SOFT EMISSIONS AT THE LHC: FROM TOP QUARK PRODUCTION TO PHOTON SPECTRA

ANNA KULESZA (UNIVERSITY OF MÜNSTER)

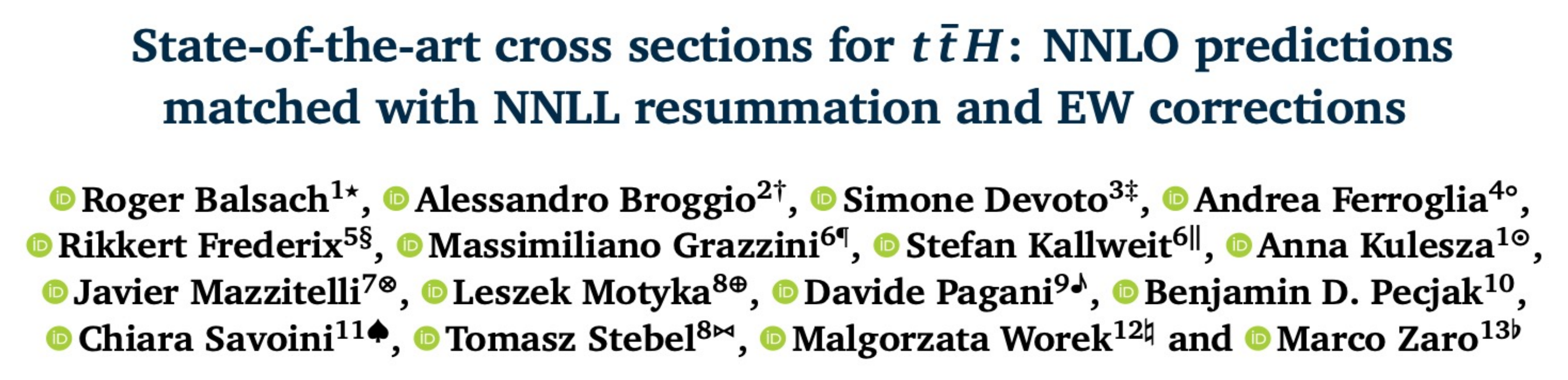
IFJ KRAKÓW, 29.01.2026

Universität
Münster

DFG Deutsche
Forschungsgemeinschaft

Top Quarks And Soft Emission

arXiv:2503.15043



State-of-the-art cross sections for $t\bar{t}H$: NNLO predictions matched with NNLL resummation and EW corrections

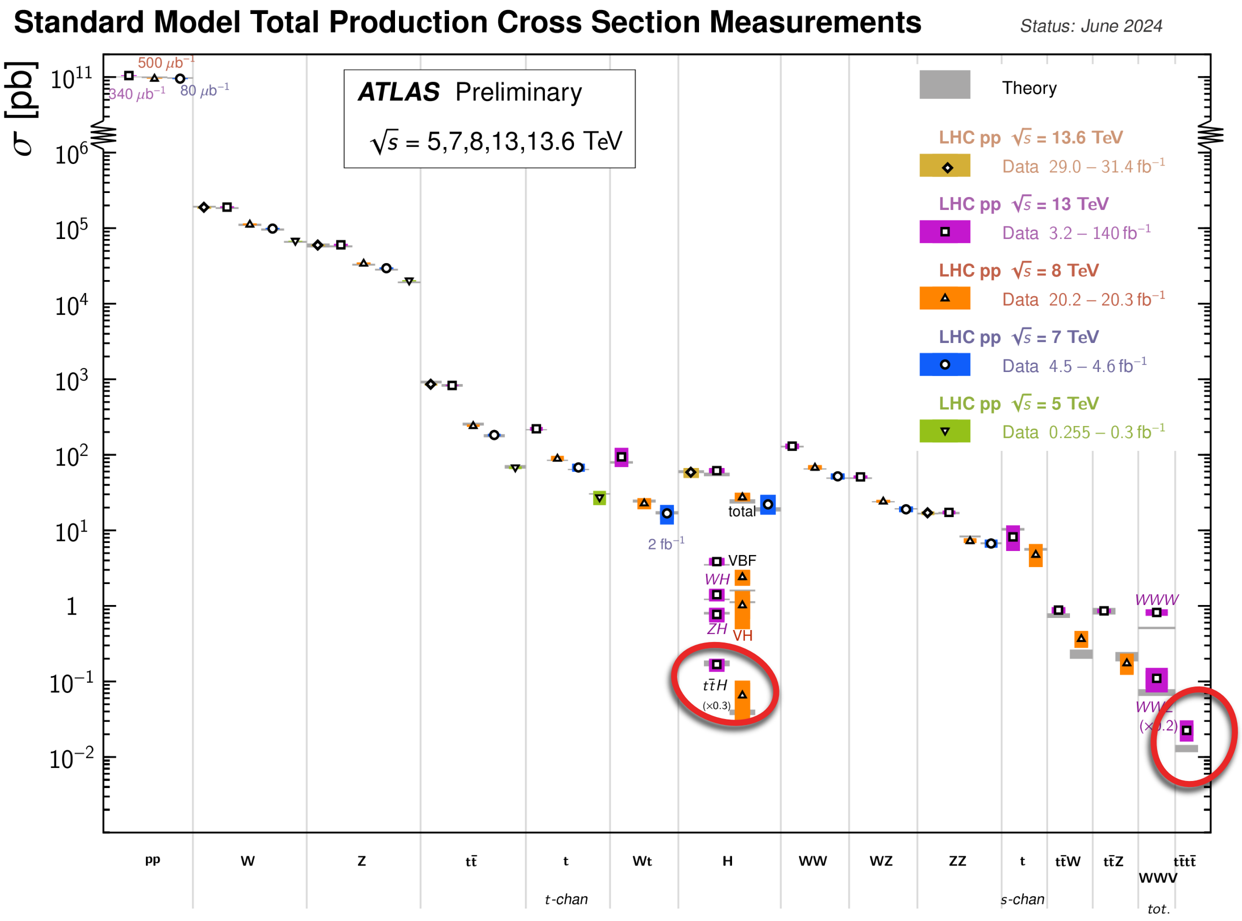
Roger Balsach[1⋆], Alessandro Broggio[2†], Simone Devoto[3‡], Andrea Ferroglia[4∘], Rikkert Frederix[5§], Massimiliano Grazzini[6¶], Stefan Kallweit[6‖], Anna Kulesza[1⊙], Javier Mazzitelli[7⊗], Leszek Motyka[8⊕], Davide Pagani[9♪], Benjamin D. Pecjak[10], Chiara Savoini[11♠], Tomasz Stebel[8⋈], Malgorzata Worek[12♮] and Marco Zaro[13♭]

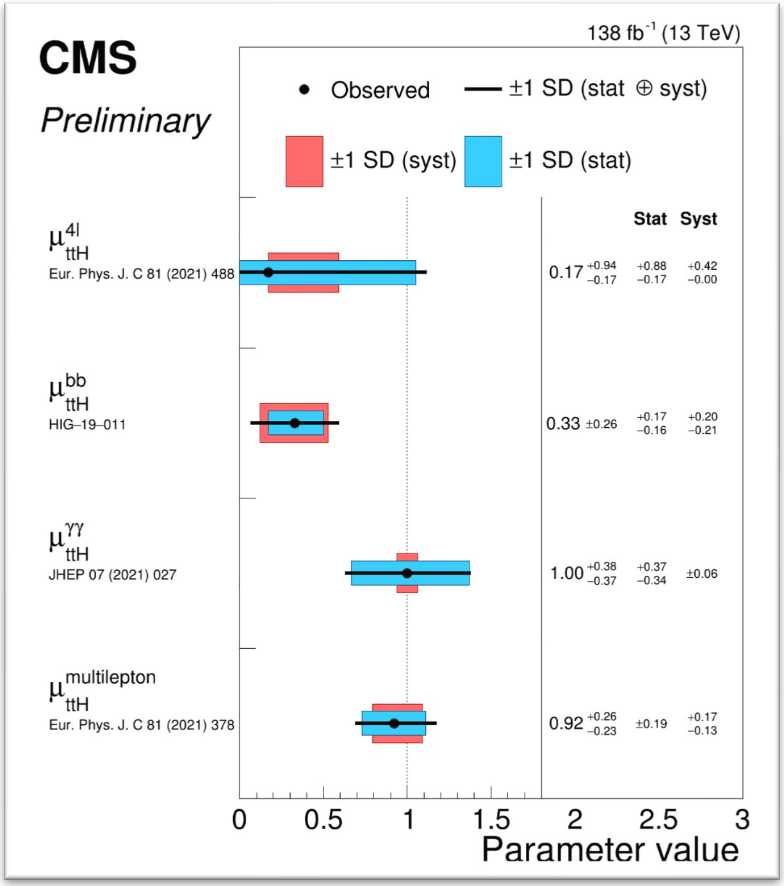
arXiv:2505.1038

Invariant-mass threshold resummation for the production of four top quarks at the LHC

Melissa van Beekveld,[a] Anna Kulesza,[b] Michele Lupattelli[b]
and Tommaso Saracco[a]

TOP QUARK AT THE LHC





$t\bar{t}\,H$

production

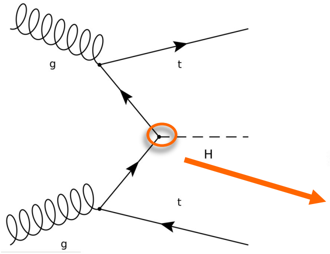
with

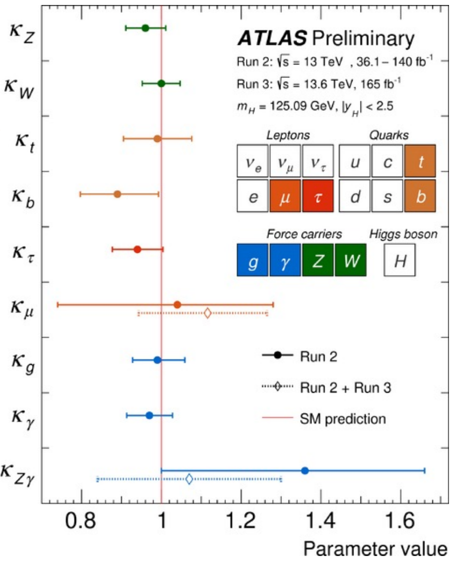
R. Balsach (DESY)

L. Motyka (UJ) and T. Stebel (UJ)

ASSOCIATED HIGGS PRODUCTION WITH A TOP-QUARK PAIR

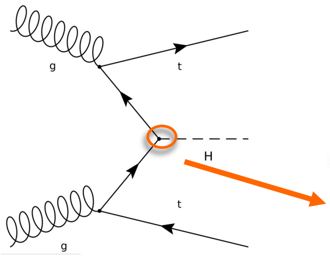
- Direct probe of the strength of the top-Yukawa coupling without making any assumptions regarding its nature

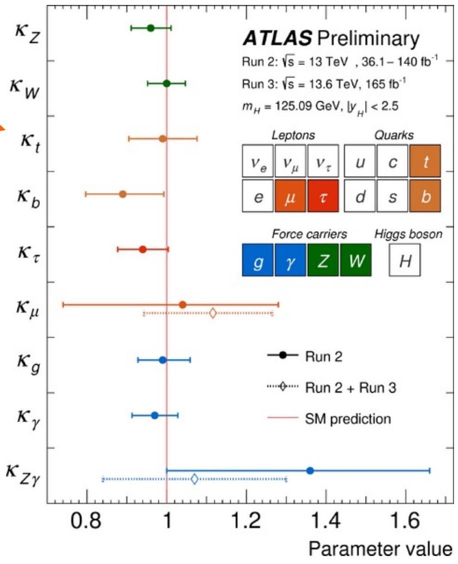


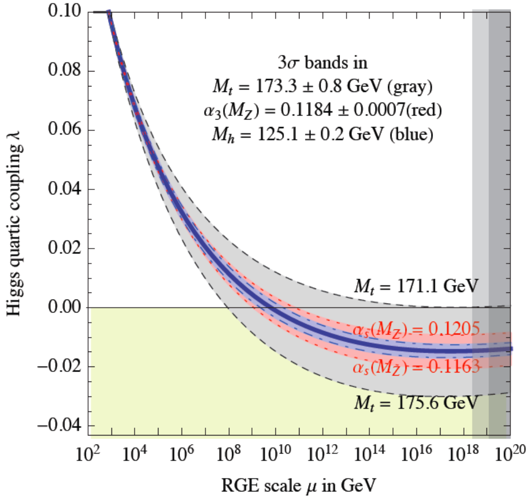


ASSOCIATED HIGGS PRODUCTION WITH A TOP-QUARK PAIR

- Direct probe of the strength of the top-Yukawa coupling without making any assumptions regarding its nature
- Far-reach consequences: stability of the Universe



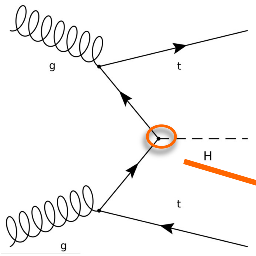


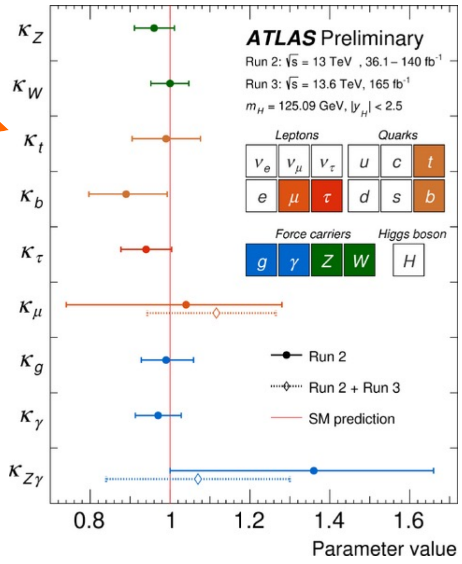


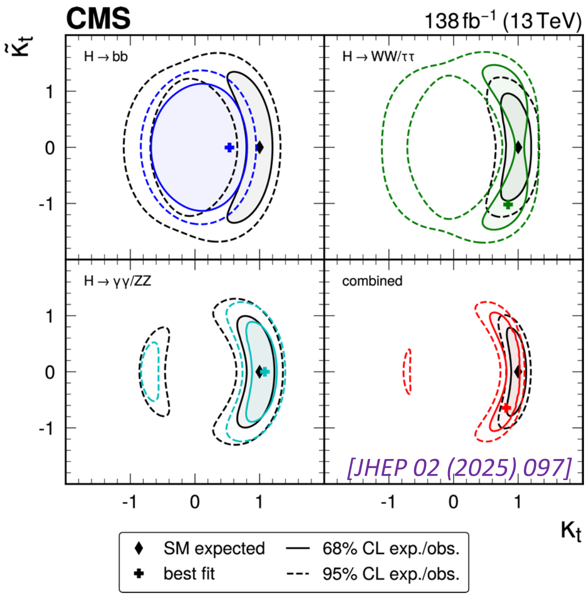
ASSOCIATED HIGGS PRODUCTION WITH A TOP-QUARK PAIR

- Direct probe of the strength of the top-Yukawa coupling without making any assumptions regarding its nature

- Probe of the CP nature of the top-Higgs Yukawa interaction



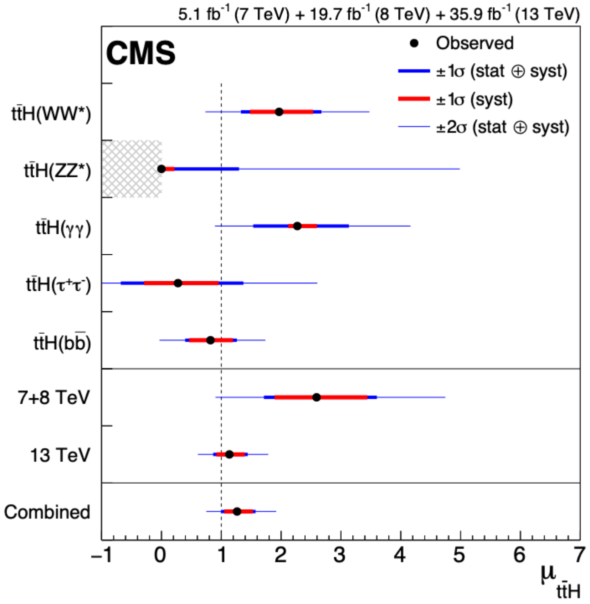


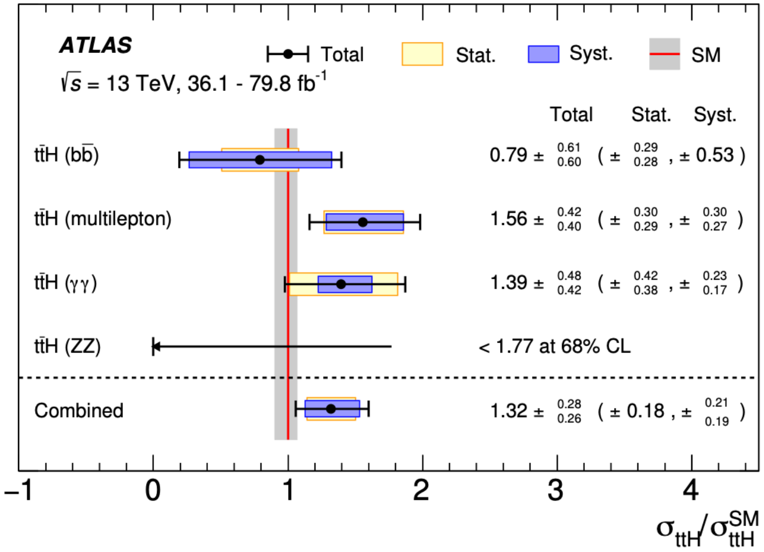


[JHEP 02 (2025) 097]

