Improving perturbative calculations for forward physics and saturation

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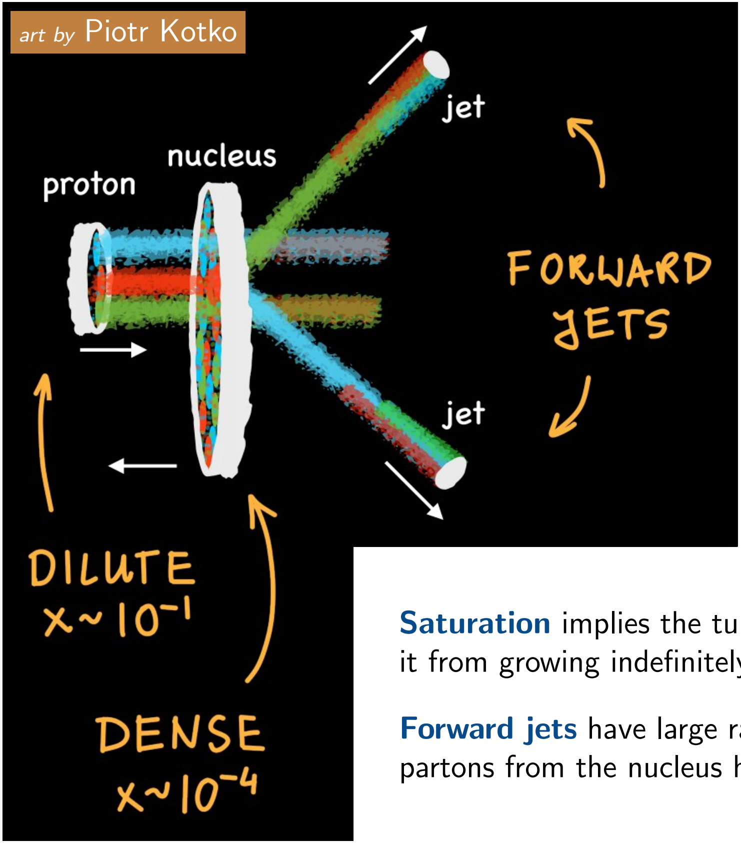
presented at

Joint ECFA-NuPECC-APPEC workshop "Synergies between EIC and the LHC"

on

23/09/2025, Kraków, Poland

QCD evolution, dilute vs. dense, forward jets



A dilute system carries a few high-x partons contributing to the hard scattering.

A dense system carries many low-x partons.

At high density, gluons are imagined to undergo recombination, and to saturate.

This is modeled with non-linear evolution equations, involving explicit non-vanishing k_T.

Saturation implies the turnover of the gluon density, stopping it from growing indefinitely for small x.

Forward jets have large rapidities, and trigger events in which partons from the nucleus have small x.

Color Glass Condensate (CGC)

McLerran, Venugopalan 1994

introduction from Morreale, Salazar 2021

The CGC is an effective field theory for high energy QCD.

Partons carrying large hadron momentum fraction x are treated as static color sources ρ.

Their color charge distribution is non-perturbative and is dictated by a gauge invariant weight functional $W_{x_0}[\rho]$. The sources generate a current $J^{\mu,a}$.

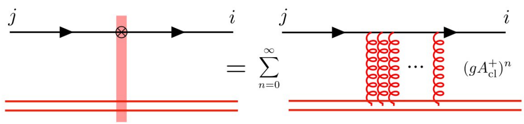
The partons carrying small x are treated as a dynamical classical field $A^{\mu,a}$.

Sources and fields are related by the Yang-Mills equations $[D_\mu, F_{\mu\nu}] = J_\nu$.

The expectation value $\langle \mathcal{O} \rangle_{x_0}$ of an observable $\mathcal{O}$ is calculated as the path integral $\mathcal{O}[\rho]$ in the presence of sources from $W_{x_0}[\rho]$, averaged over all possible configurations ρ.

The interaction of a highly energetic color charged particle with the classical field A in the eikonal approximation is encoded in the light-like Wilson lines

$$U(x_T) = \mathrm{P}\exp\left\{ ig \int_{-\infty}^{\infty} dx^+ A^{-,a}(x^+, x_T) t^a \right\}$$



Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

Evolution in x of $W_x[\rho]$ implies an infinite hierarchy (known as the B-JIMWLK hierarchy) of non-linear coupled equations dictating the evolution of n-point Wilson line correlators.

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Cross section calculations involve particle wave functions and Wilson line correlators.

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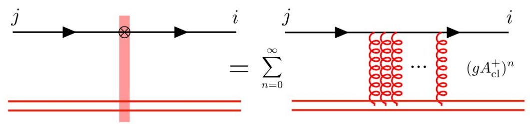

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ITMD Factorization

For forward dijet production
in dilute-dense hadronic collisions

Improved TMD factorization (Kotko, Kutak, Marquet, Petreska, Sapeta, AvH 2015)

$$d\sigma_{AB\to X} = \int dk_T^2 \int dx_A \sum_i \int dx_B \sum_b \phi_{gb}^{(i)}(x_A, k_T, \mu)\, f_{b/B}(x_B, \mu)\, d\hat{\sigma}_{gb\to X}^{(i)}(x_A, x_B, k_T, \mu)$$

target-side TMD — projectile-side PDF — parton-level matrix element

ITMD Factorization

For forward dijet production
in dilute-dense hadronic collisions

Generalized TMD factorization (Dominguez, Marquet, Xiao, Yuan 2011)

$$d\sigma_{AB\to X} = \int dk_T^2 \int dx_A \sum_i \int dx_B \sum_b \Phi_{gb}^{(i)}(x_A, k_T, \mu)\, f_{b/B}(x_B, \mu)\, d\hat{\sigma}_{gb\to X}^{(i)}(x_A, x_B, \mu)$$

For $x_A \ll 1$ and $P_T \gg k_T \sim Q_s$ (jets almost back-to-back).
TMD gluon distributions $\Phi_{gb}^{(i)}(x_A, k_T, \mu)$ satisfy non-linear evolution equations.
Partonic cross section $d\hat{\sigma}_{gb}^{(i)}$ is on-shell, but depends on color-structure i.

