Higher-order QCD calculations for hard scattering processes

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Joint ECFA-NuPECC-APPEC Workshop “Synergies between the EIC and the LHC”

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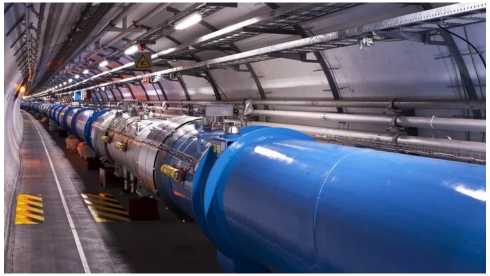
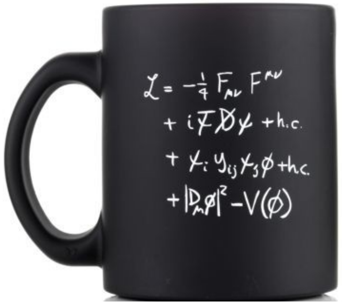
What are the fundamental building blocks of matter?

Scattering experiments

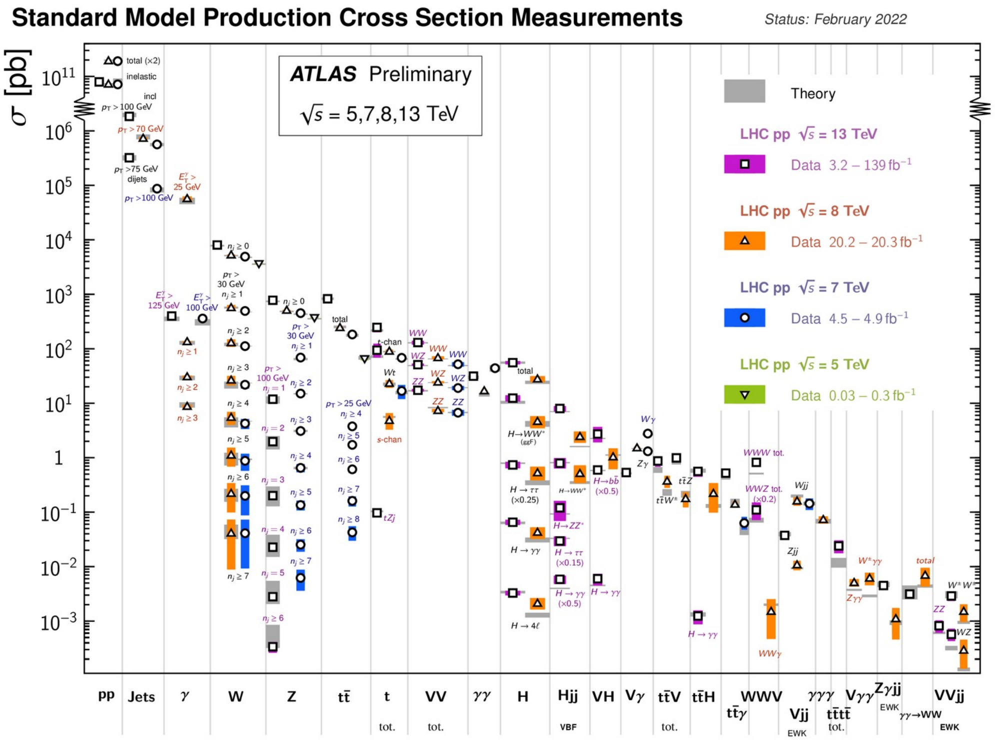
Large Hadron Collider (LHC)



