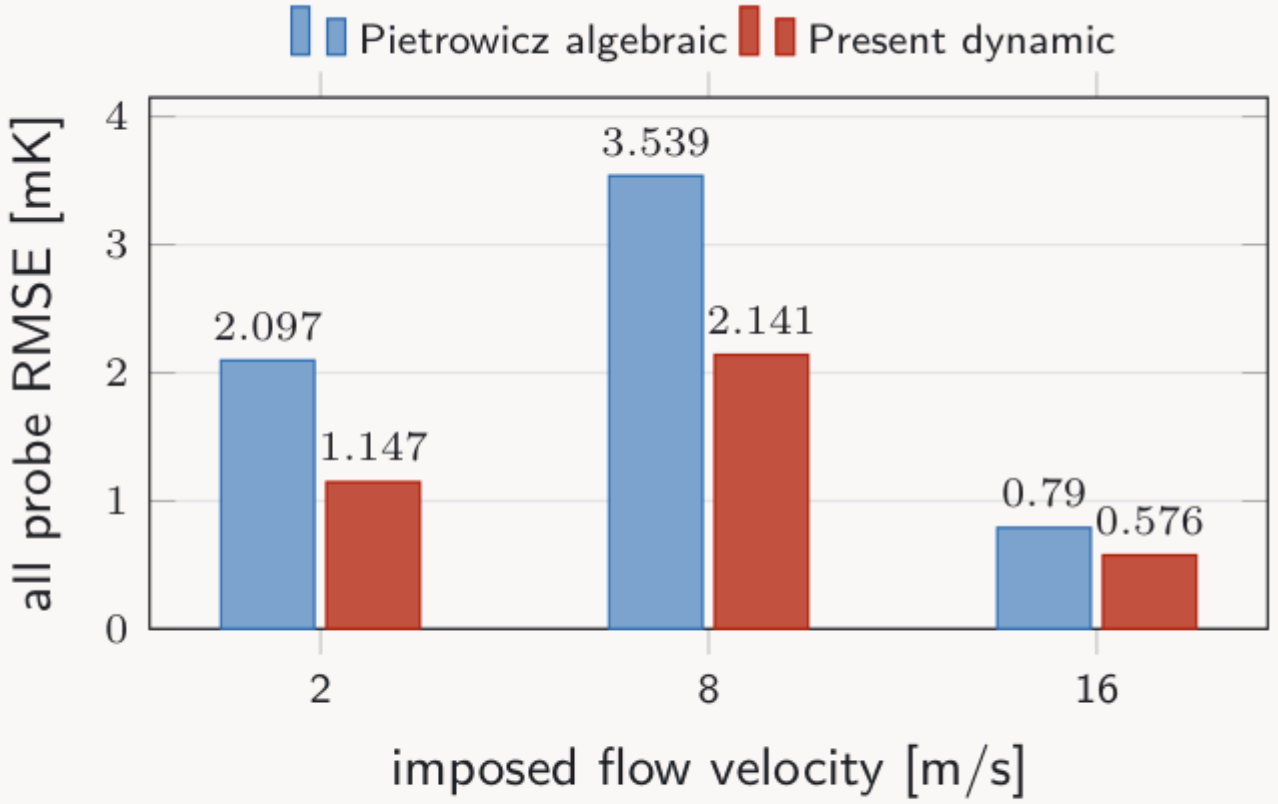


- Fuzier and Pietrowicz nearly overlap because both reconstruct counterflow algebraically.
- The dynamic formulation recovers most of the measured amplitude.

All probe RMSE
Pietrowicz: 3.539 mK
Present model: 2.141 mK

Current manuscript analysis. Experimental curves digitised from Fuzier and Van Sciver, Cryogenics (2008).

Performance across imposed flow velocities



- At **2 m/s**, the algebraic model captures the main pulse. The largest gain concerns the downstream decay.
- At **8 m/s**, it gives its largest absolute error and visibly over-attenuates the response.
- At **16 m/s**, all formulations reproduce the plateau and the remaining errors are small.

Each comparison uses identical experimental samples. The result concerns complete formulations rather than one isolated closure.

Why 8 m/s is difficult

$$\frac{\partial T}{\partial x} - \frac{1}{\rho s}\frac{\partial p}{\partial x}$$

2 m/s

$$\left|\frac{\partial T}{\partial x}\right| > \left|\frac{1}{\rho s}\frac{\partial p}{\partial x}\right|$$

Temperature-gradient contribution dominates.

8 m/s

$$\frac{\partial T}{\partial x} \approx \frac{1}{\rho s}\frac{\partial p}{\partial x}$$

The contributions are comparable and partly cancel.