TTH REVEALED

- Observed first in 2018 *[Phys. Rev. Lett. 120, 231801 (2018)][Phys.Lett.B 784 (2018) 173]*





ttH Revealed

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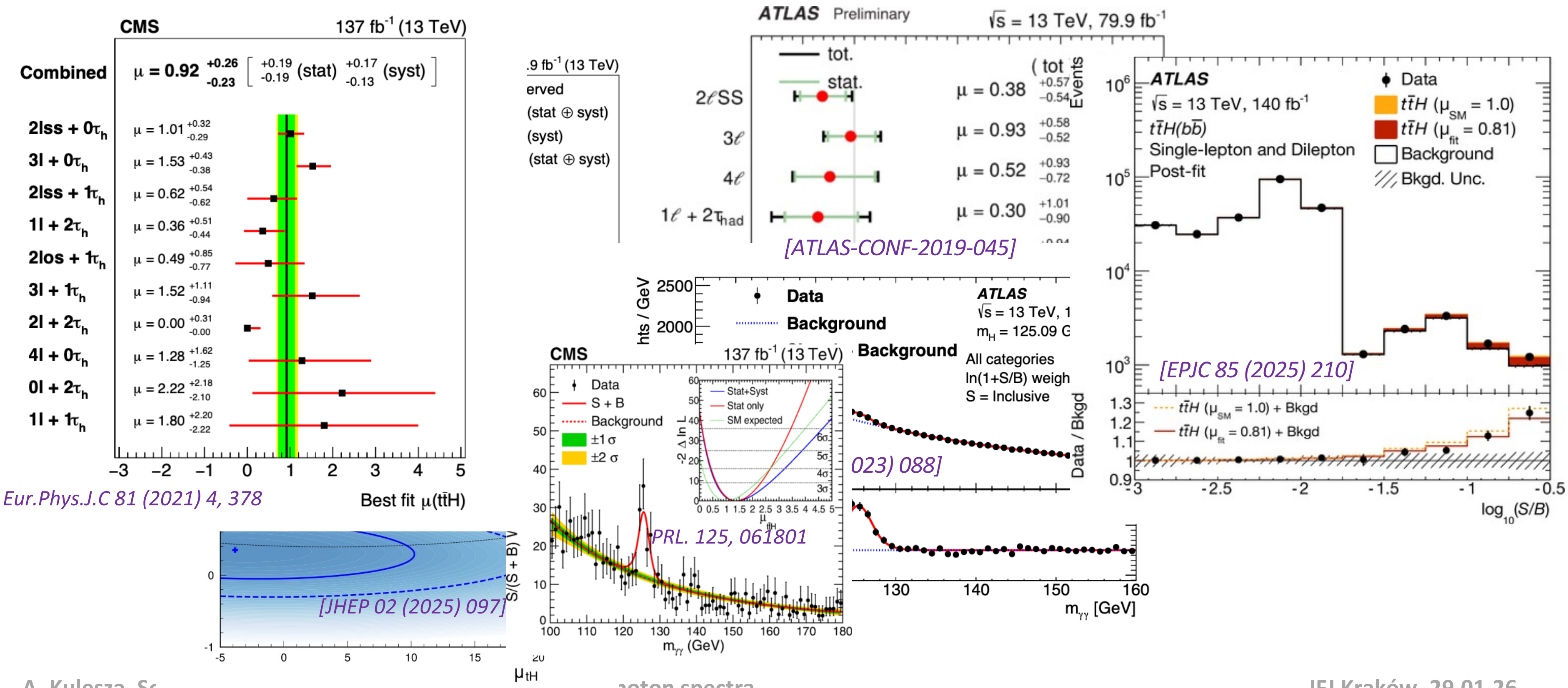
- Since then, many dedicated measurements in various Higgs decay channels



[ATLAS-CONF-2019-045]

[EPJC 85 (2025) 210]

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Eur.Phys.J.C 81 (2021) 4, 378

PRL. 125, 061801

[JHEP 02 (2025) 097]

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- Since then, many dedicated measurements in various Higgs decay channels
- Reference total cross section from the Higgs Cross Section Working Group YR4 *[arXiv:1610.07922]*

→ NLO QCD + EW accuracy

$\sqrt{S} = 13$ TeV, M_H=125 GeV

$\sigma = 507.2 \text{ fb } {}^{+5.8\%}_{-9.2\%} (\text{scale}) \pm 3.6\%$ (PDF+α_S)

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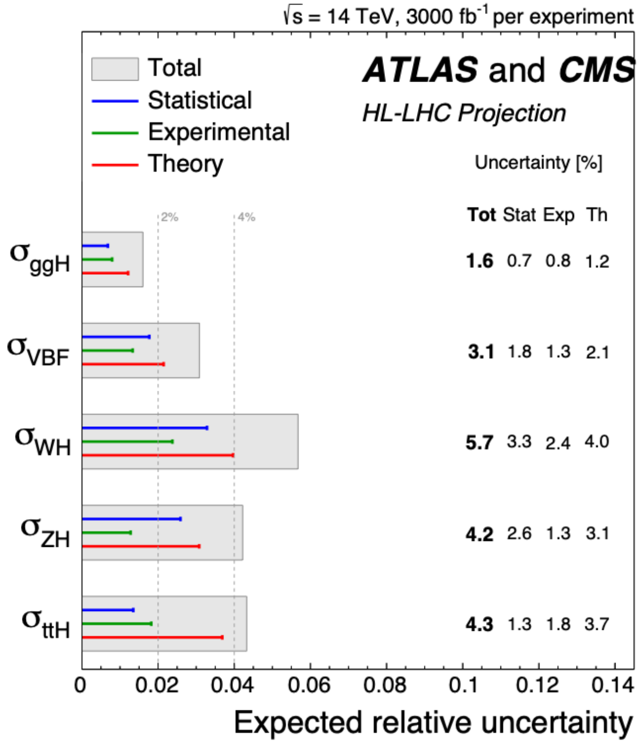
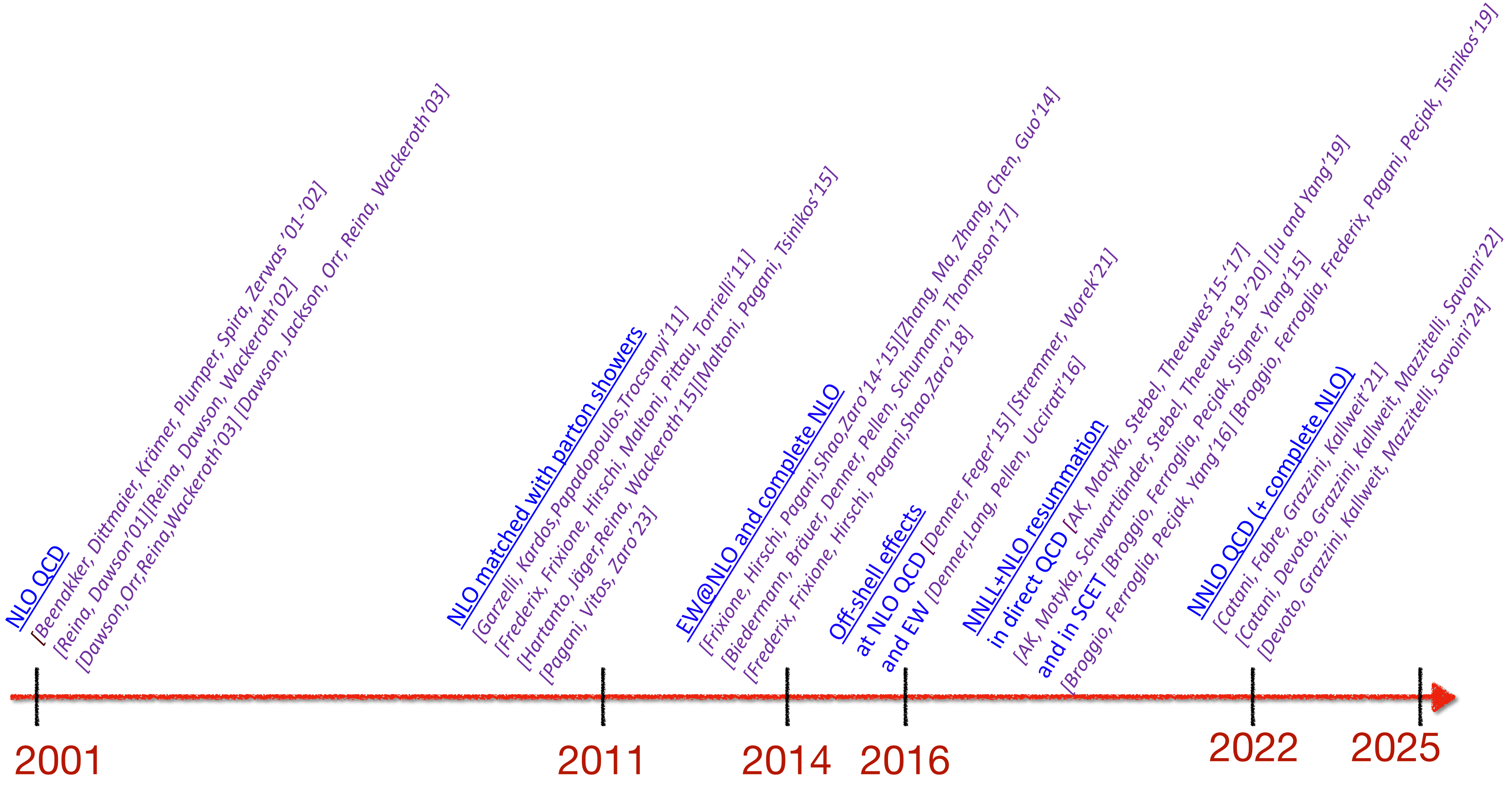
- Expect future reductions of the exp. error



[Cepeda et al.'19]

A BRIEF HISTORY OF TTH THEORY PREDICTIONS IN THE SM



NLO QCD
[Beenakker, Dittmaier, Krämer, Plumper, Spira, Zerwas '01-'02]
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NLO matched with parton showers
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Off-shell effects
at NLO QCD [Denner, Feger'15] [Stremmer, Worek'21]
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NNLL+NLO resummation
in direct QCD [AK, Motyka, Stebel, Theeuwes'15-'17]
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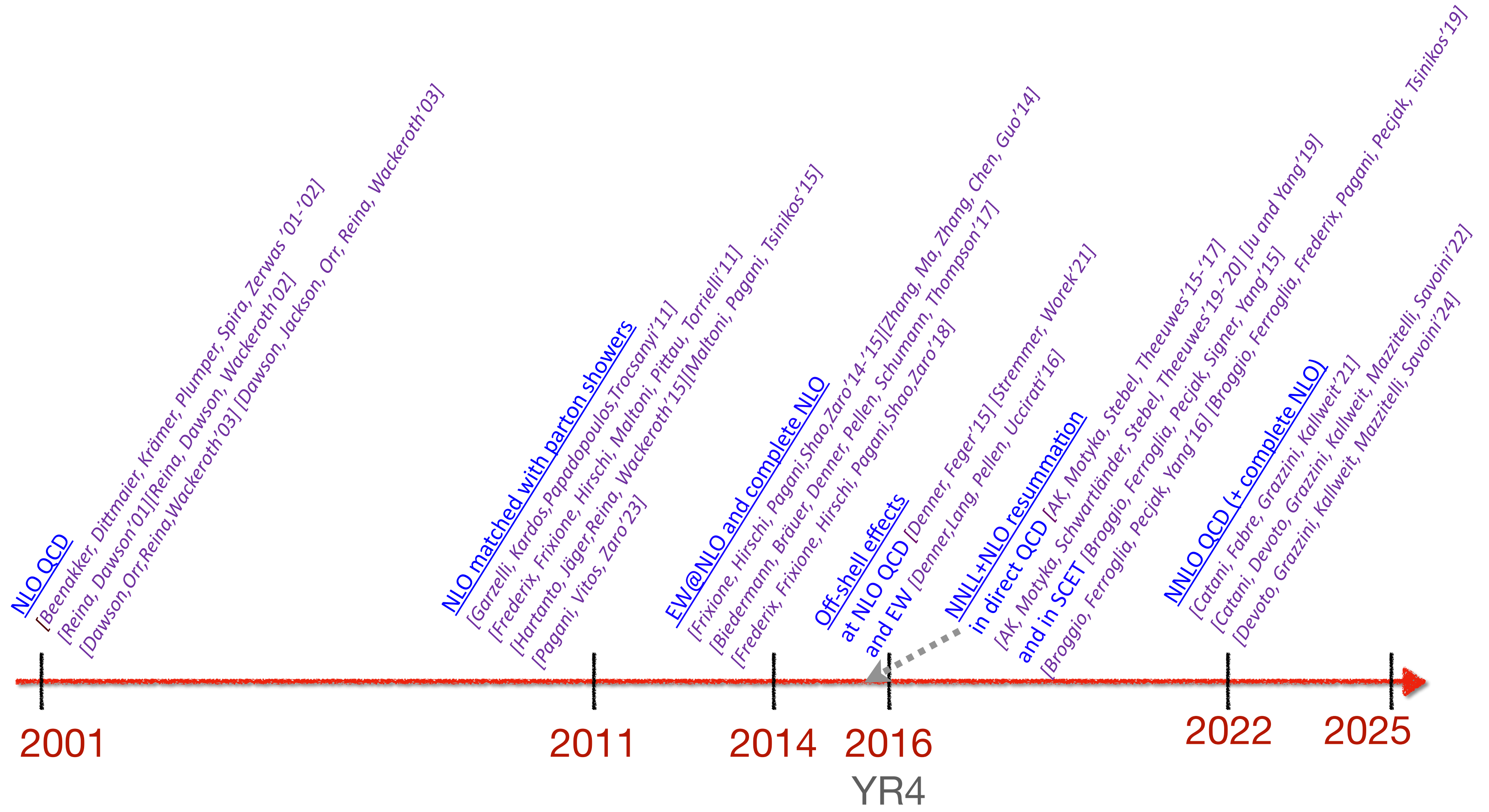
A Brief History of ttH Theory Predictions in the SM



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2001
2011
2014
2016
YR4
2022
2025