Credit: CERN



Theory/ Standard Model



Theory picture of hadron collision events

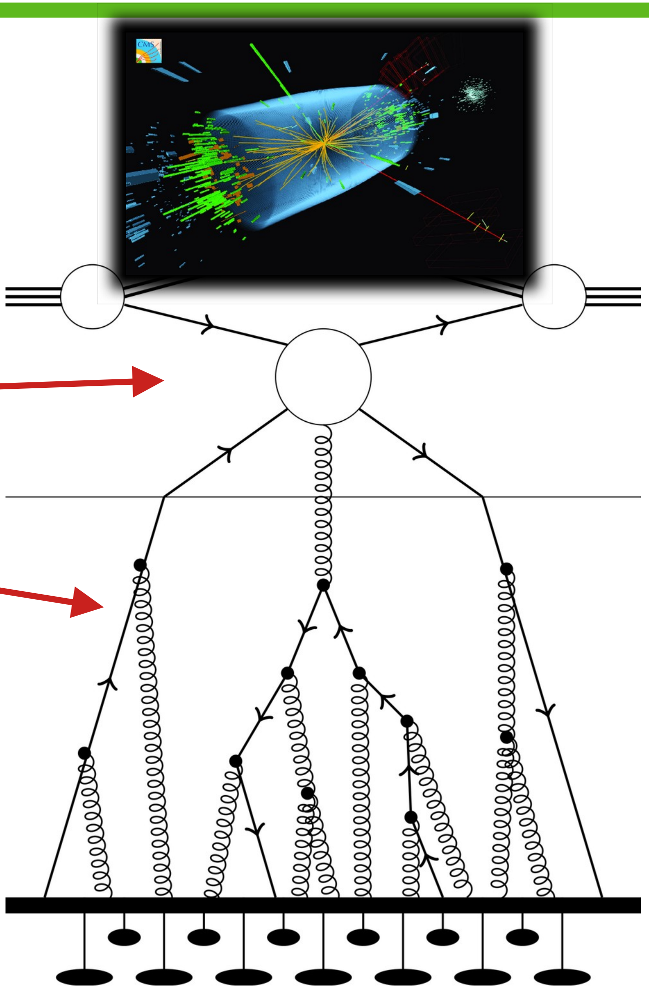
Guiding principle: factorization
"What you see depends on the energy scale"

In Quantum Chromodynamics (QCD):

$Q \gg \Lambda_{\mathrm{QCD}}$ **Fixed-order perturbation theory**
scattering of individual partons

$Q \gtrsim \Lambda_{\mathrm{QCD}}$ **Parton-shower/Resummation**
all-order bridge between perturbative and non-perturbative physics

$Q \sim \Lambda_{\mathrm{QCD}}$ **"Hadronization"/MPI/...**
non-perturbative physics



Precision predictions

Fixed order perturbation theory

→ **Core element of event simulation**
→ **Describes high Q regime**

Resummation

Parton-showers

Precision theory predictions

Parametric input:
PDFs, couplings (α_s), ...

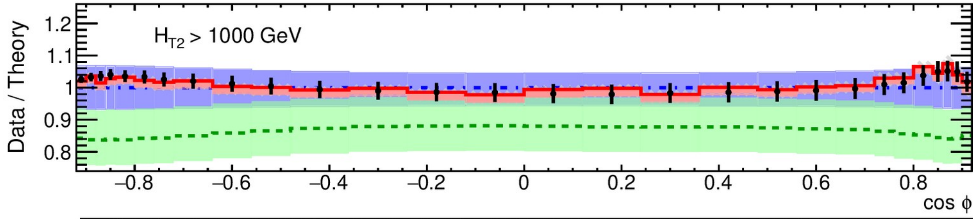
Soft physics:
MPI, colour reconnection, ...

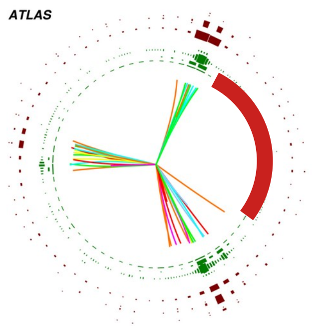
Fragmentation/hadronisation

Precision through higher-order perturbation theory

Example: ATLAS
multi-jet measurements

[ATLAS 2301.09351]