16 m/s

$$\left|\frac{\partial T}{\partial x}\right| < \left|\frac{1}{\rho s}\frac{\partial p}{\partial x}\right|$$

Pressure loss and the Joule–Thomson contribution become dominant.

partial cancellation $\Longrightarrow$ every fixed, calibrated m reacts immediately $\Longrightarrow$ the previous counterflow state is discarded $\Longrightarrow$ excessive pulse attenuation and broadening

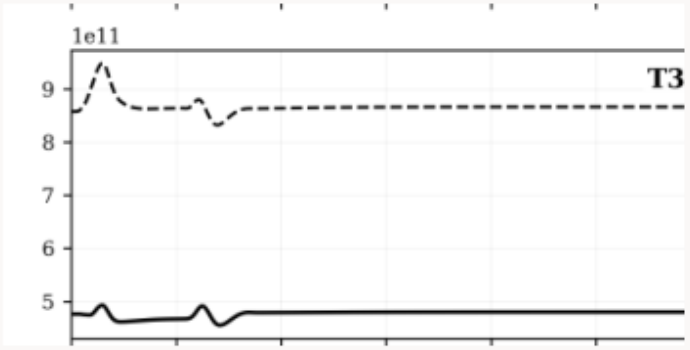
Tuning m changes the nonlinearity, but it cannot restore the state information missing from the algebraic closure.

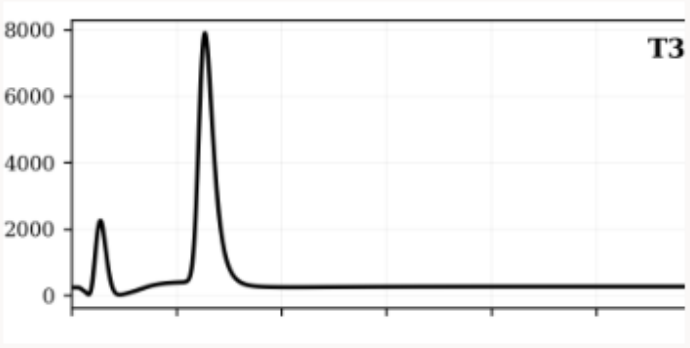
What is hidden inside the Gorter–Mellink coefficient?

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Physical state revealed by the dynamic model

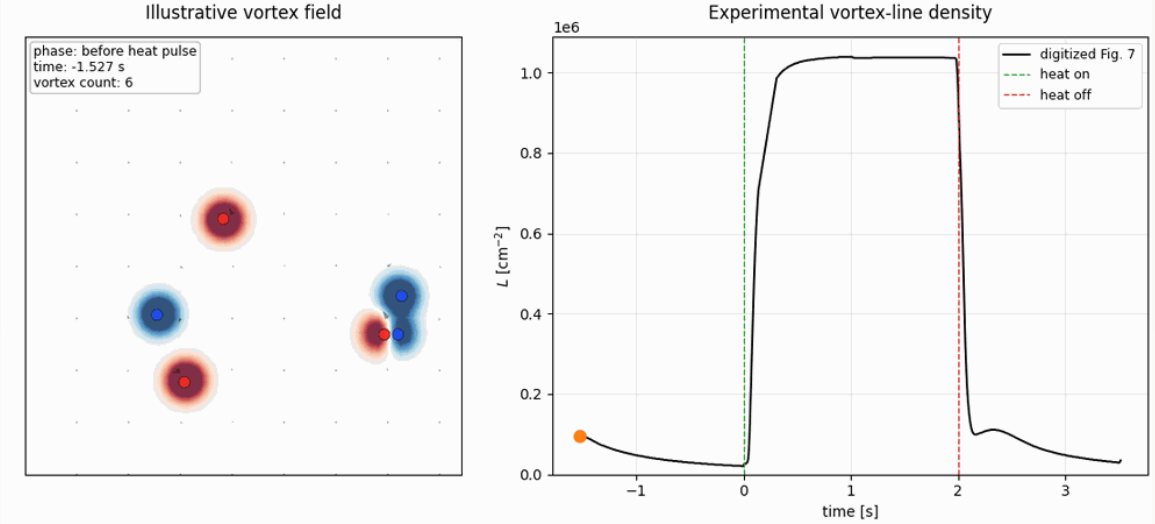
T3 at 8 m/s





Vortex line density L | Mutual-friction heating Φ_{mf}

Solid: before the pulse. Dashed: during the transient.



Visualization of superfluid to normal vortexes based on E. Varga and L. Skrbek [9]

$$\boldsymbol{F}_{ns} = A_{\mathrm{GM}}\rho_n|\boldsymbol{u}_n - \boldsymbol{u}_s|^2(\boldsymbol{u}_n - \boldsymbol{u}_s)$$

- Gorter and Mellink represented the nonlinear resistance phenomenologically.
- Vinen connected this resistance with a disordered tangle of quantised vortex lines.
- Thus, A_{GM} compresses the effect of a physical state that the algebraic model does not evolve.