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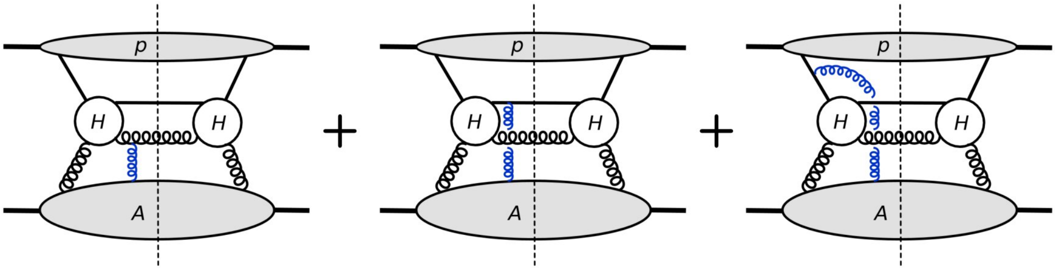
Originally a model interpolating between High Energy Factorization and Generalized TMD factorization: $P_T \gtrsim k_T \gtrsim Q_s$.
Partonic cross section $d\hat{\sigma}_{gb}^{(i)}$ is off-shell and depends on color-structure i.

ITMD formalism is obtained from the CGC formalism, by including so-called kinematic twist corrections (Antinoluk, Boussarie, Kotko 2019).

Definition of gluon TMDs



$+$ similar diagrams with 2, 3, ... gluon exchanges

Resummation of gluon exchanges leads to Wilson line $U_\gamma = \mathcal{P}\exp\left\{-ig\int_\gamma dz\cdot A(z)\right\}$ acting as a gauge link for the gauge invariant definition of a TMD

$$\mathcal{F}_{g/A}(x, k_T) = 2\int \frac{d^4\xi\,\delta(\xi^+)}{(2\pi)^3\,p_A^+}\exp\{ixp_A^+\xi^- - i\vec{k}_T\cdot\vec{\xi}_T\}\,\langle A|\mathrm{Tr}\{\hat{F}^{i+}(\xi)U_{\gamma(\xi,0)}\hat{F}^{i+}(0)\}|A\rangle$$

TMD-valued color matrix

Schematic hybrid (non-ITMD) factorization formula

$$d\sigma = \sum_{y=g,u,d,\ldots} \int dx_1 d^2k_T \int dx_2 \, d\Phi_{g^*y\to n} \, \frac{1}{\mathrm{flux}_{gy}} \, \mathcal{F}_g(x_1,|k_T|) \, f_y(x_2,\mu) \sum_{\mathrm{color}} \left|\mathcal{M}^{(\mathrm{color})}_{g^*y\to n}\right|^2$$

To get the ITMD formula: replace TMD times color matrix in

$$\mathcal{F}_g(x_1,|k_T|) \sum_{\mathrm{color}} \left|\mathcal{M}^{(\mathrm{color})}\right|^2 = \mathcal{F}_g(x_1,|k_T|) \sum_{\sigma\in S_{n+2}} \sum_{\tau\in S_{n+2}} \mathcal{A}^*_\sigma \, \mathcal{C}_{\sigma\tau} \, \mathcal{A}_\tau \quad , \quad \mathcal{C}_{\sigma\tau} = N_c^{\lambda(\sigma,\tau)}$$

with "TMD-valued color matrix" as

$$(N_c^2-1) \sum_{\sigma\in S_{n+2}} \sum_{\tau\in S_{n+2}} \mathcal{A}^*_\sigma \, \tilde{\mathcal{C}}_{\sigma\tau}(x,|k_T|) \, \mathcal{A}_\tau \quad , \quad \tilde{\mathcal{C}}_{\sigma\tau}(x,|k_T|) = N_c^{\bar{\lambda}(\sigma,\tau)} \tilde{\mathcal{F}}_{\sigma\tau}(x,|k_T|)$$

where each function $\tilde{\mathcal{F}}_{\sigma\tau}$ is one of 10 functions

$$\mathcal{F}^{(1)}_{qg} \quad , \quad \mathcal{F}^{(2)}_{qg} \quad , \quad \mathcal{F}^{(3)}_{qg}$$

$$\mathcal{F}^{(1)}_{gg} \quad , \quad \mathcal{F}^{(2)}_{gg} \quad , \quad \mathcal{F}^{(3)}_{gg} \quad , \quad \mathcal{F}^{(4)}_{gg} \quad , \quad \mathcal{F}^{(5)}_{gg} \quad , \quad \mathcal{F}^{(6)}_{gg} \quad , \quad \mathcal{F}^{(7)}_{gg}$$

List of TMDs

$$\mathcal{F}_{qg}^{(1)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle \quad , \quad \left\langle \cdots \right\rangle = 2\int \frac{d^4\xi\,\delta(\xi_+)}{(2\pi)^3 P^+}\, e^{ik\cdot\xi}\left\langle P\middle|\cdots\middle|P\right\rangle$$

$$\mathcal{F}_{qg}^{(2)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\Box]}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{qg}^{(3)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[\Box]}\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(1)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\Box]\dagger}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(2)}(x,k_T) = \frac{1}{N_c}\left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[\Box]\dagger}\right]\mathrm{Tr}\left[\hat{F}^{i+}(0)\,\mathcal{U}^{[\Box]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(3)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(4)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[-]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(5)}(x,k_T) = \left\langle \mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[\Box]\dagger}\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[\Box]}\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(6)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\Box]}\right]}{N_c}\frac{\mathrm{Tr}\left[\mathcal{U}^{[\Box]\dagger}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

$$\mathcal{F}_{gg}^{(7)}(x,k_T) = \left\langle \frac{\mathrm{Tr}\left[\mathcal{U}^{[\Box]}\right]}{N_c}\mathrm{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[\Box]\dagger}\mathcal{U}^{[+]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$$