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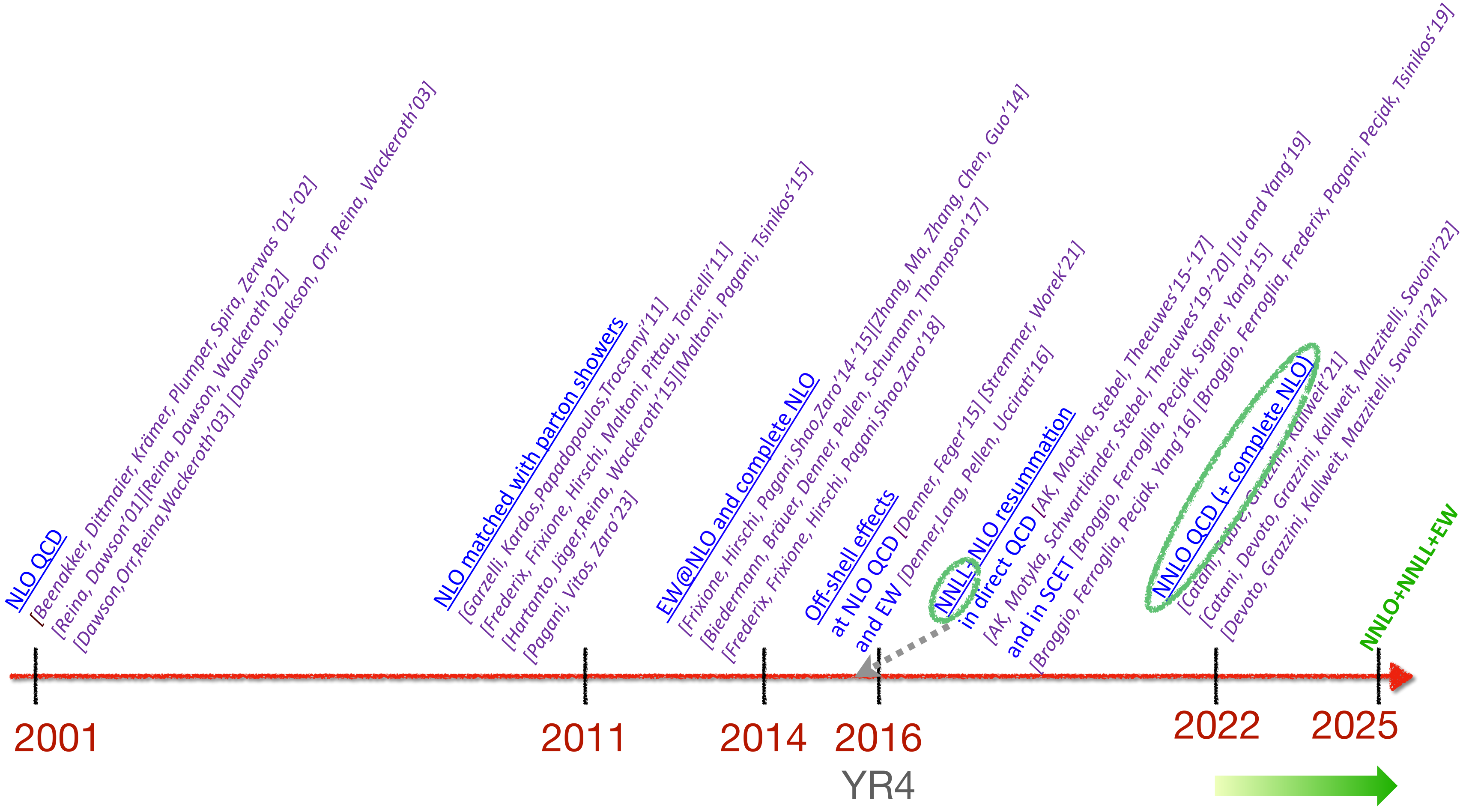

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NNLO+NNLL+EW

2001 2011 2014 2016 2022 2025

YR4

NNLO FOR TTH

[Devoto, Grazzini, Kallweit, Mazzitelli, Savoini’24]

- NNLO computation in the q_T-subtraction formalism *[Catani, Grazzini’07]* extended from the ttbar case *[Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Sargsyan’19] [Catani, Devoto, Grazzini, Kallweit, Mazzitelli’19]] [Catani, Devoto, Grazzini, Mazzitelli’23][Devoto, Mazzitelli’25]* and implemented in the MATRIX framework *[Grazzini, Kallweit, Wiesemann’17]*
  - Off-diagonal partonic channels in *[Catani, Fabre, Grazzini, Kallweit’21]*

- The two-loop contribution obtained by combining the soft Higgs and small top mass approximations to the finite part of the two-loop amplitude

$$d\sigma_{H^{(2)}} \equiv \left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^2 \underset{\uparrow}{H^{(2)}}(Q)\, d\sigma_{\rm LO}\,, \qquad H^{(2)}(\mu_{IR}) = \frac{2\mathrm{Re}\left(\mathcal{M}^{(2),\mathrm{fin}}(\mu_{IR},\mu_R)\mathcal{M}^{(0)*}\right)}{|\mathcal{M}^{(0)}|^2}\Bigg|_{\mu_R=Q}$$

two-loop hard-virtual coefficient

IR-subtracted part of the amplitude

- Uncertainty of the approximation contributes to the theory error

NNLO FOR TTH

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two-loop hard-virtual c

work towards full two-loop amplitudes ongoing *[Febres Cordero, Figueiredo, Kraus, Page, Reina '23] [Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson'24] [Buccioni, Kreer, Liu, Tancredi'23][Wang, Xia, Yang,Ye'24][Becchetti, Canco, Chestnov, Peraro, Pozzoli, Zoia'25]*

- Uncertainty of the approxim

NNLO FOR TTH

- Soft Higgs approximation *[Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Savoini'22]*
  - In the limit $q_H^0, m_H \ll Q_{t\bar{t}H}$ and $m_H \ll m_t$, Higgs emission is factorized out (in analogy to soft gluon emission)

$$\mathcal{M}^{\rm fin}(\{p_i\}, q; \mu_R, \mu_{IR}) \simeq F(\alpha_s(\mu_R), \mu_R/m_t)\, \frac{m_t}{v} \left(\frac{m_t}{p_3 \cdot q} + \frac{m_t}{p_4 \cdot q} \right) \mathcal{M}^{\rm fin}_{t\bar{t}}(\{p_i\}; \mu_R, \mu_{IR})$$

soft limit of the heavy-quark scalar form-factor
[Bernreuther et al.'05][Ablinger et al. '17]

from two-loop ttbar amplitude
[Bärnreuther, Czakon, Fiedler'13]

- Small top mass (high-energy) approximation $m_t \ll Q_{t\bar{t}H}$ *[Devoto, Grazzini, Kallweit, Mazzitelli, Savoini'24]*
  - relies on the so-called massification procedure *[Penin'05][Mitov, Moch'06, Becher, Melnikov'07][Engel, Gnendiger, Signer, Ulrich'17][Wang,Xia,Yang,Ye'23]*

$$\mathcal{M}^{\rm fin}(\{p_i\}, q; \mu) \simeq \boldsymbol{\mathcal{F}}_{[c]} \left(\alpha_s(\mu), \frac{\mu^2}{m_t^2}, \frac{\mu^2}{2p_i \cdot p_j} \right) \mathcal{M}^{\rm fin}_{(m_t=0)}(\{p_i\}, q; \mu)$$

operator in colour space, encodes all mass logarithms

from two-loop $H + 4$-parton amplitude *[Badger et al.'21]*

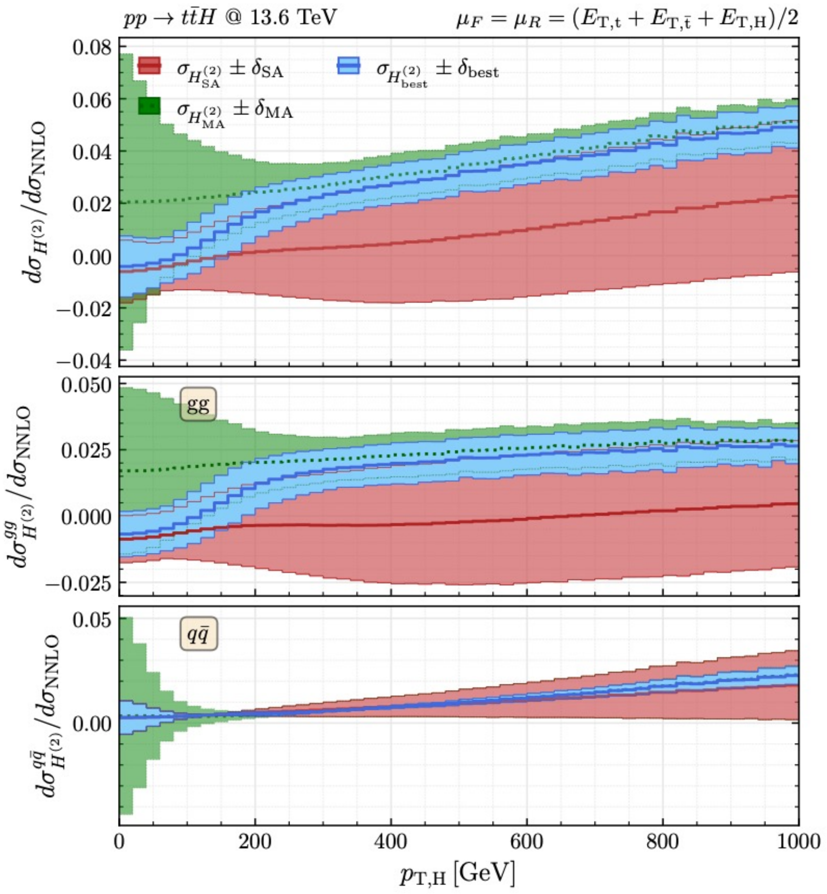
NNLO: COMBINED ESTIMATE OF THE 2-LOOP CONTRIBUTION

[Devoto, Grazzini, Kallweit, Mazzitelli, Savoini'24]

- The error of each approximation $\delta_{SA,}\delta_{MA}$ is estimated on the basis of 1) the relative difference between the full and the approximated one-loop result and 2) the dependence on the subtraction scale μ_{IR}

- The two-loop contribution is then constructed as a weighted average of the soft Higgs and massification approximation, with weights $\varpi_i = \delta_i^{-2}$ $(i = SA,MA)$, so that the more precise approximation drives the result

$$\sigma_{H^{(2)}_{\rm best}} = \frac{1}{\omega_{\rm SA}+\omega_{\rm MA}}\left(\omega_{\rm SA}\sigma_{H^{(2)}_{\rm SA}} + \omega_{\rm MA}\sigma_{H^{(2)}_{\rm MA}}\right)$$



NNLO: RESULTS

[Devoto, Grazzini, Kallweit, Mazzitelli, Savoini'24]

$\mu_R = \mu_F = m_t + m_H/2$

	σ [fb]
$\mathrm{LO}_{\mathrm{QCD}}$	$423.9\ ^{+30.7\%}_{-21.9\%}(\mathrm{scale})$
$\mathrm{NLO}_{\mathrm{QCD}}$	$528.9\ ^{+5.7\%}_{-9.0\%}(\mathrm{scale})$
$\mathrm{NNLO}_{\mathrm{QCD}}$	$550.7(5)\ ^{+0.9\%}_{-3.1\%}(\mathrm{scale}) \pm 0.9\%(\mathrm{approx})$
$\mathrm{NNLO}_{\mathrm{QCD}} + \mathrm{NLO}_{\mathrm{EW}}$	$562.3(5)\ ^{+1.1\%}_{-3.2\%}(\mathrm{scale}) \pm 0.9\%(\mathrm{approx})$

~4% correction wrt. NLO QCD

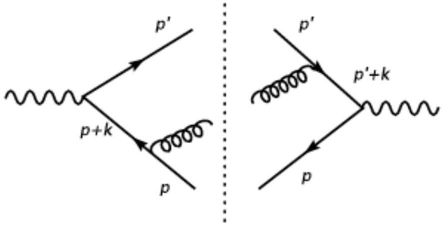
~2% correction wrt. NNLO QCD

uncertainty of the estimate of the 2-loop virtual corrections

WHAT ELSE IF NOT N^{X}LO?

- QCD: IR singularities due to emission of real and virtual gluons
  - cancellation of IR poles between real and virtual corrections (KLN theorem)

- Sudakov logarithms from soft and collinear gluon emission -> cancellations between real and virtual corrections leave out finite logarithmic remnants -> systematic structure of logarithmic terms at all orders of perturbation theory



$$\alpha_s \int \frac{d^4k}{2\pi} \frac{p \cdot p'}{(p \cdot k)\,(p' \cdot k)} \sim \alpha_s \int \frac{dk_0}{k_0} \int \frac{d\theta}{\theta} \sim \alpha_s \log^2(\ldots)$$

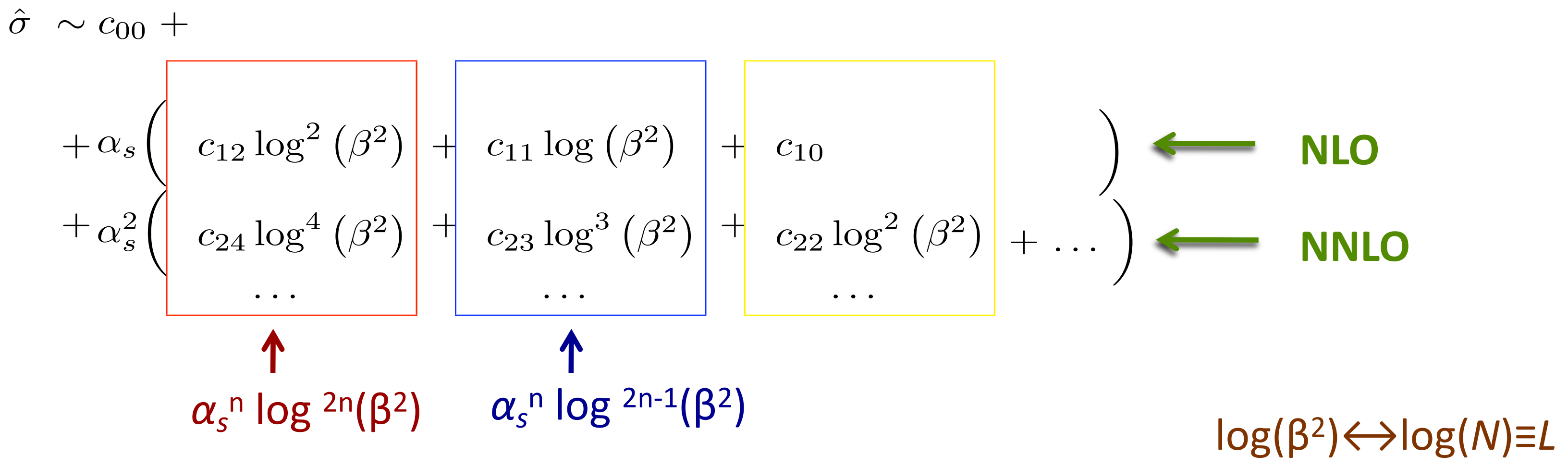
soft k_0->0 collinear ϑ->0

  - can get large, spoiling predictive power of the theory
  - same effect, appearing in many disguises, depending on the process, observable etc.

- Resummation extends accuracy of the perturbative prediction beyond fixed-order by introducing a systematic treatment of the logarithmic contributions to all orders

SOFT GLUON RESUMMATION

Systematic reorganization of perturbative series



$$\hat{\sigma} \sim c_{00} +$$

$$+\alpha_s\left(c_{12}\log^2(\beta^2) + c_{11}\log(\beta^2) + c_{10} \right)$$

$$+\alpha_s^2\left(c_{24}\log^4(\beta^2) + c_{23}\log^3(\beta^2) + c_{22}\log^2(\beta^2) + \ldots \right)$$

$$\ldots \qquad \ldots \qquad \ldots$$

NLO

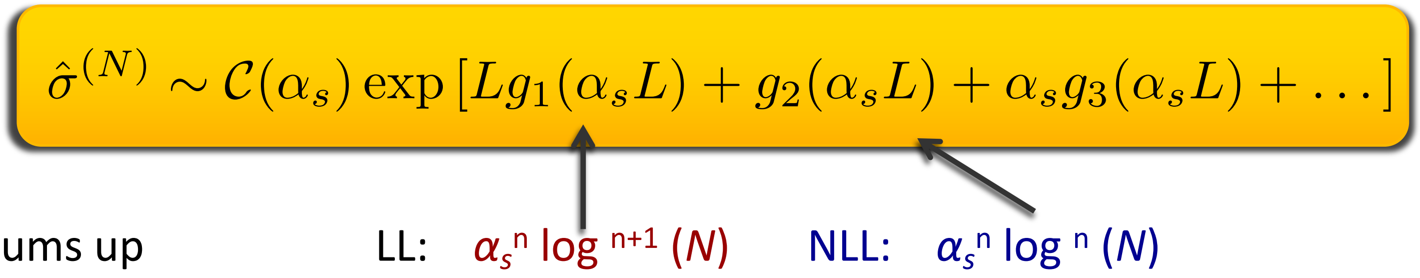
NNLO

$\alpha_s^n \log^{2n}(\beta^2)$

$\alpha_s^n \log^{2n-1}(\beta^2)$

log(β²)↔log(*N*)≡*L*

Factorization at threshold: space of Mellin moments *N*



$$\hat{\sigma}^{(N)} \sim \mathcal{C}(\alpha_s)\exp\left[Lg_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \ldots\right]$$

sums up LL: $\alpha_s^n \log^{n+1}(N)$ NLL: $\alpha_s^n \log^{n}(N)$

THRESHOLD RESUMMATION FOR TTH

- Threshold limit in the invariant mass kinematics $\hat{s} \to Q^2 = (m_t + m_{\bar{t}} + m_H)^2$
- Large logarithmic corrections at each order in α_s of the form $\alpha_s^n \left(\frac{\log^m(1-\hat{\rho})}{1-\hat{\rho}}\right)_+$; $\hat{\rho} = Q^2/\hat{s}$ due to soft gluon emission
- The underlying principle behind resummation is factorization in the soft limit

direct QCD (dQCD)

- derived from the properties of scattering amplitudes and cross sections in full QCD

[Catani, Trentadue, Parisi, Petronzio, Dokshitzer, Altarelli, Mangano, Nason,..]
[Sterman, Collins, Soper,..]