$\cos\phi$

Cross section = **LO** + **NLO** + **NNLO** + **O(α_s^3)**

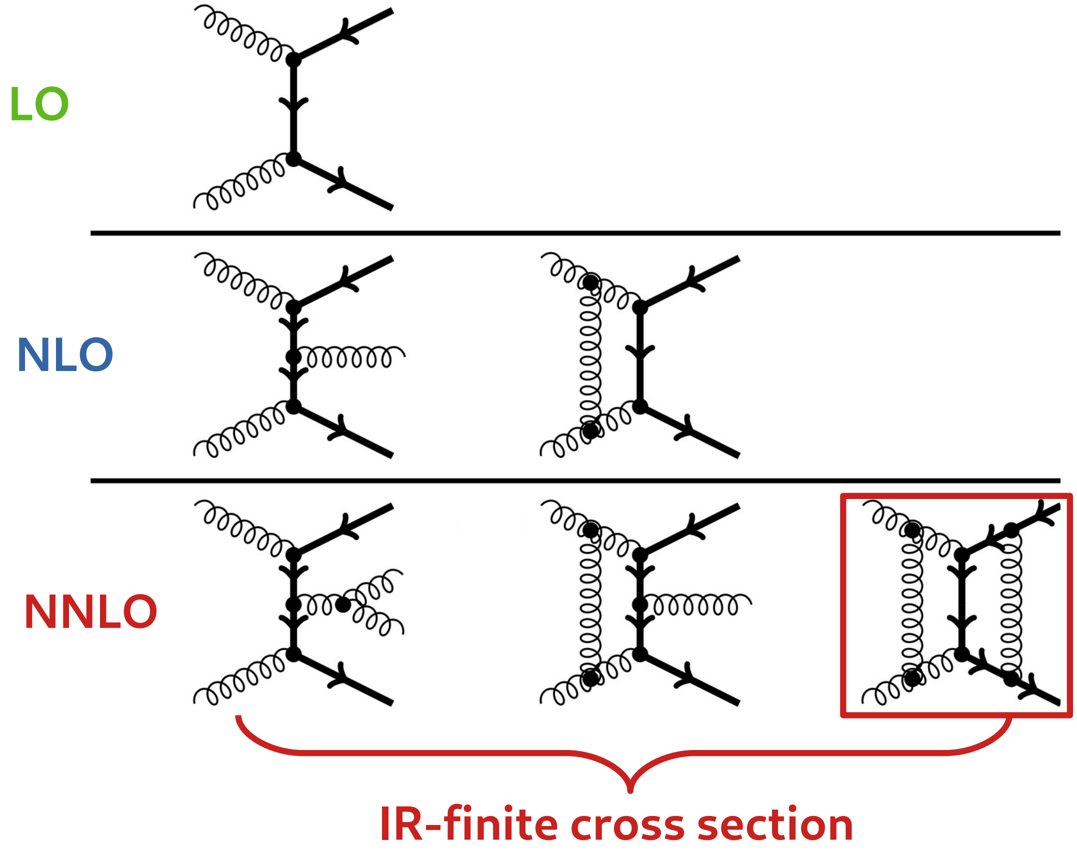
$\sim (\alpha_s)^1$ $\sim (\alpha_s)^2$

Theory uncertainty: **Order of magnitude** **O(10%)** **O(1%)**

Fixed-order expansion in the strong coupling $\alpha_s(m_Z) \approx 0.118$

Experimental precision reaches percent-level already at LHC
next-to-next-to-leading order QCD needed on theory side!

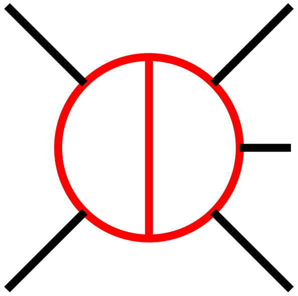
NNLO QCD challenges



1) How to compute **multi-scale two-loop amplitudes**?
   → **fast growing complexity: rational** and **transcendental**
   → deeper understanding of the analytical properties
   → refinement of computational tools

2) How to achieve **infrared finite differential** cross sections at NNLO QCD?
   ~20 years to solve this problem
   → highly non-trivial IR structure
   → plethora of schemes