TMDs relevant for dijets

Start with dipole distribution $\mathcal{F}_{qg}^{(1)}(x, k_T) = \left\langle \text{Tr}\left[\hat{F}^{i+}(\xi)\,\mathcal{U}^{[-]\dagger}\hat{F}^{i+}(0)\,\mathcal{U}^{[+]}\right]\right\rangle$ evolved via the BK equation formulated in momentum space supplemented with subleading corrections and fitted to F_2 data (Kutak, Sapeta 2012)

All other distribution appearing in dijet production, $\mathcal{F}_{qg}^{(2)}, \mathcal{F}_{gg}^{(1)}, \mathcal{F}_{gg}^{(2)}, \mathcal{F}_{gg}^{(6)}$, in the mean-field approximation (AvH, Marquet, Kotko, Kutak, Sapeta, Petreska 2016).

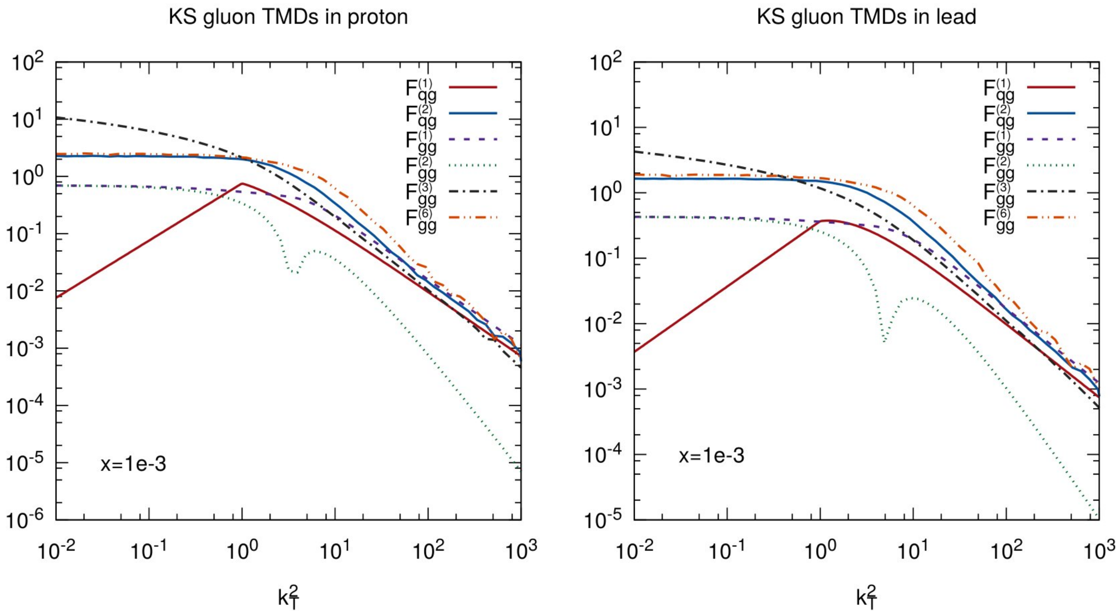
This is, at leading order in $1/N_c$. In this approximation, the same distributions suffice for trijets.

For DIS one only needs $\mathcal{F}_{gg}^{(3)}$

$$\mathcal{F}_{gg}^{(3)}(x, k_T) = \frac{\pi\alpha_s}{N_c k_T^2 S_\perp}\int_{k_T^2} dr_T^2 \ln\frac{r_T^2}{k_T^2}\int\frac{d^2q_T}{q_T^2}\,\mathcal{F}_{qg}^{(1)}(x, q_T)\,\mathcal{F}_{qg}^{(1)}(x, r_T - q_T)$$

where $S_\perp$ is the target's transverse area.

ITMDs

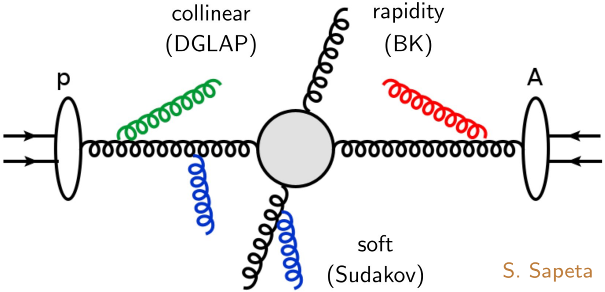


Dependence of $\mathcal{F}_{qg}^{(1)}$ on k_T below 1GeV approximated by power-like fall-off. For higher values of $|k_T|$ it is a solution to the BK equation.

TMDs decrease as $1/|k_T|$ for increasing $|k_T|$, except $\mathcal{F}_{gg}^{(2)}$, which decreases faster (even becomes negative, absolute value shown here).

Sudakov resummation for dijets

Having hard jets in the final state, large logarithms associated with the hard scale have to be resummed. This resummation can be accounted for by inclusion of the Sudakov factor.



Within the small-x saturation formalism, Sudakov effects are most conveniently included in b-space, via an "initial-state luminosity" (Mueller, Xiao, Yuan 2013)

$$\mathcal{L}^{ag\to cd}_{g^*/B}(x_p, x, k_T, \mu) = \int db_T\, b_T\, J_0(b_T k_T)\, e^{-S^{ag\to cd}_{\text{Sud}}(\mu, b_T)} \times f_{a/p}(x_p, \mu_b) \int dk'_T\, k'_T\, J_0(b_T k'_T)\, \mathcal{F}_{g^*/B}(x, k'_T)$$

with $\mu_b = 2e^{-\gamma_E}/b_*$, $b_* = b_T/\sqrt{1 + b_T^2/b_{\text{max}}^2}$. The scale choice μ_b eliminates threshold logarithms, but "breaks" factorization between initial-state variables, which complicates the Monte Carlo approach, or requires expensive 4-dim luminosity grids.