SCET

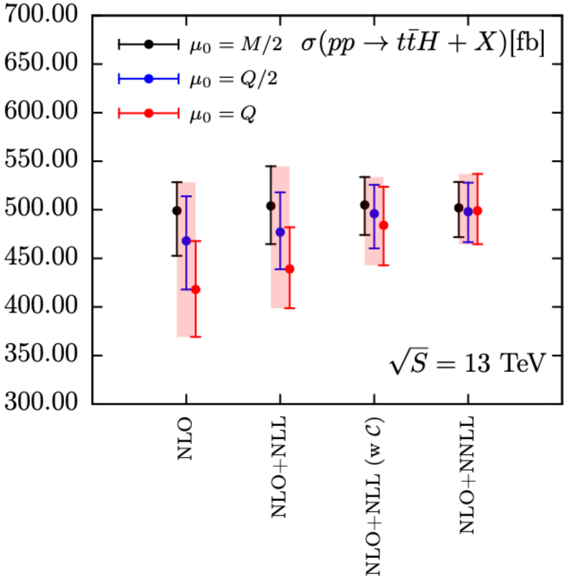
- soft and collinear limit of the full theory already at the level of the Lagrangian

[Bauer, Becher, Beneke, Chapovsky, Diehl, Feldmann, Fleming, Hill, Lee, Luke, Manohar, Neubert, Pirjol, Rothstein, Stewart,..]

- Two distinct theoretical frameworks → different organization of the expressions, nevertheless theoretically equivalent

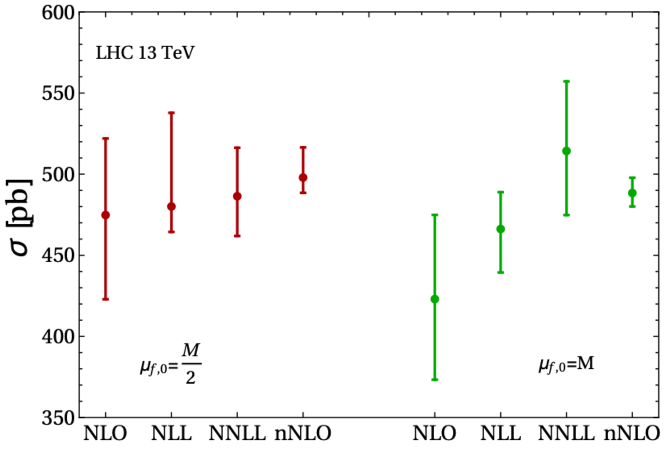
NLO+NNLL RESUMMATION FOR TTH

[AK, Motyka, Stebel, Theeuwes'17]



(*Q* is the invariant mass of the ttbarH system)

[Broggio, Ferroglia, Pecjak, Yang'16]



(*M* is the invariant mass of the ttbarH system)

differences also present at NNLO+NNLL, for the same input parameters

THE DIRECT QCD APPROACH

[AK, Motyka, Stebel, Theeuwes'17]

- For threshold resummation in dQCD, factorization takes place in the space of Mellin moments *N* taken w.r.t. $\hat{\rho}$ → logs in $\hat{\rho}$ become logs of *N*

- Resummed $ij \to t\bar{t}H$ cross section can be written as *[Kidonakis, Sterman'97] [Kidonakis, Oderda, Sterman'98][Bonciani, Catani, Mangano, Nason'98]*

$$\frac{d\tilde{\hat{\sigma}}^{(\mathrm{NNLL})}_{ij\to t\bar{t}H}}{dQ^2}(N, Q^2, \{m^2\}, \mu_R^2, \mu_F^2) = \mathrm{Tr}\left[\overset{\text{hard}}{\mathbf{H}_R(Q^2, \{m^2\}, \mu_R^2, \mu_F^2)}\right.$$
$$\left. \times\, \bar{\mathbf{U}}_R(N+1, Q^2, \{m^2\}, \mu_R^2)\tilde{\mathbf{S}}_R(N+1, Q^2, \{m^2\})\, \mathbf{U}_R(N+1, Q^2, \{m^2\}, \mu_R^2)\right] \quad \text{soft}$$
$$\times\, \Delta^i(N+1, Q^2, \mu_R^2, \mu_F^2)\; \Delta^j(N+1, Q^2, \mu_R^2, \mu_F^2). \quad \text{jets}$$

At NNLL

$$\Delta^i(N, Q^2, \mu_R^2, \mu_F^2) = \exp[Lg^{(1)}(\lambda) + g^{(2)}(\lambda, Q^2, \mu_R^2, \mu_F^2) + g^{(3)}(\lambda, Q^2, \mu_R^2, \mu_F^2)]$$

with $L = \ln N,\ \lambda = b_0\, \alpha_s(\mu_R^2) \ln N$ and $g^{(k)}$ functions for colour singlet production

Similarly, $\mathbf{U}$ is a function of λ and $\alpha_s(\mu_R^2)$

THE DIRECT QCD APPROACH

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$$\frac{d\tilde{\hat{\sigma}}^{(\mathrm{NNLL})}_{ij\to t\bar{t}H}}{dQ^2}(N, Q^2, \{m^2\}, \mu_R^2, \mu_F^2) =$$

$$\times\, \bar{\mathbf{U}}_R(N+1, Q^2, \{m^2\}$$

$$\times\, \Delta^i(N+1, Q^2, \mu_R^2, \mu$$

→ matching to the fixed-order result at a given $(\mu_R,\ \mu_F)$

$$\sigma^{\mathrm{dQCD}}_{\mathrm{NNLO+NNLL}} = \sigma_{\mathrm{NNLO}} + \sigma^{\mathrm{dQCD}}_{\mathrm{NNLL}} - \sigma^{\mathrm{dQCD}}_{\mathrm{NNLL}}\Big|_{\alpha_s^2}$$

7 –point scale variation

At NNLL

$$\Delta^i(N, Q^2, \mu_R^2, \mu_F^2) = \exp[Lg^{(1)}(\lambda) + g^{(2)}(\lambda, Q^2, \mu_R^2, \mu_F^2) + g^{(3)}(\lambda, Q^2, \mu_R^2, \mu_F^2)]$$

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Similarly, $\mathbf{U}$ is a function of λ and $\alpha_s(\mu_R^2)$

THE SCET APPROACH

[Broggio, Ferroglia,Pecjak, Yang'16]

- In momentum space

$$\sigma\left(S,m_t,m_H\right) = \frac{1}{2S}\int_{\tau_{min}}^{1} d\tau \int_{\tau}^{1}\frac{dz}{\sqrt{z}}\sum_{ij} ff_{ij}\left(\frac{\tau}{z},\mu\right)\int d\mathrm{PS}_{t\bar{t}H}\,\mathrm{Tr}\left[\mathbf{H}_{ij}\left(\{p\},\mu\right)\mathbf{S}_{ij}\left(\frac{Q(1-z)}{\sqrt{z}},\mu\right)\right]$$

$$\tau_{\min} = \frac{\left(2m_t+m_H\right)^2}{S}\,, \qquad \tau = \frac{Q^2}{S}$$

- In Mellin space

$$d\widetilde{\hat{\sigma}}_{ij}\left(N,\mu_F\right) = \mathrm{Tr}\left[\underset{\text{evolution}}{\widetilde{\mathbf{U}}_{ij}}\left(\bar{N},\{p\},\mu_F,\mu_h,\mu_s\right)\overset{\text{hard}}{\mathbf{H}_{ij}\left(\{p\},\mu_h\right)}\,\underset{\text{evolution}}{\widetilde{\mathbf{U}}^{\dagger}_{ij}}\left(\bar{N},\{p\},\mu_F,\mu_h,\mu_s\right)\;\overset{\text{soft}}{\widetilde{\mathbf{s}}_{ij}\left(\ln\frac{Q^2}{\bar{N}^2\mu_s^2},\mu_s\right)}\right]$$

$$\widetilde{\mathbf{U}}_{ij}\left(\bar{N},\{p\},\mu_F,\boxed{\mu_h,\mu_s}\right) = \exp\left\{\frac{4\pi}{\alpha_s(\mu_h)}g_1\left(\lambda_s,\lambda_f\right)+g_2\left(\lambda_s,\lambda_f\right)+\frac{\alpha_s(\mu_h)}{4\pi}g_3\left(\lambda_s,\lambda_f\right)+\cdots\right\}\mathbf{u}_{ij}\left(\{p\},\mu_h,\mu_s\right)$$

$$\lambda_s \equiv \frac{\boxed{\alpha_s(\mu_h)}}{2\pi}\beta_0\ln\frac{\mu_h}{\mu_s}\,, \qquad \lambda_f \equiv \frac{\boxed{\alpha_s(\mu_h)}}{2\pi}\beta_0\ln\frac{\mu_h}{\mu_F}$$

3 intrinsic scales in the formalism: $\mu_F, \mu_h\,,\ \mu_s$

- When matching, fixed-order results only for $\mu_R = \mu_F$ used; errors estimated by varying $\mu_F, \mu_h\,,\ \mu_s$ independently by factors of 2 and adding upper and lower variations in quadrature

THE SCET APPROACH

[Broggio, Ferroglia,Pecjak, Yang'16]

- In momentum space

$$\sigma\left(S,m_t,m_H\right)=\frac{1}{2S}\int_{\tau_{min}}^{1}d\tau\int_{\tau}^{1}\frac{dz}{\sqrt{z}}\sum_{ij}ff_{ij}\left(\frac{\tau}{z},\mu\right)\int d\mathrm{PS}_{t\bar{t}H}\mathrm{Tr}\left[\mathbf{H}_{ij}\left(\{p\},\mu\right)\mathbf{S}_{ij}\left(\frac{Q(1-z)}{\sqrt{z}},\mu\right)\right]$$

$$\tau_{\min}=\frac{\left(2m_t+m_H\right)^2}{S},\qquad \tau=\frac{Q^2}{S}$$

→ reintroduce μ_R in the SCET approach

$$\alpha_s(\mu_h)=\frac{\alpha_s(\mu_R)}{X}\left[1-\frac{\alpha_s(\mu_R)}{4\pi}\frac{\beta_1}{\beta_0}\frac{\ln X}{X}+\left(\frac{\alpha_s^2(\mu_R)}{4\pi}\right)^2\left(\frac{\beta_1^2}{\beta_0^2}\frac{\ln^2 X-\ln X-1}{X^2}+\frac{\beta_2}{\beta_0}\frac{1-X}{X}\right)+\cdots\right]$$

$$X=1-\frac{\alpha_s(\mu_R)}{2\pi}\beta_0\ln\frac{\mu_R}{\mu_h}$$

→ re-expand perturbatively in $\alpha_s(\mu_R)$ treating $\alpha_s(\mu_R)\ln\left(\mu_i/\mu_j\right)\sim\mathcal{O}(1)$

→ choose $\mu_s=Q/\bar{N}$, calculate the corrections for the same μ_R, μ_F choices as in the fixed-order for both $\mu_h=\mu_F$ and $\mu_h=\mu_R$ and estimate the scale uncertainty from the envelope over results for

$$(\mu_F/\mu_0,\mu_R/\mu_0,\mu_h/\mu_0)\in\{(1,1,1),(2,1,2),(2,1,1),(1/2,1,1/2),(1/2,1,1),(1,2,1),$$
$$(1,2,2),(1,1/2,1/2),(1,1/2,1),(2,2,2),(1/2,1/2,1/2)\}$$

RESIDUAL DIFFERENCES

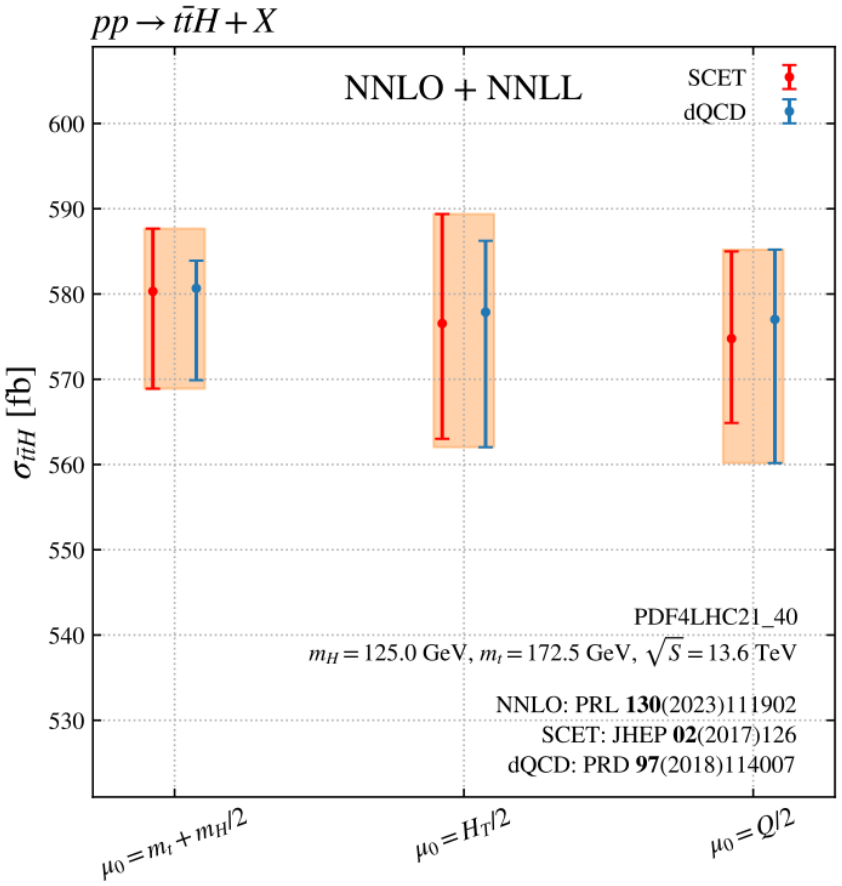
- Intrinsic remaining differences between the two formalisms:
  - set of scales: μ_F, μ_R (dQCD) vs μ_F, μ_R , μ_h (SCET)
  - dQCD: exponentials are functions of $\alpha_s(\mu_R)\ln(N)$, SCET: exponentials are functions of $\alpha_s(\mu_R)\ln(\mu_i/\mu_j)$ with $\mu_{i(j)}$ being one of the four scales μ_F, μ_R , $\mu_h, \mu_s = Q/\bar{N}$

- Differences due to choices made in the implementation:
  - dQCD convention: only terms that vanish in the limit $\lambda \to 0$ included in the exponential
  - approximation in SCET: $\exp[\alpha_s(\mu_r)\, g_3(\lambda)] \to 1 + \alpha_s(\mu_r)\, g_3(\lambda)$

- We have checked analytically that the two approaches agree up to NNLL

- The expressions used for numerical evaluation differ by N^3 LL terms → indication of the size of corrections beyond the formal accuracy of the NNLL approximation

COMPARISON



- Results for central scale choices agree within a few permille → O(N^3LL) effects negligible
- First comparison of SCET- and dQCD-resummed results at NNLL accuracy for a process with four coloured partons at the Born level

[NNLL dQCD: Balsach, AK, Motyka, Stebel] [NNLL SCET: Broggio, Ferroglia, Pecjak] [NNLO: Devoto, Grazzini, Kallweit, Mazzitelli, Savoini]

COMBINATION OF ALL CORRECTIONS

- Matching of NNLL to NNLO

$$\sigma^{\rm dQCD}_{\rm NNLO+NNLL} = \sigma_{\rm NNLO} + \sigma^{\rm dQCD}_{\rm NNLL} - \sigma^{\rm dQCD}_{\rm NNLL}\Big|_{\alpha_s^2} \qquad \sigma^{\rm SCET}_{\rm NNLO+NNLL} = \sigma_{\rm NNLO} + \sigma^{\rm SCET}_{\rm NNLL} - \sigma^{\rm SCET}_{\rm NNLL}\Big|_{\alpha_s^2}$$

- Central NNLO+NNLL value taken as an average of values obtained in the two resummation formalisms:

$$\sigma_{\rm NNLO+NNLL} = \frac{\sigma^{\rm SCET}_{\rm NNLO+NNLL} + \sigma^{\rm dQCD}_{\rm NNLO+NNLL}}{2}$$

- Uncertainties determined from the envelope over the dQCD and SCET scale variation error bands
  - In this way, the uncertainties do not only account for scale variation, but also for O(N^{3}LL) intrinsic differences between the two formalisms

- NNLO+NNLL with complete EW corrections

$$\sigma_{\rm NNLO+NNLL+EW} = \sigma_{\rm NNLO+NNLL} + \sum_{i=2}^{3} \sigma_{\rm LO,i} + \sum_{j=2}^{4} \sigma_{\rm NLO,j}$$

$\mathcal{O}(\alpha_s^2\alpha^2), \mathcal{O}(\alpha_s\alpha^3), \mathcal{O}(\alpha^4)$