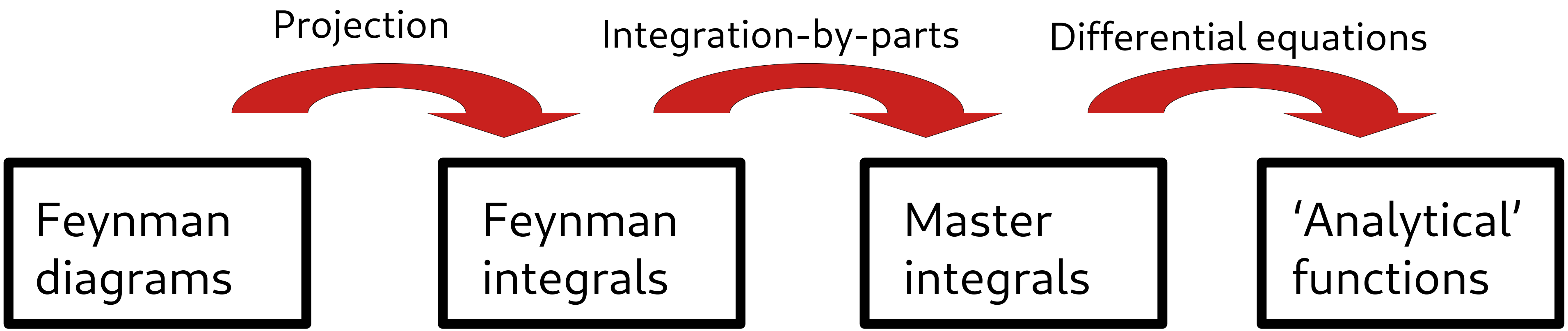
Two-loop amplitudes



Massless:
[Chawdry'19'20'21] (3A+2j,2A+3j)
[Abreu'20'21] (3A+2j,5j)
[Agarwal'21] (2A+3j)
[Badger'21'23] (5j,gggAA,jjjjA)

With external masses:
[Abreu'21] (W+4j)
[Badger'21'22] (Hqqgg,W4q,Wajjj)
[Hartanto'22] (W4q)
[Hartanto'23] (WAjjj)
[Hartanto'24] (Hbbjj, ttggg)

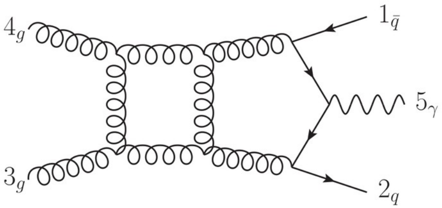
"Old school" approach:



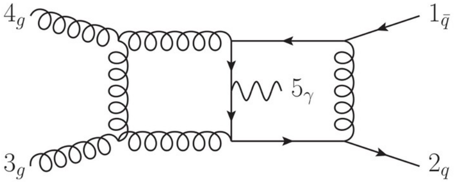
Virtual amplitudes

[Hartanto'23]

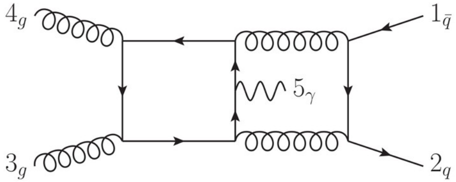
Example diagrams $0 \to q\bar{q}\gamma g g$



$A^{(2),N_c^2}_{34;q}, A^{(2),N_c}_{\delta;q}$



$A^{(2),1}_{34;q}, A^{(2),N_c}_{\delta;q}, A^{(2),1/N_c}_{\delta;q}$



$A^{(2),N_c}_{34;l}, A^{(2),1/N_c}_{34;l}, A^{(2),1/N_c^2}_{\delta;l}$

Decomposition:

Colour structures

$$\mathcal{M}^{(L)}(1_{\bar{q}}, 2_q, 3_g, 4_g, 5_\gamma) = \sqrt{2}\, e\, g_s^2\, n^L \Big\{ (t^{a_3} t^{a_4})_{i_2}^{\bar{i}_1} \mathcal{A}^{(L)}_{34}(1_{\bar{q}}, 2_q, 3_g, 4_g, 5_\gamma) + (t^{a_4} t^{a_3})_{i_2}^{\bar{i}_1} \mathcal{A}^{(L)}_{43}(1_{\bar{q}}, 2_q, 3_g, 4_g, 5_\gamma) + \delta_{i_2}^{\bar{i}_1} \delta^{a_3 a_4} \mathcal{A}^{(L)}_{\delta}(1_{\bar{q}}, 2_q, 4_g, 3_g, 5_\gamma) \Big\}$$