Sudakov resummation for dijets

The Sudakov receives perturbative and non-perturbative contributions for each cannel

$$S_{Sud}^{ab\to cd}(\mu, b_T) = \sum_{i=a,b,c,d} S_p^i(\mu, b_T) + \sum_{i=a,c,d} S_{np}^i(\mu, b_T)$$

Perturbative part (Mueller, Xiao, Yuan 2013)

$$S_p^i(Q, b_T) = \frac{\alpha_s}{2\pi}\int_{\mu_b^2}^{Q^2} \frac{d\mu^2}{\mu^2}\left[A^i \ln\frac{Q^2}{\mu^2} - B^i\right]$$

$$\{A, B\}^{qg\to qg} = \left\{2(C_A + C_F)\, ,\, 3C_F + 2C_A\beta_0\right\} \;,\; \{A, B\}^{gg\to gg} = \left\{4C_A\, ,\, 6C_A\beta_0\right\}$$

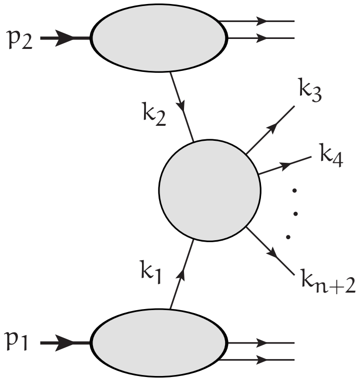
$$b_{max} = 0.5\text{GeV}^{-1}$$

Non-perturbative contribution for small-x gluon already in TMD and should be omitted in our application (Staśto, Wei, Xiao, Yuan 2018).

Parton-level cross sections

Hadron-scattering process Y with partonic processes y contributing to multi-jet final state

$$d\sigma_Y(p_1, p_2; k_3, \dots, k_{2+n}) = \sum_{y\in Y} \int d^4k_1\, \mathcal{P}_{y_1}(k_1) \int d^4k_2\, \mathcal{P}_{y_2}(k_2)\, d\hat{\sigma}_y(k_1, k_2; k_3, \dots, k_{2+n})$$



Collinear factorization:

$$\mathcal{P}_{y_i}(k_i) = \int \frac{dx_i}{x_i} f_{y_i}(x_i, \mu)\, \delta^4(k_i - x_i p_i)$$

k_T-dependent factorization factorization:

$$\mathcal{P}_{y_i}(k_i) = \int \frac{d^2\mathbf{k}_{iT}}{\pi} \int \frac{dx_i}{x_i} \mathcal{F}_{y_i}(x_i, |\mathbf{k}_{iT}|, \mu)\, \delta^4(k_i - x_i p_i - k_{iT})$$

Differential partonic cross section:

$$\begin{aligned} d\hat{\sigma}_y(k_1, k_2; k_3, \dots, k_{2+n}) &= d\Phi_Y(k_1, k_2; k_3, \dots, k_{2+n})\, \Theta_Y(k_3, \dots, k_{2+n}) \\ &\times \text{flux}(k_1, k_2) \times \mathcal{S}_y\, |\mathcal{M}_y(k_1, \dots, k_{2+n})|^2 \end{aligned}$$

Parton-level phase space:

$$d\Phi_Y(k_1, k_2; k_3, \dots, k_{2+n}) = \left(\prod_{i=3}^{n+2} d^4k_i \delta_+(k_i^2 - m_i^2) \right) \delta^4(k_1 + k_2 - k_3 - \cdots - k_{n+2})$$

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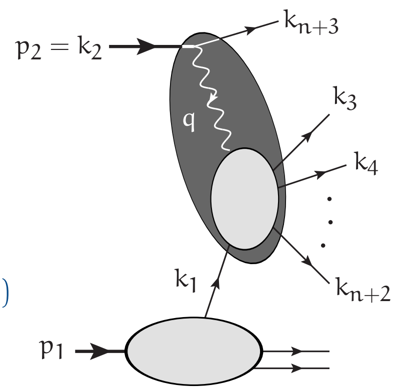

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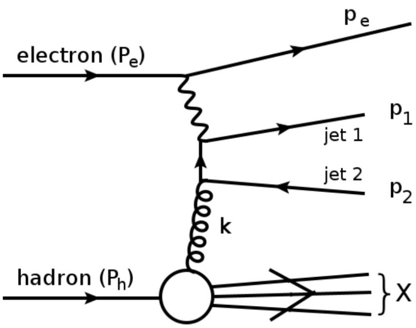
KaTie https://bitbucket.org/hameren/katie

- parton-level tree-level event generator, like Alpgen, Helac, MadGraph, etc.
- arbitrary hadron-hadron or hadron-lepton processes within the standard model (including effective Higgs-gluon coupling) with several final-state particles.
- 0, 1, or 2 space-like initial states.
- produces (partially un)weighted event files, for example in the LHEF format.
- requires LHAPDF. TMD PDFs can be provided as files containing rectangular grids, or with TMDlib (Hautmann, Jung, Krämer, Mulders, Nocera, Rogers, Signori 2014).
- a calculation is steered by a single input file.
- employs an optimization stage in which the pre-samplers for all channels are optimized.
- during the generation stage several event files can be created in parallel.
- event files can be processed further by parton-shower program like CASCADE.
- (evaluation of) matrix elements separately available.
- ITMD available.