$\mathcal{O}(\alpha_s\alpha^2), \mathcal{O}(\alpha^3)$

IMPACT OF THE NNLL CORRECTIONS

$\mu_R = \mu_F = m_t + m_H/2$

$\sqrt{S}$ [TeV]	m_H [GeV]	$\sigma_{\rm NNLO+EW}$ [fb]$^{+[\%]}_{-[\%]}$	$\sigma_{\rm NNLO+NNLL+EW}$ [fb]$^{+[\%]}_{-[\%]}$
13.0	124.60	$533.8^{+1.1}_{-3.2}$	$534.2^{+1.6}_{-2.2}$
13.0	125.00	$530.2^{+1.2}_{-3.3}$	$530.6^{+1.7}_{-2.3}$
13.0	125.09	$528.4^{+1.1}_{-3.2}$	$528.8^{+1.6}_{-2.2}$
13.0	125.38	$524.5^{+1.1}_{-3.2}$	$524.9^{+1.6}_{-2.2}$
13.0	125.60	$521.7^{+1.1}_{-3.2}$	$522.1^{+1.6}_{-2.2}$
13.0	126.00	$517.2^{+1.1}_{-3.2}$	$517.6^{+1.6}_{-2.2}$
13.6	124.60	$598.9^{+1.0}_{-3.1}$	$599.3^{+1.5}_{-2.2}$
13.6	125.00	$592.0^{+0.9}_{-3.1}$	$592.5^{+1.4}_{-2.1}$
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13.6	125.38	$588.4^{+1.0}_{-3.1}$	$588.8^{+1.5}_{-2.2}$
13.6	125.60	$585.6^{+1.0}_{-3.2}$	$586.0^{+1.6}_{-2.2}$
13.6	126.00	$580.1^{+1.1}_{-3.2}$	$580.5^{+1.6}_{-2.2}$
14.0	124.60	$642.1^{+0.8}_{-3.0}$	$642.6^{+1.3}_{-2.1}$
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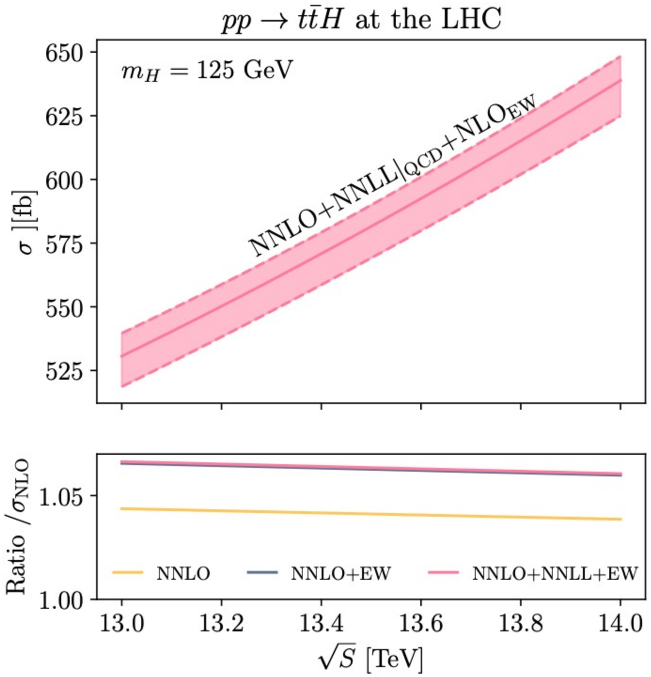
14.0	125.38	$634.6^{+1.0}_{-3.2}$	$635.1^{+1.6}_{-2.2}$
14.0	125.60	$630.2^{+1.0}_{-3.1}$	$630.7^{+1.5}_{-2.2}$
14.0	126.00	$623.6^{+0.9}_{-3.1}$	$624.1^{+1.5}_{-2.2}$



central predictions changed only at the level of 1 permille or below

IMPACT OF THE NNLL CORRECTIONS

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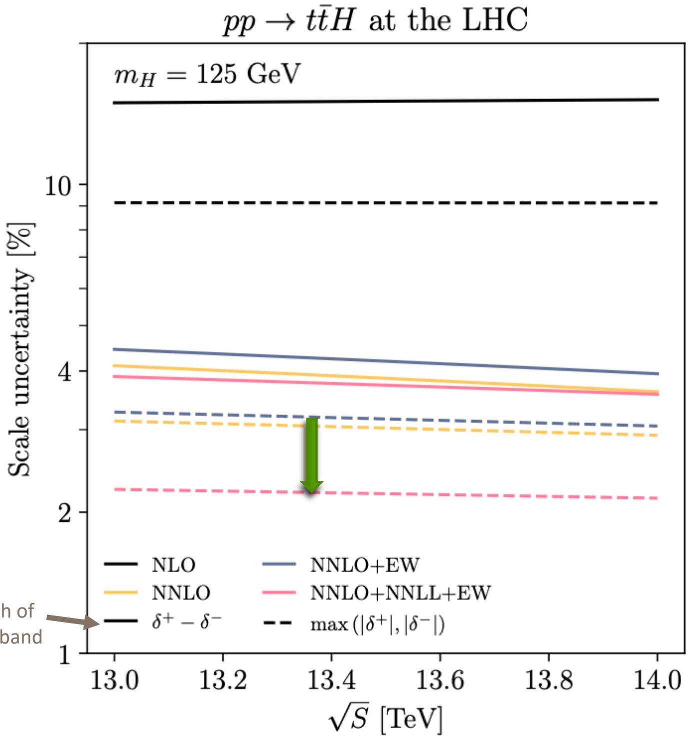
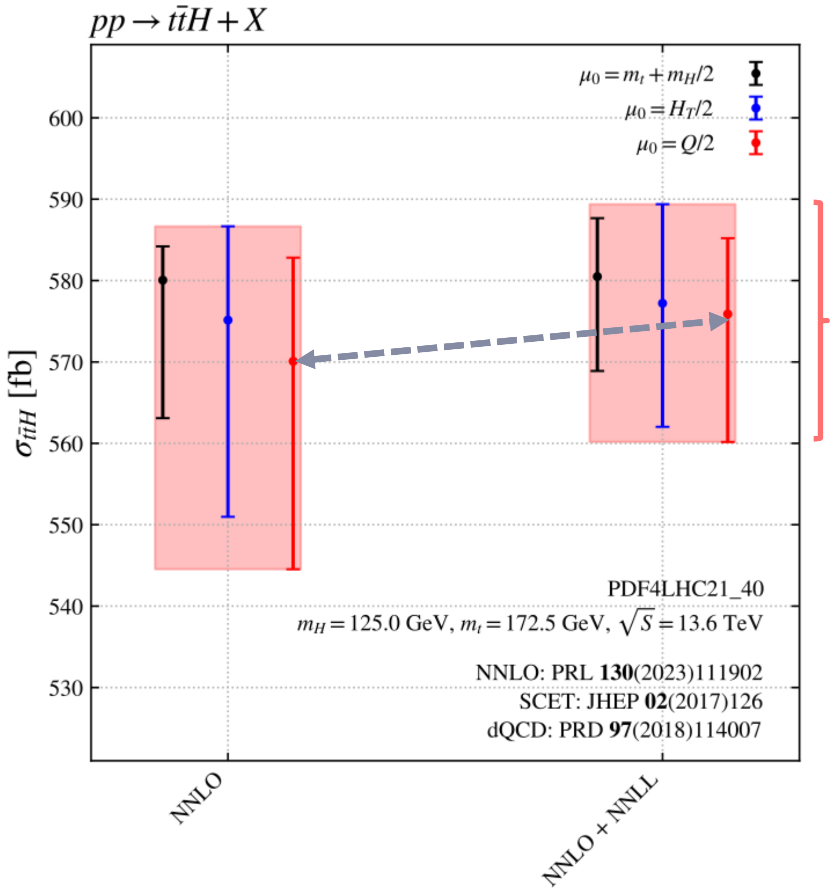


total width of the error band

size of the scale variation error (w.r.t. the central value) down to less than or around 2%

IMPACT OF THE NNLL CORRECTIONS



remarkable improvement in the stability of the predictions w.r.t. central scale choice

Note: while for the default scale choice $\mu_F = \mu_R = m_t + m_H/2$ NNLL corrections negligible for other scale choices corrections are bigger

THEORY PREDICTIONS AND UNCERTAINTIES

- Apart from the scale error $\Delta_\mu \sim 2\%$, other notable sources of theory uncertainties are due to:

$\Delta_{PDF} = 2.2\%$	PDFs; estimated according to the PDF4LHC prescription.
$\Delta_{\alpha_s} = 1.7\% \dfrac{\delta\alpha_s}{0.001}$	PDF4LHC prescription: variation of $\alpha_s(m_Z)$ by 0.001 w.r.t. $\alpha_s(m_Z)$ =0.118
$\Delta_{virt} = 0.9\%$	approximation of the 2-loop virtual contribution to the NNLO correction

- Other sources such as: *numerical uncertainties, resummation approach,* m_t *value, renormalization scheme, higher order EW corrections* are estimated subleading (at most a few permille)

- All uncertainties have a negligible dependence on m_H and $\sqrt{S}$ in the range considered

State-of-the-art predictions:

$$\sigma^{\sqrt{S}=13.6\,\mathrm{TeV},\,m_H=125.09\,\mathrm{GeV}}_{\mathrm{NNLO+NNLL+EW}} = 592.1\,\mathrm{fb}\;\underbrace{{}^{+1.5\%}_{-2.2\%}}_{\Delta_\mu}\;\underbrace{\pm 2.2\%}_{\Delta_{\mathrm{PDF}}}\;\underbrace{\pm 1.7\%}_{\Delta_{\alpha_s}}\;\underbrace{\pm 0.9\%}_{\Delta_{\mathrm{virt}}}$$

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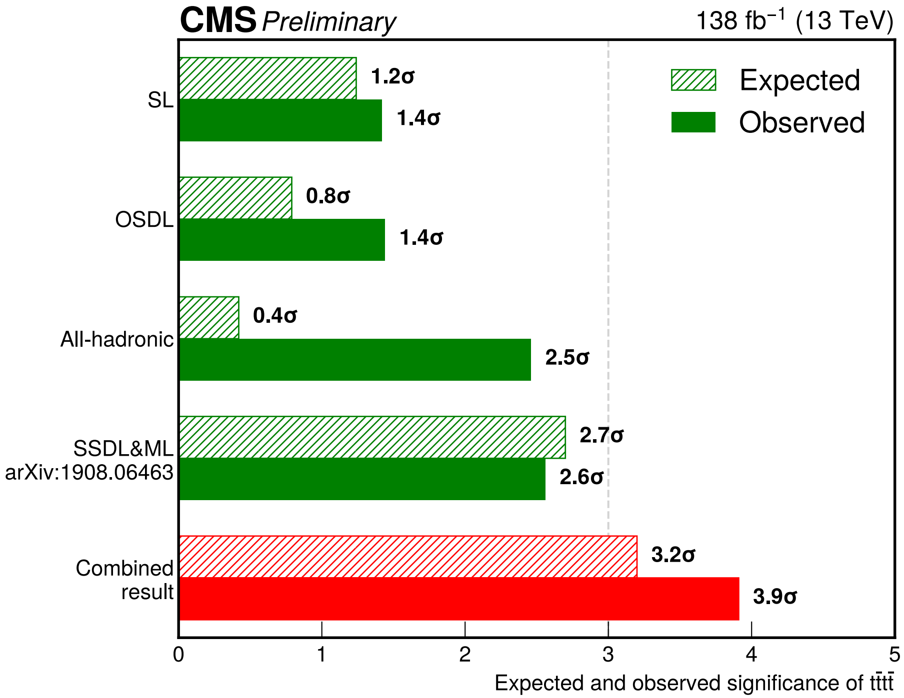


$t\bar{t}t\bar{t}$

production

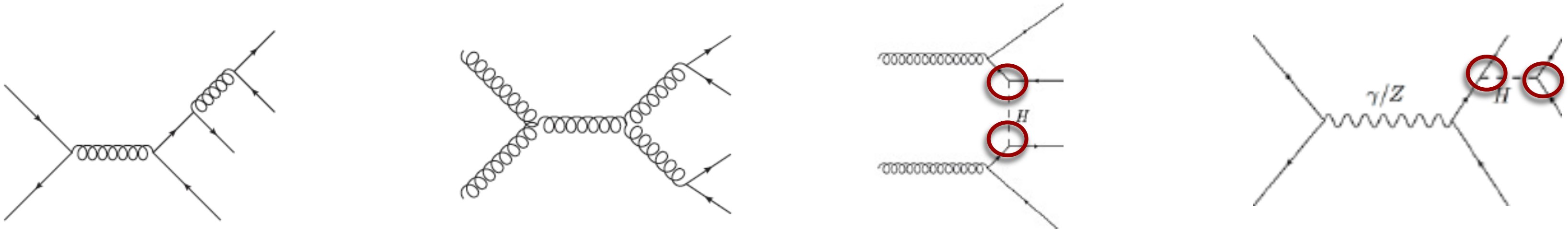
with

M. Lupattelli (Münster)

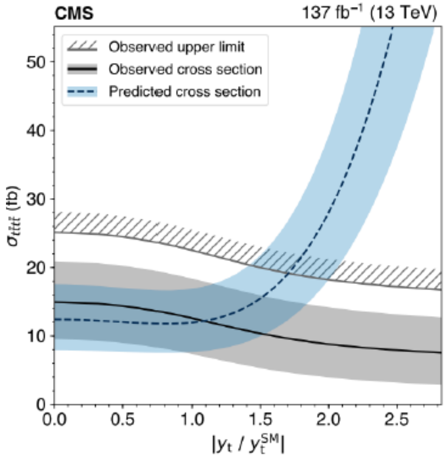
M. van Beekveld (NIKHEF) and

T. Saracco (NIKHEF)

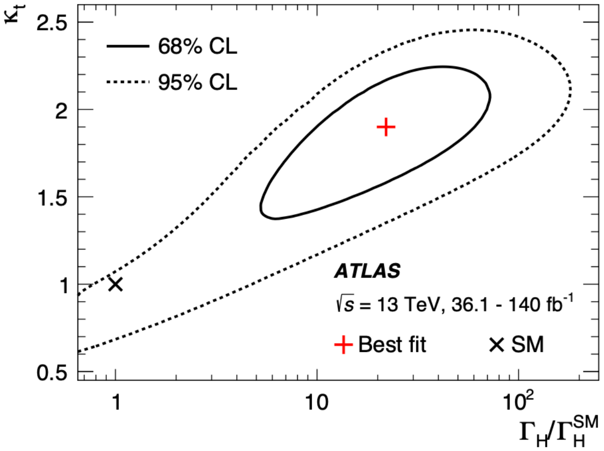
4-TOPS AS A PROBE



- Sensitive to the top Yukawa coupling *[Cao, Chen, Liu'16] [Cao, Chen, Liu, Zhang, Zhang'19]* and the Higgs width *[ATLAS'25]*

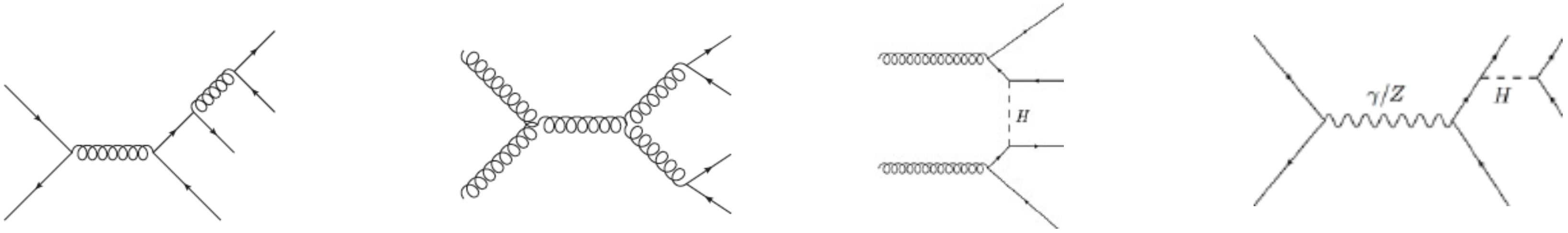


[CMS Collaboration , JHEP 11 (2021) 118]



[ATLAS Collaboration , Phys.Lett.B 861 (2025) 139277]