Independent partial amplitudes
→ different gauge couplings &
Nc/nf

$$\mathcal{A}^{(2)}_{34} = \mathcal{Q}_q N_c^2 A^{(2),N_c^2}_{34;q} + \mathcal{Q}_q A^{(2),1}_{34;q} + \mathcal{Q}_q \frac{1}{N_c^2} A^{(1),1/N_c^2}_{34;q} + \mathcal{Q}_q N_c n_f A^{(2),N_c n_f}_{34;q} + \mathcal{Q}_q \frac{n_f}{N_c} A^{(2),n_f/N_c}_{34;q} + \mathcal{Q}_q n_f^2 A^{(2),n_f^2}_{34;q} + \Big(\sum_l \mathcal{Q}_l\Big) N_c A^{(2),N_c}_{34;l} + \Big(\sum_l \mathcal{Q}_l\Big) \frac{1}{N_c} A^{(2),1/N_c}_{34;l} + \Big(\sum_l \mathcal{Q}_l\Big) n_f A^{(2),n_f}_{34;l},$$

$$A_j = \sum_i c_{j,i}(\{p\}, \epsilon) \mathcal{I}(\{p\}, \epsilon)$$ → prohibitively large number of integrals

Integration-By-Parts reduction

$$\mathcal{I}_i(\{p\},\epsilon) \equiv \mathcal{I}(\vec{n_i},\{p\},\epsilon) = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\frac{\mathrm{d}^d k_2}{(2\pi)^d}\prod_{k=1}^{11} D_k^{-n_{i,k}}(\{p\},\{k\})$$

Integration-By-Parts identities connect different integrals → system of equations
→ only a small number of independent "master" integrals

$$0 = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\frac{\mathrm{d}^d k_2}{(2\pi)^d} l_\mu \frac{\partial}{\partial l^\mu}\prod_{k=1}^{11} D_k^{-n_{i,k}}(\{p\},\{k\}) \quad \text{with} \quad l \in \{p\} \cap \{k\}$$

$$A_j = \sum_i c_{j,i}(\{p\},\epsilon)\mathcal{I}(\{p\},\epsilon) \quad \Rightarrow \quad a_i^{(L),p} = \sum_i d_{j,i}(\{p\},\epsilon)\mathrm{MI}(\{p\},\epsilon)$$

Differential Equations:

$$\mathrm{d}\vec{\mathrm{MI}} = \mathrm{d}A(\{p\},\epsilon)\vec{\mathrm{MI}}$$

[Remiddi, 97]
[Gehrmann, Remiddi, 99]

→ Direct numerical integration

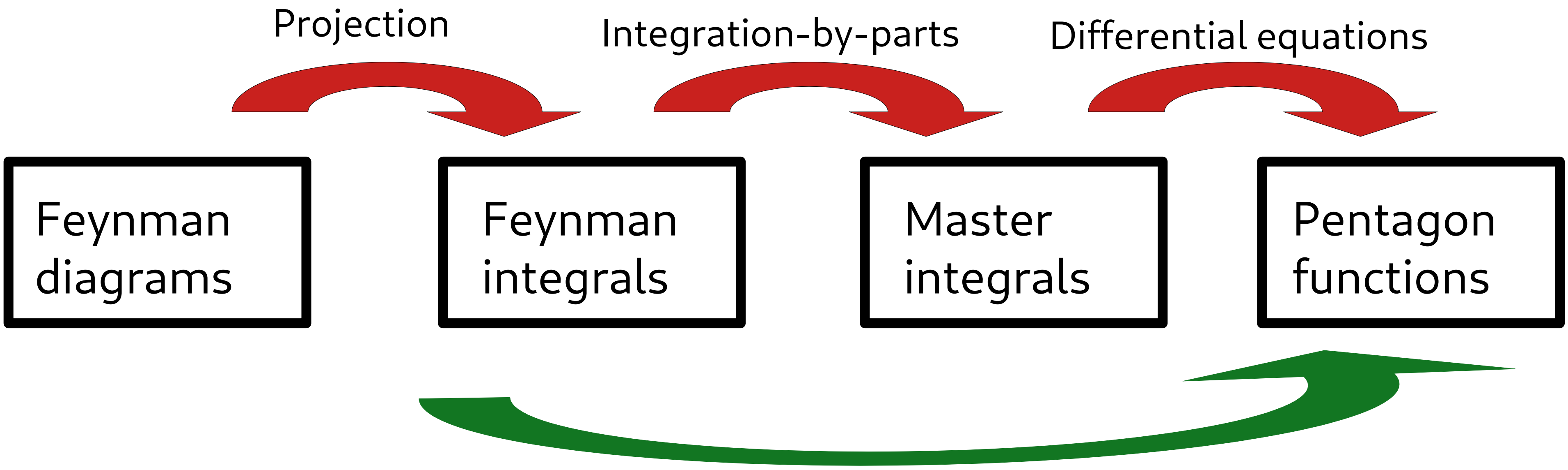
→ Canonical basis: $\mathrm{d}\vec{\mathrm{MI}} = \epsilon \mathrm{d}\tilde{A}(\{p\})\vec{\mathrm{MI}}$ [Henn, 13]