Dijets in DIS

AvH, Kotko, Kutak, Sapeta 2021

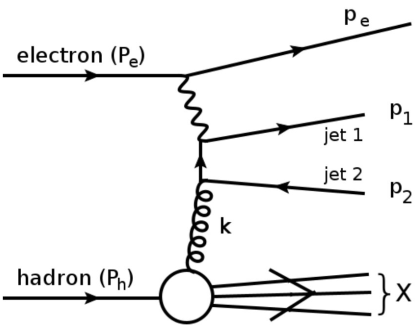


$$d\sigma_{eh\to e'+2j+X} = \int \frac{dx}{x}\frac{d^2k_T}{\pi}\,\mathcal{F}_{gg}^{(3)}(x,k_T,\mu) \times \frac{1}{4xP_e\cdot P_h}\,d\Phi(P_e,k;p_e,p_1,p_2)\,|\overline{M}_{eg^*\to e'+2j}|^2$$

ITMD for DIS only requires $\mathcal{F}_{gg}^{(3)}$,
aka the Weizsäcker-Williams density

Dijets in DIS

AvH, Kotko, Kutak, Sapeta 2021



$$
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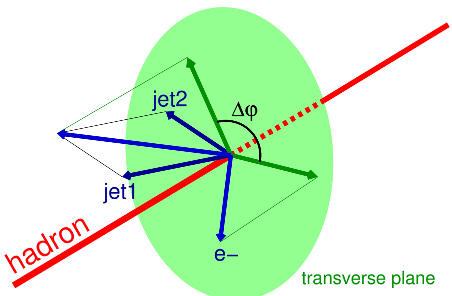
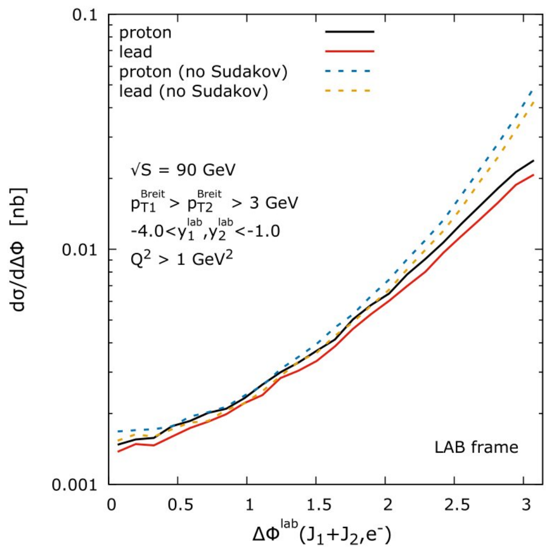
$$

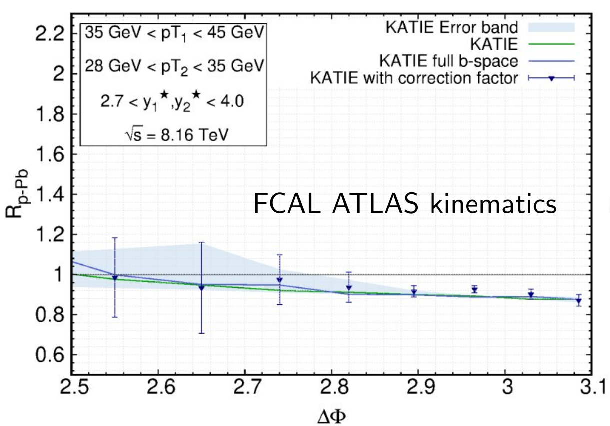


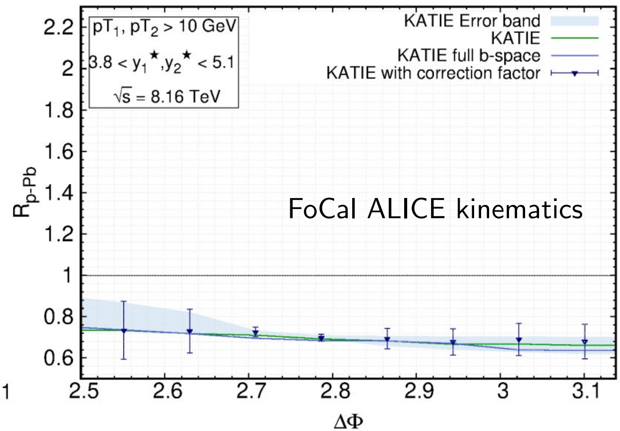


Dijet azimuthal correlations

in p-p and p-Pb collisions at forward LHC calorimeters

Abdullah Al-Mashad, AvH, Kakkad, Kotko, Kutak, van Mechelen, Sapeta
2022





Predictions for the nuclear modification ratio $R_{p-pB} = \left(d\sigma^{p+Pb}/d\Delta\Phi\right)/\left(d\sigma^{p+p}/d\Delta\Phi\right)$ as function of the azimuthal angle $\Delta\Phi$ between the two hardes jets p-p and p-Pb collisions.

Points with error bars are corrected with final-state shower effects using Pythia, and represent uncertainty both form statistics and scale dependence.

Sudakov factors, feared to wash out saturation effects, appear to cancel and the latter stay manifest.

NLO

Collinear factorization at NLO

$$d\sigma^{LO} = \sum_{i,\bar{\imath}} \int dx\, d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x})\, dB_{i\bar{\imath}}(x,\bar{x})$$

$$a_\epsilon = \frac{\alpha_s}{2\pi} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$$

$$\begin{aligned} d\sigma^{NLO} = \sum_{i,\bar{\imath}} \int dx\, d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x}) \Bigg\{ & \Big[a_\epsilon\, dV_{i\bar{\imath}}(x,\bar{x}) + a_\epsilon\, dR_{i\bar{\imath}}(x,\bar{x}) \Big]_{\text{finite}} \\ & - a_\epsilon \Bigg[\frac{1}{\epsilon} \sum_{i'} \int_x^1 \frac{dz}{z} \mathcal{P}_{ii'}(z) \frac{f_{i'}(x/z)}{f_i(x)} + \frac{1}{\epsilon} \sum_{\bar{\imath}'} \int_{\bar{x}}^1 \frac{d\bar{z}}{\bar{z}} \mathcal{P}_{\bar{\imath}\bar{\imath}'}(\bar{z}) \frac{f_{\bar{\imath}'}(\bar{x}/\bar{z})}{f_{\bar{\imath}}(\bar{x})} \Bigg] dB_{i\bar{\imath}}(x,\bar{x}) \\ & + a_\epsilon \Bigg[\frac{\delta f_i\ (x,\mu_F)}{f_i(x)} + \frac{\delta f_{\bar{\imath}}\ (\bar{x},\mu_{\bar{F}})}{f_{\bar{\imath}}(\bar{x})} \Bigg] dB_{i\bar{\imath}}(x,\bar{x}) \Bigg\} \end{aligned}$$