4-TOPS AS A PROBE



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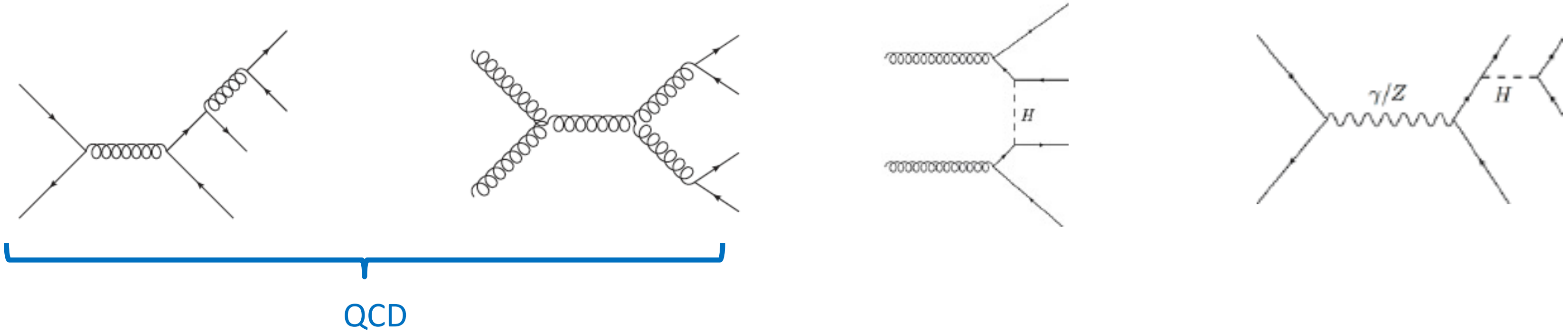
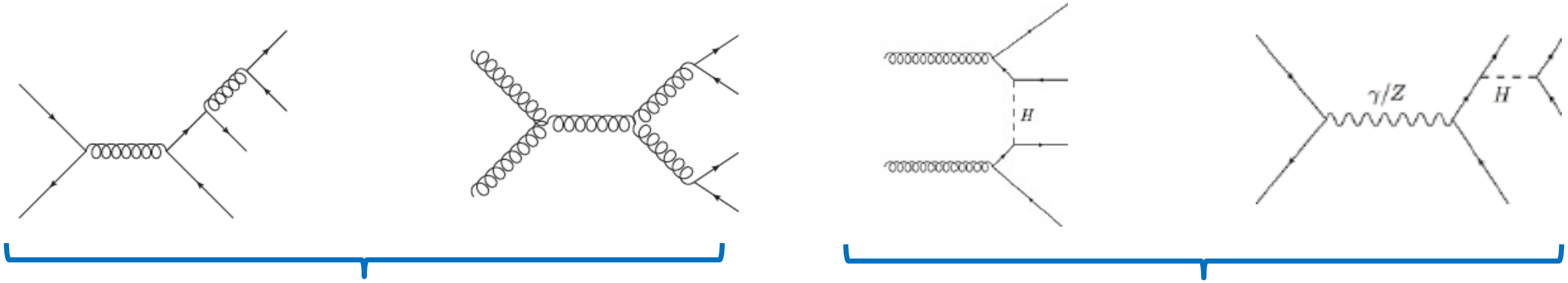
- Probe of BSM physics: gluino and sgluon production *[Farrar,Fayet'78] [Plehn, Tait'08] [Calvet, Fuks, Gris, Valery'13] [Benakli et al.'14][Beck et al.'15][Darme, Fuks, Goodsell'18], [Carpenter, Murphy, Smylie'20], [Carpenter, Murphy'20],* extended Higgs sectors *[Dicus, Stange, Willenbrock'94] [Craig et al.'15] [Craig et al.'16] [Coloretti, Crivellin,Melado'24] [Anisha et al.'23]* , top-quark composite models *[Pomarol, Serra'08] [Kumar, Tait, Vega-Morales'09] [Cacciapaglia et al'15][Belayev et al'16][Liu, Wang, Xie'19]* Z' *[Ducu, Heurtier, Maurer'16],* top-philic heavy resonances *[Darme, Fuks, Maltoni'21], [Darme et al'24'25]*
- Sensitive to four-fermion operators in SMEFT *[Degrande, Gerard, Grojean, Maltoni, Servant'11][Zhang'17],[Barducci et al.'18], [Banelli, Salvioni, Serra, Theil, Weiler'21][Hartland et al. '21]][Darme, Fuks, Maltoni'21],[SMEFiT Collaboration'21][Aoude, El Faham, Maltoni, Vrynidou'22]*

FIXED-ORDER THEORY PREDICTIONS



- First calculations of NLO QCD corrections in *[Bevilacqua, Worek'12]*
- Matched with parton shower and studied in aMC@NLO *[Alwall et al. '14][Maltoni, Pagani, Tsinikos'15]*

FIXED-ORDER THEORY PREDICTIONS



QCD + EW

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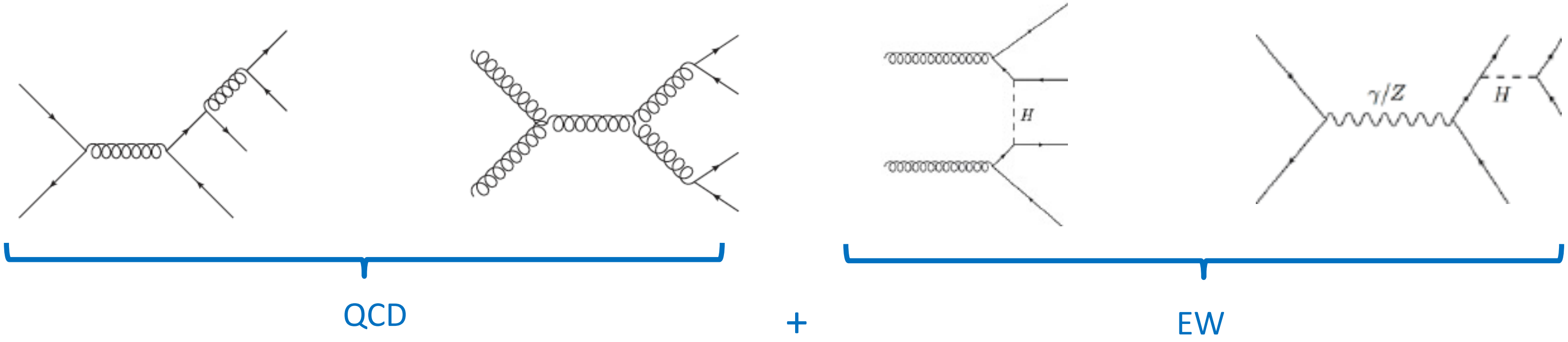
- Full set of EW corrections added in *[Frederix, Pagani, Zaro'17]*
- Spin correlations in LO top quark decays within the framework of Powheg Box *[Jezo, Krauss'21]*

➡ NLO (QCD+EW) predictions used for comparisons with data *[Frederix, Pagani, Zaro'17]*

$$\sigma^{\mathrm{NLO(QCD+EW)}}_{t\bar{t}t\bar{t}} = 12.0^{+2.2}_{-2.3}\ \mathrm{fb}$$

$$\mathrm{K}^{\mathrm{NLO(QCD+EW)}} = 1.75 \qquad \mathrm{K}^{\mathrm{NLO(QCD)}} = 1.63$$

FIXED-ORDER THEORY PREDICTIONS



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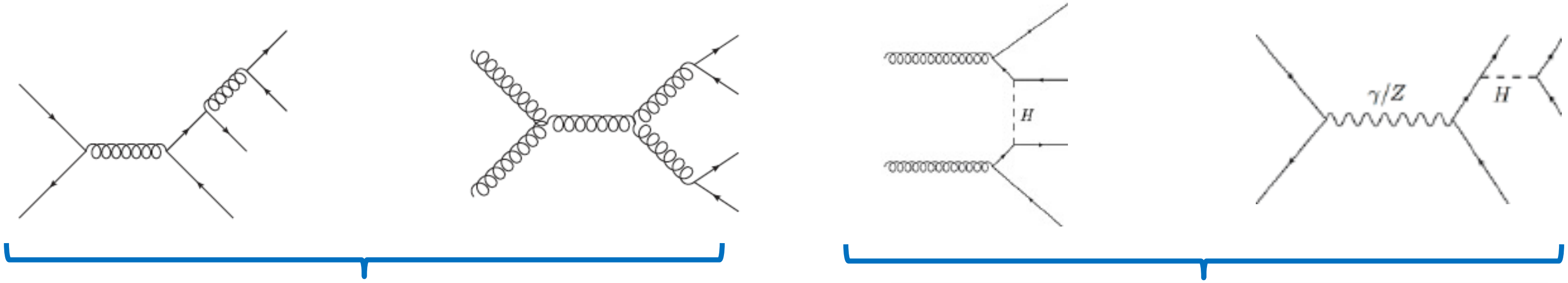
- Spin correlations in LO top quark decays within the framework of Powheg Box *[Jezo, Krauss'21]*

➡ NLO (QCD+EW) predictions used for comparisons with data *[Frederix, Pagani, Zaro'17]*

$$\sigma_{t\bar{t}t\bar{t}}^{\mathrm{NLO(QCD+EW)}} = 12.0^{+2.2}_{-2.3}\ \mathrm{fb} \qquad K^{\mathrm{NLO(QCD+EW)}} = 1.75 \qquad K^{\mathrm{NLO(QCD)}} = 1.63$$

ca. 20 % theory error ↔ exp. uncertainty

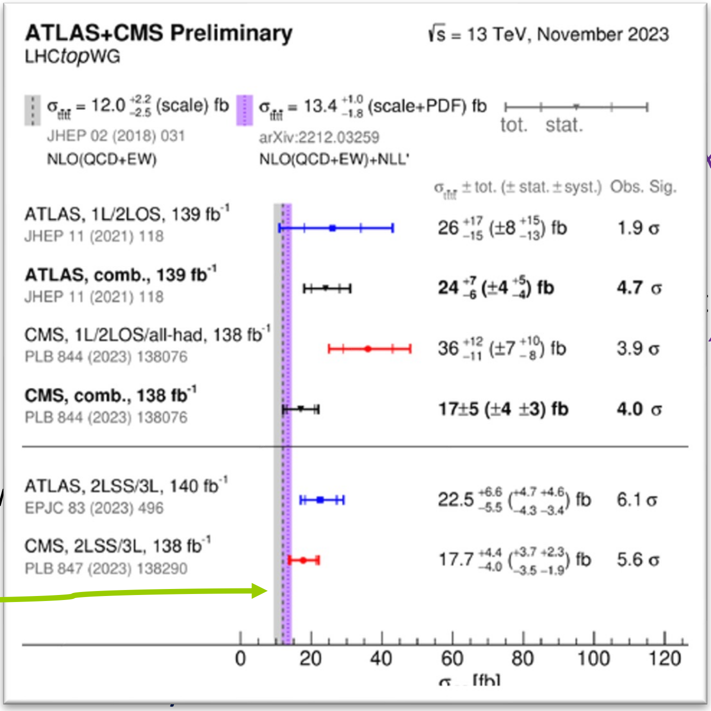
FIXED-ORDER THEORY PREDICTIONS



QCD

+

- First calculations of NLO QCD corrections in *[Bevilacqua, Worek’12]*
- Matched with parton shower and studied in aMC@NLO *[Alwall et al. ’14][Maltoni, Pagani, Tsinikos’15]*
- *x,*
- hin *’21]*



NLO (QCD+EW) predictions used for comparisons v

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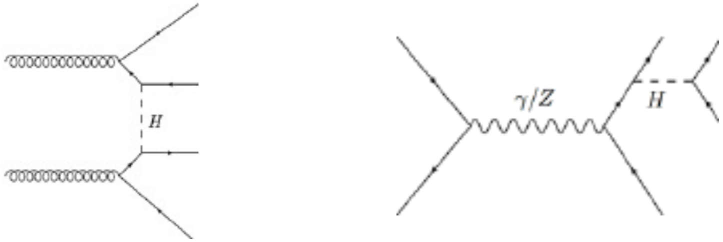

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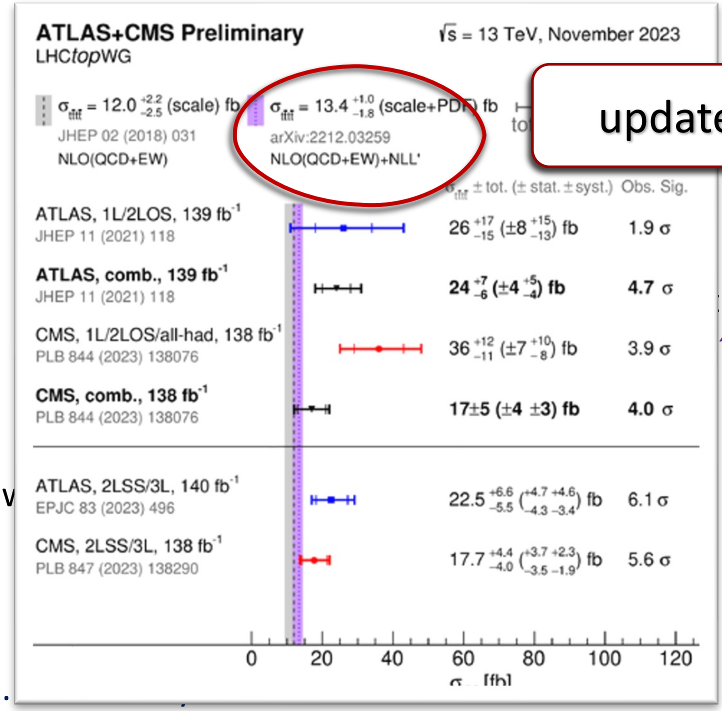

update

hin

'21]

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INVARIANT MASS VS ABSOLUTE MASS THRESHOLD RESUMMATION

- Direct QCD framework
- Factorization of phase-space at threshold occurs in space of Mellin moments taken w.r.t.

$$d\hat{\sigma}_{t\bar{t}t\bar{t}}(N,\dots) = \int_0^1 d\hat{\rho}\ \hat{\rho}^{N-1}\ d\hat{\sigma}_{t\bar{t}t\bar{t}}(\hat{\rho},\dots) \qquad \rightarrow \qquad L = \log N$$

Absolute mass threshold (AMT) resummation	Invariant mass threshold (IMT) resummation
$\hat{\rho} = \frac{(4\,m_t)^2}{\hat{s}}$	$\hat{\rho} = \frac{Q^2}{\hat{s}}$
2212.03259	2505.10381

applicable only to the total cross section
adequate to choose central scale
proportional to 4 m_t

enables to calculate invariant mass distribution
and the total cross section

justified to use dynamical scales such Q or H_T

4TOP RESUMMATION

$$d\sigma^{\rm res}_{ij\to t\bar{t}t\bar{t}}(N,\mu_F,\mu_R) = {\rm Tr}\left[\mathbf{H}_{ij\to t\bar{t}t\bar{t}}(\mu_F,\mu_R)\mathbf{S}_{ij\to t\bar{t}t\bar{t}}(N,\mu_R)\right]\Delta_i(N,\mu_F,\mu_R)\Delta_j(N,\mu_F,\mu_R)$$

- Processes with four or more legs carrying colour: hard and soft functions are matrices in colour space ↔ soft radiation sensitive to the overall colour structure of the Born process

$q\bar{q}$ channel: 6-dimensional colour space

gg channel: 14-dimensional colour space

↪ same for AMT and IMT resummations

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↪ same for AMT and IMT resummations

- The soft function, as a solution of an RG equation, contains the evolution factors

$$\mathbf{U}_{ij\to t\bar{t}t\bar{t}} = P\exp\left[\frac{1}{2}\int_{\mu_R^2}^{(4m_t)^2/\bar{N}^2}\frac{dq^2}{q^2}\mathbf{\Gamma}_{ij\to t\bar{t}t\bar{t}}(\alpha_s(q^2))\right]$$

up to NLL $\mathbf{\Gamma} = \left(\frac{\alpha_s}{\pi}\right)\mathbf{\Gamma}^{(1)}$

calculation requires diagonalization of the soft anomalous dimension matrix

↪ AMT: just once

↪ IMT: needs to be done for all phase-space points

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↪ IMT: needs to be done for all phase-space points

- NLL accuracy can be supplemented by including perturbative information from $\mathbf{H}^{(1)}$ and $\mathbf{S}^{(1)}$ → NLL' accuracy

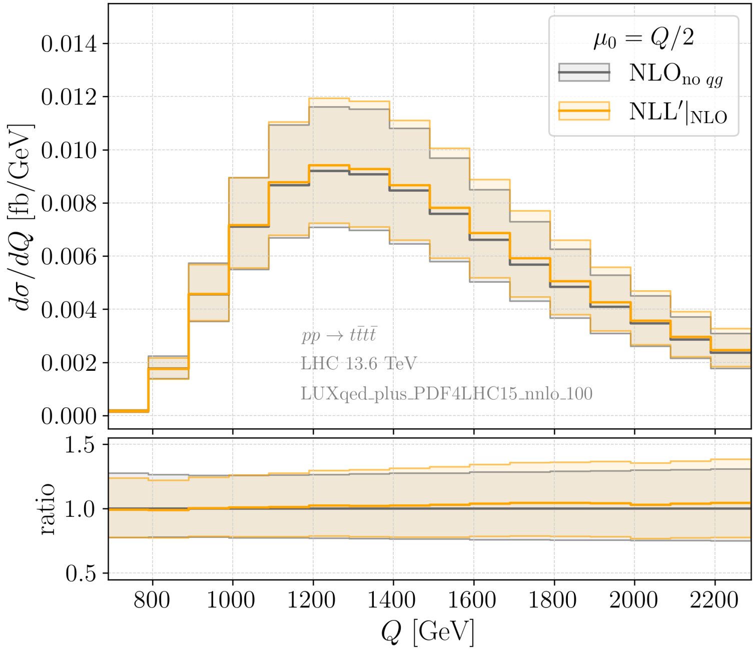
$${\rm Tr}\left[\mathbf{H}\,\mathbf{S}\right] = {\rm Tr}\left[\mathbf{H}^{(0)}\mathbf{S}^{(0)} + \frac{\alpha_s}{\pi}\left(\mathbf{H}^{(0)}\mathbf{S}^{(1)} + \mathbf{H}^{(1)}\mathbf{S}^{(0)}\right)\right]$$

$$\mathbf{H} = \mathbf{H}^{(0)} + \left(\frac{\alpha_s}{\pi}\right)\mathbf{H}^{(1)} + \ldots$$