Iterative solution: $\mathrm{MI}_i = \sum_w \epsilon^w \tilde{\mathrm{MI}}_i^w$

Iterated integrals (e.g. "Pentagon"-functions)
[Chicherin, Sotnikov, 20]
[Chicherin, Sotnikov, Zoia, 21]

Overview

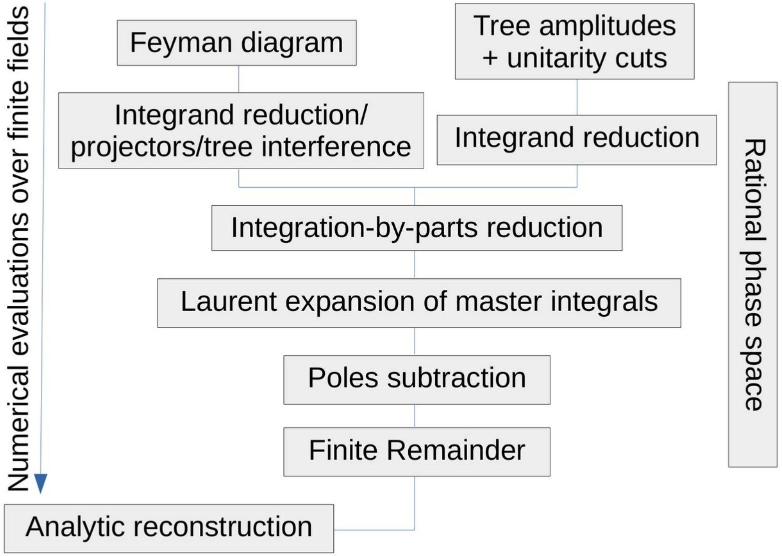
“Old school” approach:



Avoid expression swell through finite-field techniques
FiniteFlow [Peraro’19], Firefly [Klappert‘19], ...

Reconstruction of Amplitudes

Example workflow



QGRAF[Nogueira], FORM[Vermaseren,etal]
MATHEMATICA, SPINNEY[Cullen,etal]

Credit: Bayu

finite field framework: FINITEFLOW[Peraro(2019)]

IBP identities generated using LITERED[Lee(2012)]
solved numerically in FINITEFLOW using
Laporta algorithm[Laporta(2000)]

[Badger,Bronnum-Hansen,Hartanto,Moodie,Peraro,Krys,Zoia]

Mature technology + new optimizations

- Syzygy's to simplify IBPs
- Exploitation of Q-linear relations
- Denominator Ansaetze
- On-the-fly partial fractioning