Collinear factorization at NLO

$$d\sigma^{\mathrm{LO}} = \sum_{i,\bar{\imath}} \int dx\,d\bar{x}\; f_i(x)\; f_{\bar{\imath}}(\bar{x})\; dB_{i\bar{\imath}}(x,\bar{x})$$

$$a_\epsilon = \frac{\alpha_s}{2\pi}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$$

$$\begin{aligned} d\sigma^{\mathrm{NLO}} = \sum_{i,\bar{\imath}} \int dx\,d\bar{x}\; f_i(x)\; f_{\bar{\imath}}(\bar{x}) \Bigg\{ & \Big[a_\epsilon\, dV_{i\bar{\imath}}(x,\bar{x}) + a_\epsilon\, dR_{i\bar{\imath}}(x,\bar{x}) \Big]_{\text{finite}} \\ & + a_\epsilon \Bigg[\ln\frac{\mu^2}{\mu_F^2} \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, \frac{f_{i'}\big(x/z\big)}{f_i(x)} + \ln\frac{\mu^2}{\mu_{\bar{F}}^2} \sum_{\bar{\imath}'} \int_{\bar{x}}^1 \frac{d\bar{z}}{\bar{z}}\, \mathcal{P}_{\bar{\imath}\bar{\imath}'}(\bar{z})\, \frac{f_{\bar{\imath}'}\big(\bar{x}/\bar{z}\big)}{f_{\bar{\imath}}(\bar{x})} \Bigg] dB_{i\bar{\imath}}(x,\bar{x}) \\ & + a_\epsilon \Bigg[\frac{\delta f_i^{\mathrm{fin}}(x,\mu_F)}{f_i(x)} + \frac{\delta f_{\bar{\imath}}^{\mathrm{fin}}(\bar{x},\mu_{\bar{F}})}{f_{\bar{\imath}}(\bar{x})} \Bigg] dB_{i\bar{\imath}}(x,\bar{x}) \Bigg\} \end{aligned}$$

$$\frac{df_i^{\mathrm{fin}}(x,\mu_F)}{d\ln\mu_F^2} = a_\epsilon \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, f_{i'}^{\mathrm{fin}}\big(x/z,\mu_F\big)$$

Collinear factorization at NLO

$$d\sigma^{\rm LO} = \sum_{i,\bar\imath} \int dx\, d\bar x\, f_i(x)\, f_{\bar\imath}(\bar x)\, dB_{i\bar\imath}(x,\bar x)$$

$$a_\epsilon = \frac{\alpha_s}{2\pi} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$$

$$\begin{aligned} d\sigma^{\rm NLO} = \sum_{i,\bar\imath} \int dx\, d\bar x\, f_i(x)\, f_{\bar\imath}(\bar x) \Bigg\{ & \Big[a_\epsilon\, dV_{i\bar\imath}(x,\bar x) + a_\epsilon\, dR_{i\bar\imath}(x,\bar x) \Big]_{\rm finite} \\ & + a_\epsilon \left[\ln\frac{\mu^2}{\mu_F^2} \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, \frac{f_{i'}(x/z)}{f_i(x)} + \ln\frac{\mu^2}{\mu_{\bar F}^2} \sum_{\bar\imath'} \int_{\bar x}^1 \frac{d\bar z}{\bar z}\, \mathcal{P}_{\bar\imath\bar\imath'}(\bar z)\, \frac{f_{\bar\imath'}(\bar x/\bar z)}{f_{\bar\imath}(\bar x)} \right] dB_{i\bar\imath}(x,\bar x) \\ & + a_\epsilon \left[\frac{\delta f_i^{\rm fin}(x,\mu_F)}{f_i(x)} + \frac{\delta f_{\bar\imath}^{\rm fin}(\bar x,\mu_{\bar F})}{f_{\bar\imath}(\bar x)} \right] dB_{i\bar\imath}(x,\bar x) \Bigg\} \end{aligned}$$

$$\frac{df_i^{\rm fin}(x,\mu_F)}{d\ln\mu_F^2} = a_\epsilon \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, f_{i'}^{\rm fin}(x/z,\mu_F)$$

Establish the same for $\quad d\sigma^{\rm LO} = \sum_{\bar\imath} \int dx \int \frac{d^2k_\perp}{\pi} \int d\bar x\, F(x,k_\perp)\, f_{\bar\imath}(\bar x)\, dB_{\star\bar\imath}(x,k_\perp,\bar x)$

Hybrid k_T-factorization at NLO

A. van Hameren, L. Motyka and G. Ziarko,
"Hybrid k_T -factorization and impact factors at NLO,"
JHEP **11** (2022), 103, arXiv:2205.09585,
doi:10.1007/JHEP11(2022)103

E. Blanco, A. Giachino, A. van Hameren and P. Kotko,
"One-loop gauge invariant amplitudes with a space-like gluon,"
Nucl. Phys. B **995** (2023), 116322, arXiv:2212.03572,
doi:10.1016/j.nuclphysb.2023.116322

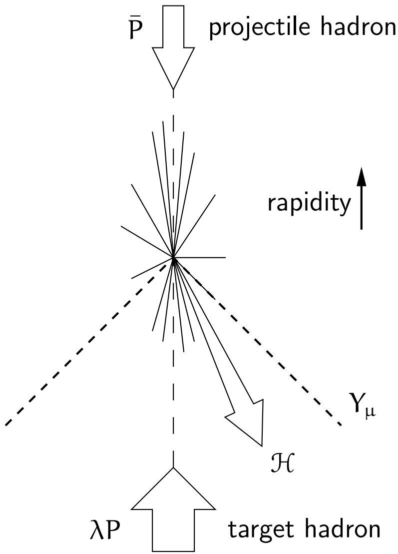
A. Giachino, A. van Hameren and G. Ziarko,
"A new subtraction scheme at NLO exploiting the privilege of k_T-factorization,"
JHEP **06** (2024), 167, arXiv:2312.02808,
doi:10.1007/JHEP06(2024)167