$\mathbf{H}^{(1)}$ includes one-loop virtual contributions (extracted from OpenLoops *[Buccioni et al.'19]*)

IMT: EXPANSION OF NLL' VS NLO

[van Beekveld, AK, Lupattelli, Saracco'25]



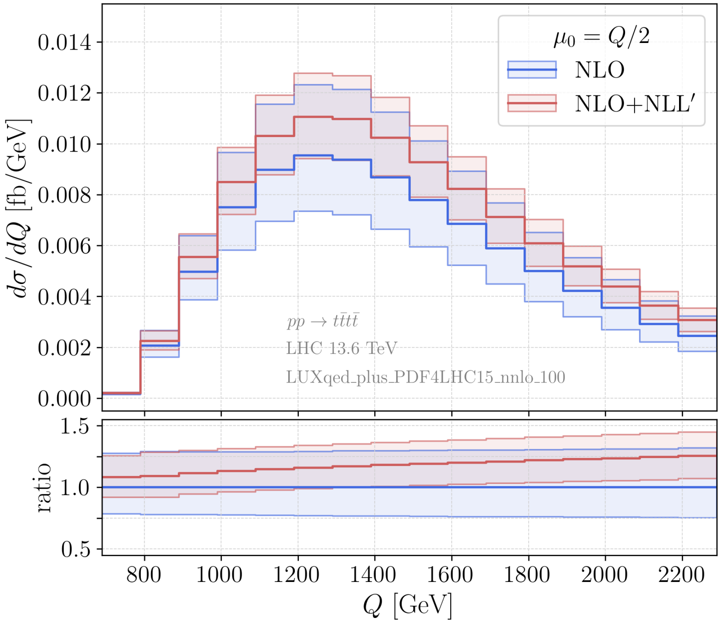
μ_0	NLO_{QCD} [fb]	$\text{NLO}_{\text{no }qg}$ [fb]	$\text{NLL}'\vert_{\text{NLO}}$ [fb]
$M/2$	$13.13^{+25.2\%}_{-24.5\%}$	$13.05^{+20.2\%}_{-21.1\%}$	$13.45^{+21.6\%}_{-21.9\%}$
$Q/2$	$9.38^{+33.3\%}_{-25.8\%}$	$9.77^{+28.1\%}_{-23.9\%}$	$9.92^{+28.7\%}_{-24.1\%}$
$H_T/2$	$10.88^{+32.3\%}_{-25.8\%}$	$11.22^{+26.0\%}_{-23.7\%}$	$11.44^{+27.0\%}_{-24.0\%}$

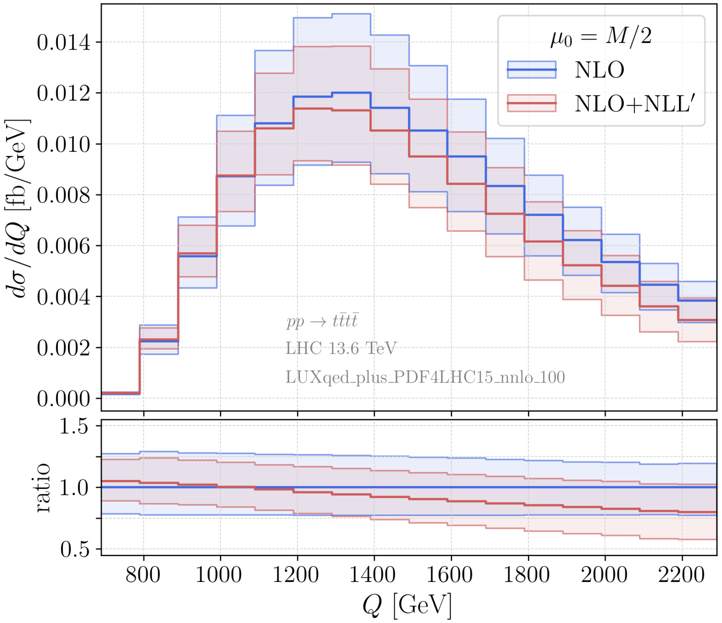
The shown errors are the 7-point scale variation errors

- NLL' expanded up to $\mathcal{O}(\alpha_s^5)$ returns a very good approximation of the NLO(no qg), i.e. contributions from $q\bar{q}$ and gg channels

INV. MASS DISTRIBUTION AT NLO+NLL’

[van Beekveld, AK, Lupattelli, Saracco’25]

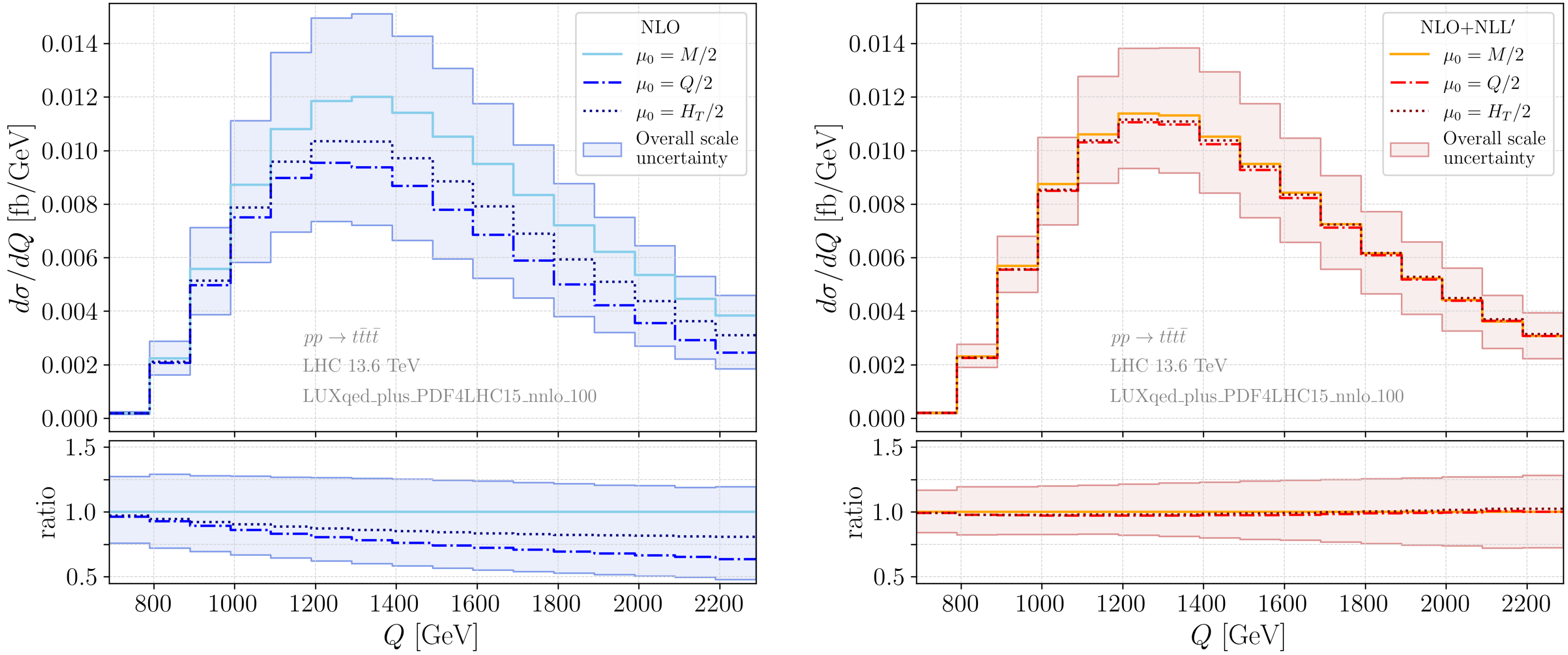




- NLL’ corrections can substantially change the shape. The effect depends on the central scale choice.

INV. MASS DISTRIBUTION AT NLO+NLL′

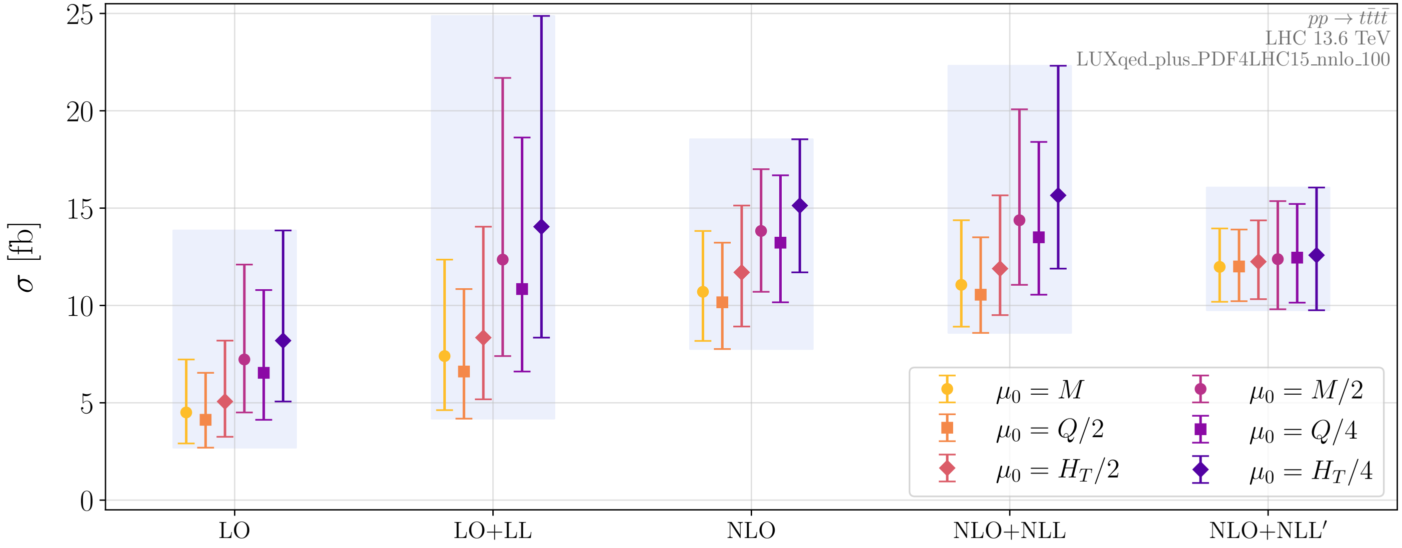
[van Beekveld, AK, Lupattelli, Saracco'25]



- Soft gluon corrections significantly reduce the dependence on the central scale choice, as well as individual scale variation errors for each central scale choice

IMT: TOTAL XSECTION AT NLO+NLL'

[van Beekveld, AK, Lupattelli, Saracco'25]

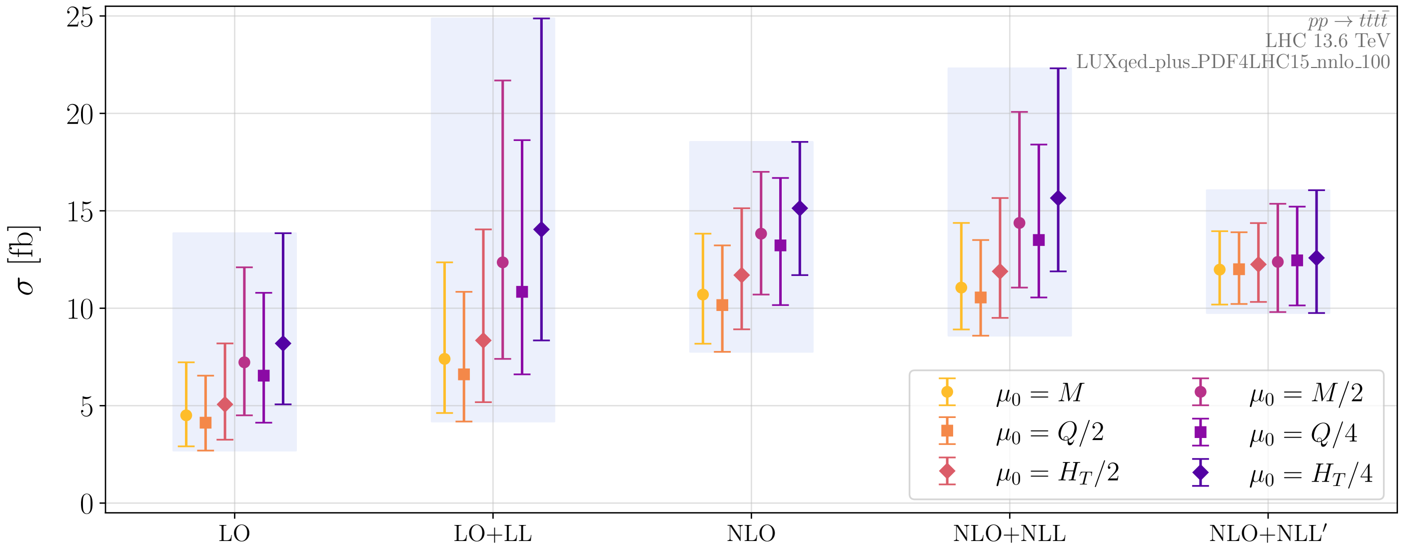


$\sqrt{S}$ [TeV]	μ_0	NLO [fb]	NLO+NLL [fb]	$\mathcal{K}^{\mathrm{NLL}}$	NLO+NLL' [fb]	$\mathcal{K}^{\mathrm{NLL}'}$
13.6	$M/2$	$13.83^{+23.0\%}_{-22.6\%}$	$14.38^{+39.6\%}_{-23.1\%}$	1.04	$12.38^{+24.1\%}_{-20.8\%}$	0.90
	$Q/2$	$10.16^{+30.1\%}_{-23.6\%}$	$10.55^{+27.9\%}_{-18.6\%}$	1.04	$12.00^{+15.8\%}_{-14.9\%}$	1.18
	$H_T/2$	$11.70^{+29.3\%}_{-23.8\%}$	$11.89^{+31.7\%}_{-20.1\%}$	1.02	$12.25^{+17.3\%}_{-15.7\%}$	1.05

- NLL' resummation substantially reduces the dependence on the central scale choice and individual scale errors.

IMT: TOTAL XSECTION AT NLO+NLL'

[van Beekveld, AK, Lupattelli, Saracco'25]



$\sqrt{S}$ [TeV]	μ_0	NLO [fb]	NLO+NLL [fb]	$\mathcal{K}^{\mathrm{NLL}}$	NLO+NLL$'$ [fb]	$\mathcal{K}^{\mathrm{NLL}'}$
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- NLL' resummation substantially reduces the dependence on the central scale choice and individual scale errors. The size of the corrections depends on the scale choice.