Massive reduction of complexity

amplitude	helicity	original	stage 1	stage 2	stage 3	stage 4
$A^{(2),1}_{34;q}$	$-++-+$	94/91	74/71	74/0	22/18	22/0
$A^{(2),1}_{34;q}$	$-+-++$	93/89	90/86	90/0	24/14	18/0
$A^{(2),1/N_c^2}_{34;q}$	$-++-+$	90/88	73/71	73/0	23/18	22/0
$A^{(2),1/N_c^2}_{34;q}$	$-+-++$	90/86	86/82	86/0	24/14	19/0
$A^{(2),1/N_c}_{34;l}$	$-+-++$	89/82	74/67	73/0	27/14	20/0
$A^{(2),1/N_c}_{34;l}$	$-++-+$	85/81	61/58	60/0	27/18	20/0
$A^{(2),N_c^2}_{34;q}$	$-+-++$	58/55	54/51	53/0	20/16	20/0

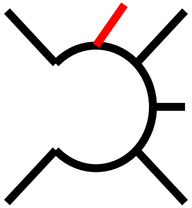
Next-to-leading order case

$$\hat{\sigma}_{ab}^{(1)} = \hat{\sigma}_{ab}^{\mathrm{R}} + \hat{\sigma}_{ab}^{\mathrm{V}} + \hat{\sigma}_{ab}^{\mathrm{C}}$$

KLN theorem
sum is finite for sufficiently inclusive observables and regularization scheme independent

Each term separately infrared (IR) divergent:

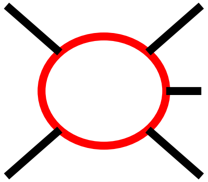
Real corrections:



$$\hat{\sigma}_{ab}^{\mathrm{R}} = \frac{1}{2\hat{s}} \int \boxed{\mathrm{d}\Phi_{n+1}} \left\langle \mathcal{M}_{n+1}^{(0)} \middle| \mathcal{M}_{n+1}^{(0)} \right\rangle \mathrm{F}_{n+1}$$

Phase space integration over unresolved configurations

Virtual corrections:

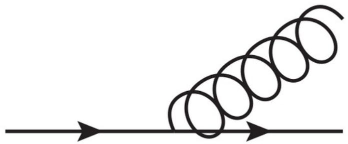


$$\hat{\sigma}_{ab}^{\mathrm{V}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_n \, 2\mathrm{Re} \left\langle \mathcal{M}_n^{(0)} \middle| \mathcal{M}_n^{(1)} \right\rangle \mathrm{F}_n$$

Integration over loop-momentum
(UV divergences cured by renormalization)

IR singularities in real radiation

$$\hat{\sigma}_{ab}^{\mathrm{R}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}_{n+1}^{(0)} \middle| \mathcal{M}_{n+1}^{(0)} \right\rangle \mathrm{F}_{n+1}$$

$\ni \mathrm{d}\Phi_1$ 

$$\sim \int_0 \mathrm{d}E\mathrm{d}\theta \frac{1}{E(1-\cos\theta)} f(E, \cos(\theta))$$

Finite function

Divergent

Regularization in Conventional Dimensional Regularization (CDR) $d = 4 - 2\epsilon$

$$\to \int_0 \mathrm{d}E\mathrm{d}\theta \frac{1}{E^{1-2\epsilon}(1-\cos\theta)^{1-\epsilon}} f(E, \cos(\theta)) \sim \frac{1}{\epsilon^2} + \ldots$$

Cancellation against similar divergences in

$$\hat{\sigma}_{ab}^{\mathrm{V}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_n \, 2\mathrm{Re} \left\langle \mathcal{M}_n^{(0)} \middle| \mathcal{M}_n^{(1)} \right\rangle \mathrm{F}_n$$

How to extract these poles? Slicing and Subtraction

Central idea: Divergences arise from infrared (IR, soft/collinear) limits → Factorization!

Slicing

$$\hat{\sigma}^{\mathrm{R}}_{ab} = \frac{1}{2\hat{s}} \int_{\delta(\Phi)\geq\delta_c} \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1} + \frac{1}{2\hat{s}} \int_{\delta(\Phi)<\delta_c} \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1}$$

$$\approx \frac{1}{2\hat{s}} \int_{\delta(\Phi)\geq\delta_c} \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1} + \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_n \, \tilde{M}(\delta_c) \mathrm{F}_n + \mathcal{O}(\delta_c)$$