A. van Hameren and M. Nefedov,
"Hybrid high-energy factorization and evolution at NLO from the high-energy limit of collinear factorization,"
JHEP **02** (2025), 160, arXiv:2501.02619,
doi:10.1007/JHEP02(2025)160

Embedding in collinear factorization

Consider hadron collisions, with production of the final state of interest $\mathcal{H}$:

$$h(\lambda P) + h(\bar{P}) \to \mathcal{H} + \mathcal{X}$$



We assume that there is a natural rapidity Y_μ associated with $\mathcal{H}$, which separates the event into "target" and "projectile" parts. Then we can define

$$x = \sum_j \theta\big(y_j < Y_\mu\big)\, \frac{p_j \cdot \bar{P}}{P \cdot \bar{P}} \quad , \quad k_\perp = -\sum_j \theta\big(y_j < Y_\mu\big)\, p_{j\perp}$$

And the (Collins-Soper) scale μ_Y via

$$Y_\mu = \ln \frac{\nu x}{\mu_Y} \quad , \quad \nu^2 = (P + \bar{P})^2$$

Due to IRC-safety of variables x and $k_\perp$, the hadronic differential cross section

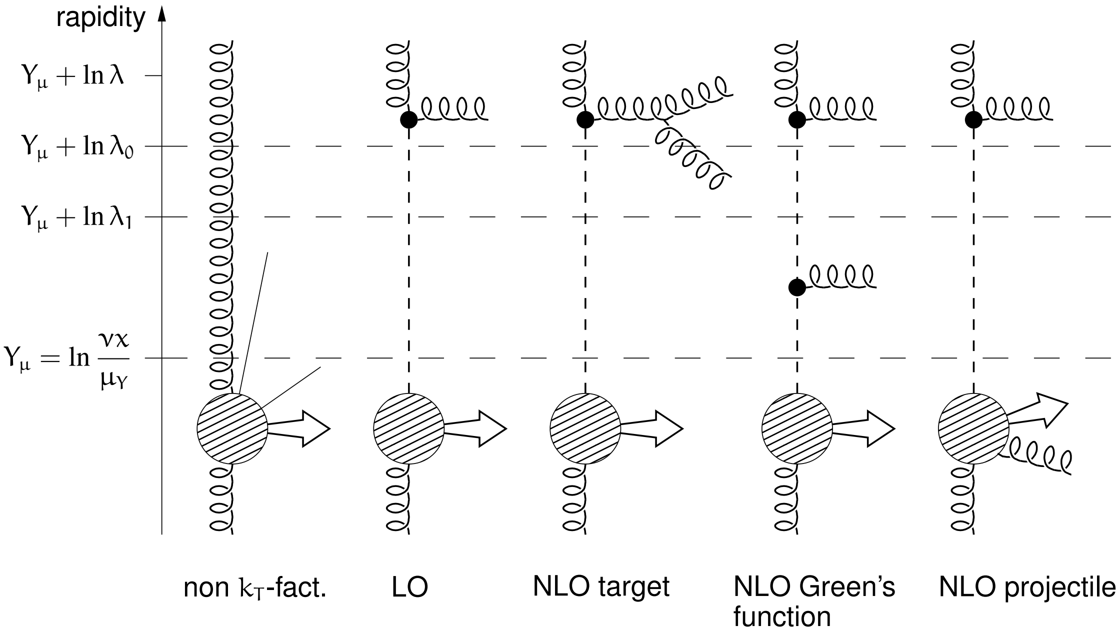
$$\frac{d\sigma_\lambda^{CF}}{dx\, d^2k_\perp}(x, k_\perp, \ldots) = \sum_{i,\bar{\imath}} \int_0^1 dX\, f_i(X) \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, \frac{d\hat{\sigma}_{i\bar{\imath}}^{CF}}{dx\, d^2k_\perp}(\lambda X, \bar{x}\,; x, k_\perp, \ldots)$$

should be computable in collinear factorization, at least up to NLO, and in the limit:

$$\lambda \to \infty \quad , \quad x, k_\perp - \text{fixed} \; .$$

Recovering high-energy factorization

Demand of consistency and cancellation of all divergences leads to a hierarchy of large rapidities separating the phase space, recovering "usual high-energy factorization".



Evolution equations

We find an equation for evolution with respect to $\mu_Y \Leftrightarrow Y_\mu$

$$\frac{d\hat{F}(x, k_\perp; \mu_Y)}{d\ln\mu_Y^2} = \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2 r_\perp}{|r_\perp|^2}\Bigg\{\hat{F}\left(x\left[1+\frac{|r_\perp|}{\mu_Y}\right], k_\perp + r_\perp; \mu_Y\right)\theta\left(|r_\perp| < \mu_Y\frac{1-x}{x}\right) - \theta\big(\mu_Y - |r_\perp|\big)\,\hat{F}(x, k_\perp; \mu_Y)\Bigg\}$$

- Very similar to the equation from Born-Oppenheimer renormalization group by Duan, Kovner, Lublinsky 2024
- Closely related to the LO Collins-Soper-Sterman equation

Evolution equations

We find an equation for evolution with respect to $\mu_Y \Leftrightarrow Y_\mu$

$$\frac{d\hat{F}(x,k_\perp;\mu_Y)}{d\ln\mu_Y^2} = \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2r_\perp}{|r_\perp|^2}\left\{\hat{F}\left(x\left[1+\frac{|r_\perp|}{\mu_Y}\right],k_\perp+r_\perp;\mu_Y\right)\theta\left(|r_\perp|<\mu_Y\,\frac{1-x}{x}\right)\right.$$
$$\left. -\,\theta\left(\mu_Y-|r_\perp|\right)\hat{F}(x,k_\perp;\mu_Y)\right\}$$

This equation resums $\ln\left(\mu_Y^2/|k_\perp|\right)$.