Photons And Soft Emission

with

R. Balsach (DESY) and D. Bonocore (TUM)

SOFT EMISSION

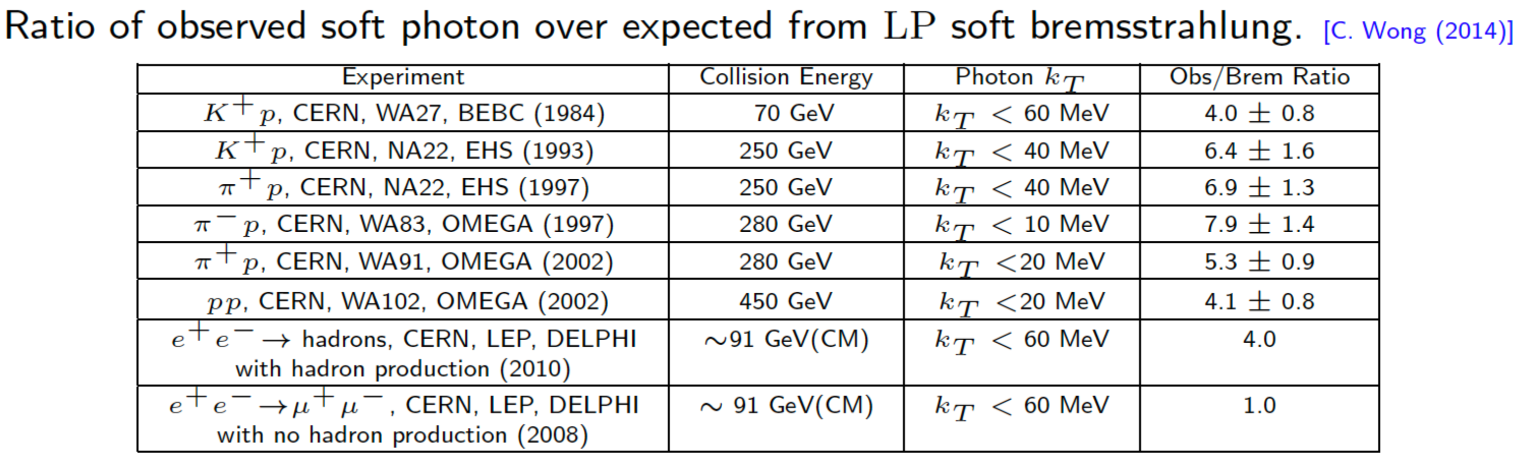
- Unobserved soft radiation: taken care by resummation (to all orders in perturbation theory), or only up to a certain order
- Inclusive approach (radiation "integrated out"), focus on the underlying process

- Observed radiation: soft photon bremsstrahlung spectrum
- Calculated in the soft approximation or fully up to a certain order in perturbation theory

SOFT PHOTONS

- ALICE3, the planned upgrade of the ALICE detector (~2035) is expected to be able to measure soft photons up to 1 MeV

- A long-standing soft photon puzzle:



Ratio of observed soft photon over expected from LP soft bremsstrahlung. [C. Wong (2014)]

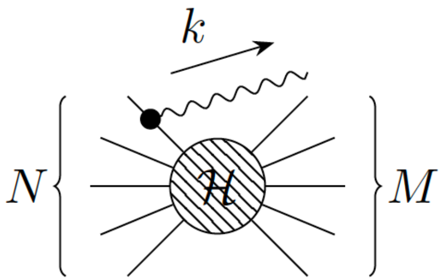
Experiment	Collision Energy	Photon k_T	Obs/Brem Ratio
K^+p, CERN, WA27, BEBC (1984)	70 GeV	$k_T < 60$ MeV	4.0 ± 0.8
K^+p, CERN, NA22, EHS (1993)	250 GeV	$k_T < 40$ MeV	6.4 ± 1.6
π^+p, CERN, NA22, EHS (1997)	250 GeV	$k_T < 40$ MeV	6.9 ± 1.3
π^-p, CERN, WA83, OMEGA (1997)	280 GeV	$k_T < 10$ MeV	7.9 ± 1.4
π^+p, CERN, WA91, OMEGA (2002)	280 GeV	k_T <20 MeV	5.3 ± 0.9
pp, CERN, WA102, OMEGA (2002)	450 GeV	k_T <20 MeV	4.1 ± 0.8
$e^+e^- \to$ hadrons, CERN, LEP, DELPHI with hadron production (2010)	$\sim$91 GeV(CM)	$k_T < 60$ MeV	4.0
$e^+e^- \to \mu^+\mu^-$, CERN, LEP, DELPHI with no hadron production (2008)	$\sim$ 91 GeV(CM)	$k_T < 60$ MeV	1.0

see also R.Bailhache et al. 2406.17959

- Current simulations only use the leading-power approximation: precision predictions for the soft photon spectra?

EIKONAL APPROXIMATION

Emission of a soft photon from a general process $N \to M + \gamma$:



$$\mathcal{A}_n = Q_n \bar{v}(p_n) \not{\varepsilon}^*(k) \frac{\not{k} - \not{p}_n + m}{(p_n - k)^2 - m^2} \mathcal{H}_n(p_1, \ldots, p_n - k, \ldots, p_{N+M})$$

In the limit of soft photons ($k \to 0$) the amplitude behaves like $\mathcal{O}\left(k^{-1}\right)$

leading power (LP)

$$\mathcal{A}_n^{\mathrm{LP}}(p, k) = \pm Q_n \frac{p_n \cdot \varepsilon^*(k)}{p_n \cdot k} \mathcal{H}(p)$$

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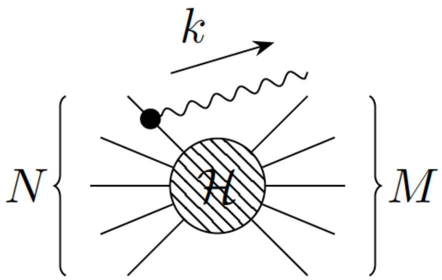

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In the limit of soft photons ($k \to 0$) the amplitude beh

$$\mathcal{A}_n^{\mathrm{LP}}(p,k) = \pm Q_n \frac{p_n \cdot \varepsilon^*(k)}{p_n \cdot k} \mathcal{H}(p)$$

$$\frac{1}{p_n \cdot k} = \frac{1}{k_0 E_n (1 - \cos\theta)}$$

LBK THEOREM

- Next-to-leading power in *k* : Low-Burnett-Kroll theorem *[Low’58][Burnett and Kroll’68]*

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -\sum_{i,j} \frac{(\eta_i Q_i p_i)\cdot(\eta_j Q_j p_j)}{(p_i\cdot k)(p_j\cdot k)} \left[\underset{\substack{\uparrow\\ \text{LP}}}{1} + \underset{\substack{\uparrow\\ \text{NLP}}}{\frac{(p_j\cdot k)\, p_{i\mu}}{p_i\cdot p_j}} G_j^{\mu\nu}\frac{\partial}{\partial p_j^{\nu}}\right] \overline{|\mathcal{H}|}^2$$

$$G_i^{\mu\nu} = g^{\mu\nu} - \frac{p_i^{\mu} k^{\nu}}{p_i\cdot k}$$

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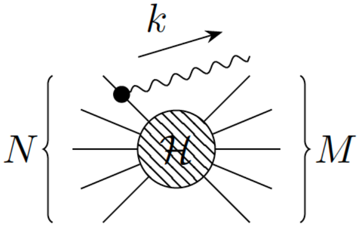
$$G_i^{\mu\nu} = g^{\mu\nu} - \frac{p_i^\mu k^\nu}{p_i\cdot k}$$

- LHS: $\overline{|A(p_{1,\dots,}\, p_{n,}\, k)|}^2$ for $N \to M + \gamma$, RHS: $\overline{|\mathcal{H} p_{1,\dots,}\, p_n\,)|}^2$ for $N \to M$

$\sum_i p_i = k$ ← difference for finite $k \neq 0$ → $\sum_i p_i = 0$



Not possible to impose momentum conservation for both amplitudes simultaneously ↔ relation between physical and unphysical quantities

LBK THEOREM, REVISITED

- Alternative formulation of the LBK theorem proposed *[Lebiedowicz, Nachtmann, Szczurek'21] [Lebiedowicz, Nachtmann, Szczurek'23]*

- It can be shown that any ambiguities due to momentum non-conservation are an NNLP effect *[Bonocore, AK, Balsach'24][Fadin, Khoze'24]*. In consequence, many equivalent forms of the LBK theorem exist, all equivalent up to NNLP

- Preferable version for numerical calculations? -> Unambiguously defined physical momenta / uniquely defined $\mathcal{H}$

LBK WITH SHIFTED KINEMATICS

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \underbrace{\sum_{ij}(-\eta_i\eta_j)\frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}}_{=S_{\mathrm{LP}}}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \to \mathbf{LP}$$

$$+\sum_{ij}(-\eta_i\eta_j)\frac{p^i_\mu}{p_i\cdot k}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \to \mathbf{NLP}$$

- Derivatives are generators of translations $f(x+\epsilon)=f(x)+\epsilon\frac{df(x)}{dx}$ → convert derivatives into small shifts (of order *k*) of the momenta arguments *[del Duca, Laenen, Magnea, Vernazza, White'17][Bonocore, AK'21]*

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \overline{|\mathcal{S}_{\mathrm{LP}}|}^2\,\overline{|\mathcal{H}(p_1+\delta p_1,\ldots,p_n+\delta p_n)|}^2$$

$$\delta p_j^\nu = Q_j\left(\sum_{k,l}\eta_k\eta_l Q_k Q_l\frac{p_k\cdot p_l}{(p_k\cdot k)(p_l\cdot k)}\right)^{-1}\sum_i\left(\frac{\eta_i Q_i p_{i\mu}}{k\cdot p_i}\right)G_j^{\mu\nu}$$

- Momentum conservation restored in the non-radiative amplitude $\mathcal{H}$

$$\sum_j \eta_j\delta p_j^\mu = -k^\mu$$

SHIFTED KINEMATICS: ON-SHELL MOMENTA

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \overline{|\mathcal{S}_{\rm LP}|}^2 \overline{|\mathcal{H}(p_1+\delta p_1,\ldots,p_n+\delta p_n)|}^2$$

- The shifts modify the mass of the particles by NNLP terms.

$$p_j \cdot \delta p_j = 0 \Longrightarrow (p_j + \delta p_j)^2 = m_j^2 + \mathcal{O}(k^2)$$

This is consistent with the approximation, but not ideal for numerical implementations.

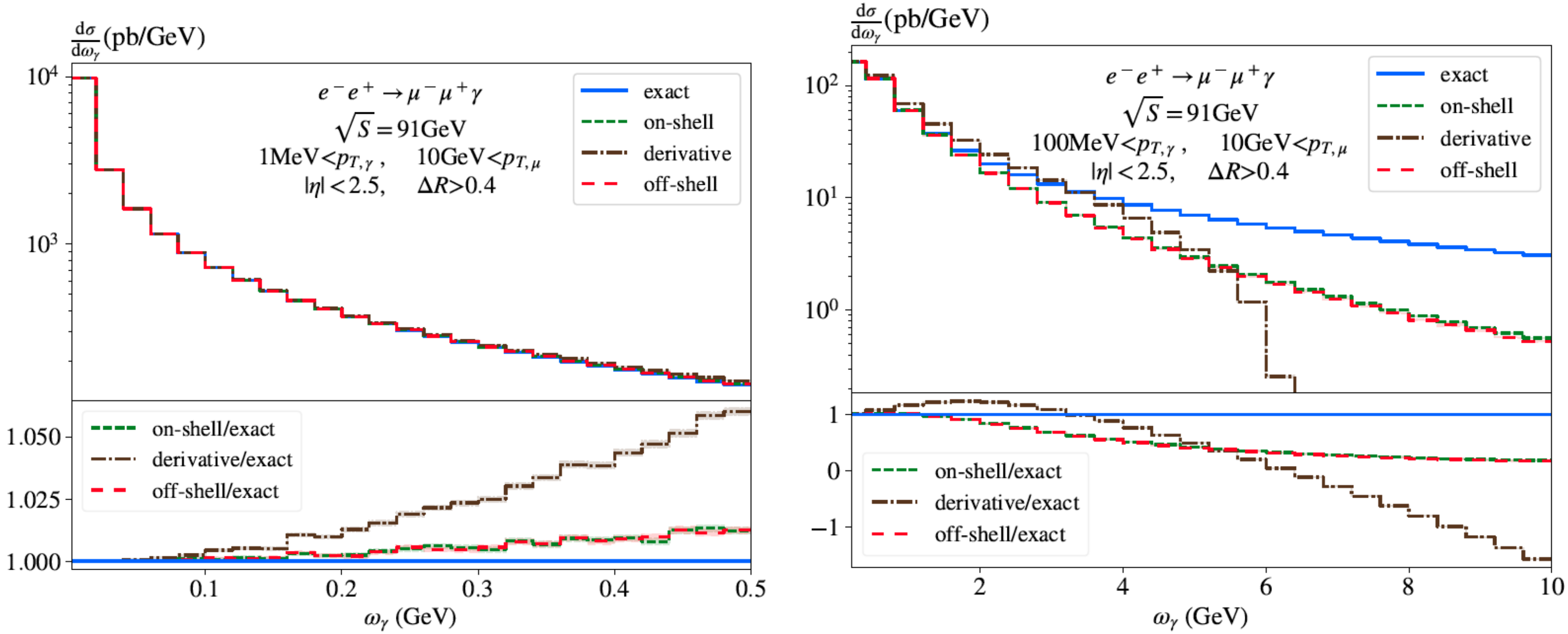
- Possible to find an alternative form of momenta shifts such that it:
  - satisfies 4-momentum conservation
  - uses on-shell momenta (exactly, not only up to NLP) $(p_j+\delta p_j)^2 = m_j^2$.

$$\delta p_i^\mu = A Q_i \sum_j \frac{\eta_j Q_j}{k\cdot p_j} p_{j\nu} G_i^{\nu\mu} + \frac{1}{2}\frac{A^2 Q_i^2 \overline{|\mathcal{S}_{\rm LP}|}^2}{p_i\cdot k} k^\mu$$

[Balsach, Bonocore, AK'23]

QUALITY OF THE NLP APPROXIMATIONS

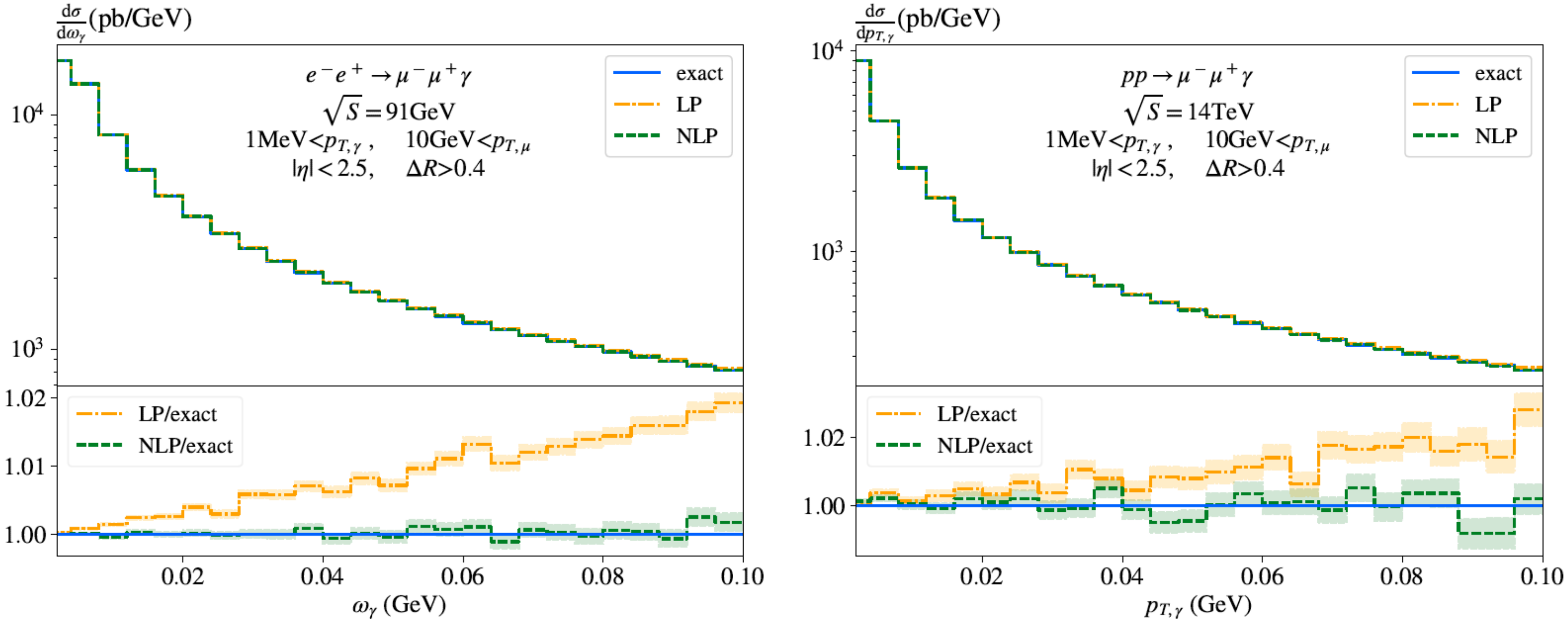
[Balsach, Bonocore, AK’24]



- Test case: $e^+ e^- \to \mu^+ \mu^- \gamma$. Simple analytical expressions for the radiative and non-radiative amplitudes
- “exact” = tree level with no soft expansion
- No discernible difference between result obtained off-shell and on-shell shifted momenta

TEST CASE: $\mu^+\mu^-\gamma$, LP VS NLP VS EXACT

[Balsach, Bonocore, AK'24]



- Dedicated numerical implementation of the NLP approximation
- Results for arbitrary processes, provided the non-radiative amplitude $\mathcal{H}$

SUMMARY

- Control over effects due to soft emissions (QCD and QED) necessary to increase precision of the theoretical predictions, together with calculation of the fixed-order corrections

- Soft gluon corrections play an important role in processes of associated top quark pair-production @ the LHC → most precise predictions for these class of production processes involve resummation and reach N(N)LL+N(N)LO +EW accuracy -- first applications of threshold resummation to a class of 2 → 3 and 2 → 4 processes
  - Remarkable stability of the total and differential and cross sections w.r.t. scale variation
  - Reduction of the theory error due to scale variation with the increasing logarithmic order of the calculations

- New formulation of the LBK theorem using on-shell momenta shifting enables an efficient implementation of the NLP approximation of soft photon spectrum