L: total vortex-line length per unit volume. It can preserve the memory of an earlier counterflow state.

Dynamic component momentum

Pietrowicz model: current gradients set the relative velocity

$$s\frac{\partial T}{\partial x} = \frac{1}{\rho}\frac{\partial p}{\partial x} - A_{\mathrm{GM}}\rho_n|u_n - u_s|^2(u_n - u_s).$$

$$\frac{\partial T}{\partial x}, \frac{\partial p}{\partial x} \quad \Longrightarrow \quad u_n - u_s \quad \text{algebraically and instantaneously}$$

Present model: each component retains its own momentum

$$\frac{\partial u_n}{\partial t} + u_n\frac{\partial u_n}{\partial x} + v_{n,r}\frac{\partial u_n}{\partial r} = a - \frac{\rho_s}{\rho_n}s\frac{\partial T}{\partial x} - \frac{F_{ns,x}}{\rho_n} + \frac{1}{\rho_n r}\frac{\partial}{\partial r}\left(r\eta_{\mathrm{eff}}\frac{\partial u_n}{\partial r}\right),$$

$$\frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial x} + v_{s,r}\frac{\partial u_s}{\partial r} = a + s\frac{\partial T}{\partial x} + \frac{F_{ns,x}}{\rho_s}.$$

Subtracting the equations makes $u_n - u_s$ a dynamic state. The common acceleration $a(x,t)$, used to enforce the prescribed mass flow, cancels from the relative momentum balance.

Vortex line density closes mutual friction

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The evolving vortex population sets the mutual-friction force

$$\boxed{\boldsymbol{F}_{ns} = \rho_s \kappa \alpha(T) \Omega L (\boldsymbol{u}_n - \boldsymbol{u}_s)}$$

The Vinen equation supplies $L(x, r, t)$

$$\underbrace{\frac{\partial L}{\partial t}}_{\text{local evolution}} + \underbrace{\frac{\partial}{\partial x}(u_s L) + \frac{1}{r}\frac{\partial}{\partial r}(r v_{s,r} L)}_{\text{transport}} = \underbrace{C_1 w_{\text{eff}} L^{3/2}}_{\text{production}} - \underbrace{\frac{C_1}{\gamma(T)} L^2}_{\text{decay}}.$$

$$w_{\text{eff}} = \left[(u_n - u_s)^2 + w_r^2 + u_{\text{sh}}^2 \right]^{1/2}$$

κ: circulation quantum, $\alpha(T)$: friction coefficient, Ω: orientation factor.

Where mutual friction enters

normal momentum: $\cdots - \dfrac{F_{ns,x}}{\rho_n}$,

superfluid momentum: $\cdots + \dfrac{F_{ns,x}}{\rho_s}$,

$$\Phi_{\text{mf}} = \boldsymbol{F}_{ns} \cdot \boldsymbol{w} \geq 0,$$

temperature: $\cdots + \dfrac{\Phi_{\text{mf}}}{\rho c_p}$.

The opposite signs transfer momentum between the components. The scalar product converts the lost relative-motion energy into heat.

Relative motion controls vortex production. L sets the mutual-friction force, which changes both component velocities and returns dissipation to the temperature equation.

Open questions at the next model level

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Vortex production

Which Vinen production law remains valid in an inhomogeneous channel with imposed mass flow?

A scalar L cannot describe vortex orientation, curvature or polarisation.

Spatial structure

How much radial resolution is required when the heat pulse originates at the wall?

Cross sectional averaging may hide different near wall and core vortex states.

Independent validation

Which measurement can constrain $L(t)$ independently of temperature?

Time resolved pressure and second sound attenuation would add information unavailable from $T(t)$ alone.

Conclusions

1. The algebraic Gorter–Mellink reduction provides an efficient representation of macroscopic counterflow heat transport.

2. At 2 and 16 m/s, the Pietrowicz model already captures the principal pulse shape. Its main limitation appears at 8 m/s, where the thermal and pressure-gradient contributions become comparable.

3. At 8 m/s, the complete dynamic formulation reduces the all probe RMSE from 3.539 to $2.141\ \mathrm{mK}$, supporting the direction without assigning the gain to one mechanism.

4. Vortex line density provides a physically motivated state variable between nonlinear diffusion and the complete HVBK continuum description.

The central transition is from instantaneous counterflow to history dependent mutual friction.

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Presented by Sławomir Pietrowicz

Thank you

Questions and
discussion



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