The initial-condition to this equation is found to be

$$F\left(x,k_\perp,\mu_Y=|k_\perp|\right) = \sum_i\int_x^1 dX\, f_i(X,\mu_F)\int d^{2-2\epsilon}k'_\perp\, I_i\left(k'_\perp,\mu_F\right)\, G\left(k'_\perp,k_\perp,\frac{X}{x},\mu_F\right)$$

reproducing known expressions for target-side inpact factor corrections $I_i\left(k'_\perp,\mu_F\right)$

and recovering the Green's function $G\left(k'_\perp,k_\perp,\frac{X}{x},\mu_F\right)$ in terms of the LO BFKL kernel, resumming $\ln\left(X/x\right)$.

Summary

- ITMD factorization provides a momentum-space formula suitable to study saturation in QCD within an automatable Monte Carlo approach

- collinear embedding appears to be be a powerfull strategy to extract NLO formulas for cross sections and evolution equations within k_T-type factorization

Backup

Augmented TMD evolution

Kwieciński, Martin, Staśto 1997
Kwieciński, Kutak 2003

linear BFKL with kinematic constraint

$$\phi(x,k^2) = \phi^{(0)}(x,k^2)$$

$$+\frac{\alpha_s(k^2)N_c}{\pi}\int_x^1\frac{dz}{z}\int_{k_0^2}^{\infty}\frac{dl^2}{l^2}\left\{\frac{l^2\phi(\frac{x}{z},l^2)\theta(\frac{k^2}{z}-l^2)-k^2\phi(\frac{x}{z},k^2)}{|l^2-k^2|}+\frac{k^2\phi(\frac{x}{z},k^2))}{\sqrt{|4l^4+k^4|}}\right\}$$

$$+\frac{\alpha_s(k^2)}{2\pi k^2}\int_x^1 dz\left(P_{gg}(z)-\frac{2N_c}{z}\right)\int_{k_0^2}^{k^2}dl^2\,\phi\left(\frac{x}{z},l^2\right)+\frac{\alpha_s(k^2)}{2\pi}\int_x^1 dz\,P_{gq}(z)\Sigma\left(\frac{x}{z},k^2\right)$$

$$-\frac{2\alpha_s^2(k^2)}{R^2}\left[\left(\int_{k^2}^{\infty}\frac{dl^2}{l^2}\phi(x,l^2)\right)^2+\phi(x,k^2)\int_{k^2}^{\infty}\frac{dl^2}{l^2}\ln\left(\frac{l^2}{k^2}\right)\phi(x,l^2)\right]$$

non-linear term from triple-pomeron vertex, with $R_A = R\,A^{1/3}$

DGLAP corrections

Kutak, Sapeta 2012:

Starting distribution $\phi^{(0)}(x,k^2)=\frac{\alpha_s(k^2)}{2\pi k^2}\int_x^1 dz P_{gg}(z)\frac{x}{z}g\left(\frac{x}{z}\right)$, $xg(x)=N(1-x)^{\beta}(1-Dx)$

fitted to combined HERA F_2 data, and with $\phi(x,k^2<1)=k^2\phi(x,1)$.

ITMD* factorization for more than 2 jets

We want to establish a similar factorization for more than 2 jets.

However, the ITMD formalism does not account for linearly polarized gluons in unpolarized target.

Such a contribution is absent for massless 2-particle production in CGC theory, but does appear in heavy quark production (Marquet, Roiesnes, Taels 2018), in the correlation limit for 3-parton final-states (Altinoluk, Boussarie, Marquet, Taels 2020), and can be concluded to be present from 3-jet formulae in CGC (Iancu, Mulian 2019).

This contribution cannot staightforwardly be formulated in terms of gauge-invariant off-shell hard scattering amplitudes

$$\sum_{i,j} \mathcal{M}_i^* \left(\frac{k_T^{(i)} k_T^{(j)}}{2|\vec{k}_T|^2} (\mathcal{F} + \mathcal{H}) + \frac{q_T^{(i)} q_T^{(j)}}{2|\vec{q}_T|^2} (\mathcal{F} - \mathcal{H}) \right) \mathcal{M}_j \quad , \quad \vec{q}_T \cdot \vec{k}_T = 0$$

$\sum_i \mathcal{M}_i k_T^{(i)}$ is gauge invariant while $\sum_i \mathcal{M}_i q_T^{(i)}$ is not. For dijets, it happens that $\mathcal{F} = \mathcal{H}$.

In the following only the manifestly gauge-invariant contribution is included, hence the designation ITMD*.

ITMD* factorization for more than 2 jets

We want to establish a similar factorization for more than 2 jets.

Howe ... ıpolarized
target

Such ... but does
appea ... tion limit
for 3- ... concluded
to be

This ... ariant off-
shell

Using the axial gauge with gluon propagator

$$\frac{-i}{K^2}\left(g^{\mu\nu} - \frac{P^\mu K^\nu + K^\mu P^\nu}{P\cdot K}\right) \qquad P^\mu \text{ hadron momentum}$$

the amplitude $\mathcal{M}$ for a process involving an off-shell gluon with momentum $xP^\mu + k_T^\mu$ can be written as

$$\mathcal{M} = k_T^\mu \mathcal{M}_\mu = -\sum_{i=1}^{2} k_T^{(i)} \mathcal{M}_i$$

where $\mathcal{M}_\mu$ is obtained from the usual Feynman graphs indeed with one gluon simply left "off-shell". The role of "polarization vector" is played by k_T^μ.

$\sum_i \mathcal{M}$ ... $\mathcal{F} = \mathcal{H}$.

In the ... ıence the
designation ITMD*.