$$\ldots + \hat{\sigma}^{\mathrm{V}}_{ab} = \text{finite}$$

Subtraction

$$\hat{\sigma}^{\mathrm{R}}_{ab} = \frac{1}{2\hat{s}} \int \left(\mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1} - \mathrm{d}\tilde{\Phi}_{n+1} \, \mathcal{S}\mathrm{F}_n \right) + \frac{1}{2\hat{s}} \int \mathrm{d}\tilde{\Phi}_{n+1} \, \mathcal{S}\mathrm{F}_n$$

$$\frac{1}{2\hat{s}} \int \mathrm{d}\tilde{\Phi}_{n+1} \, \mathcal{S}\mathrm{F}_n = \frac{1}{2\hat{s}} \int \underline{\mathrm{d}\Phi_n \, \mathrm{d}\Phi_1} \, \mathcal{S}\mathrm{F}_n$$

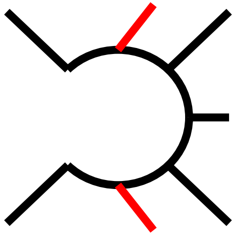
Phase space factorization
→ momentum mappings

Most popular NLO QCD schemes:
CS [hep-ph/9605323],
FKS [hep-ph/9512328]

→ Basis of modern event simulations
[MadGraph, Sherpa, Herwig, ...]

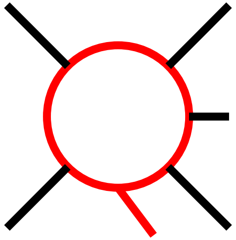
Partonic cross section beyond NLO

$$\hat{\sigma}^{(2)}_{ab} = \hat{\sigma}^{\rm VV}_{ab} + \hat{\sigma}^{\rm RV}_{ab} + \hat{\sigma}^{\rm RR}_{ab} + \hat{\sigma}^{\rm C2}_{ab} + \hat{\sigma}^{\rm C1}_{ab}$$



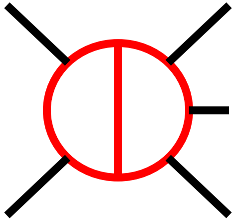
Real-Real

$$\hat{\sigma}^{\rm RR}_{ab} = \frac{1}{2\hat{s}} \int {\rm d}\Phi_{n+2} \left\langle \mathcal{M}^{(0)}_{n+2} \middle| \mathcal{M}^{(0)}_{n+2} \right\rangle {\rm F}_{n+2}$$



Real-Virtual

$$\hat{\sigma}^{\rm RV}_{ab} = \frac{1}{2\hat{s}} \int {\rm d}\Phi_{n+1}\, 2{\rm Re} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(1)}_{n+1} \right\rangle {\rm F}_{n+1}$$



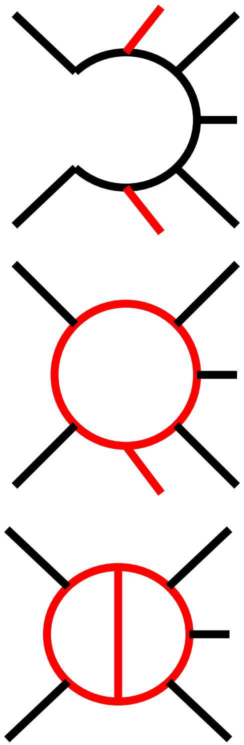
Virtual-Virtual

$$\hat{\sigma}^{\rm VV}_{ab} = \frac{1}{2\hat{s}} \int {\rm d}\Phi_{n} \left(2{\rm Re} \left\langle \mathcal{M}^{(0)}_{n} \middle| \mathcal{M}^{(2)}_{n} \right\rangle + \left\langle \mathcal{M}^{(1)}_{n} \middle| \mathcal{M}^{(1)}_{n} \right\rangle \right) {\rm F}_{n}$$

$\hat{\sigma}^{\rm C2}_{ab} = (\text{double convolution})\, {\rm F}_n$ $\quad$ $\hat{\sigma}^{\rm C1}_{ab} = (\text{single convolution})\, {\rm F}_{n+1}$

Partonic cross section beyond NLO

$$\hat{\sigma}^{(2)}_{ab} = \hat{\sigma}^{\mathrm{VV}}_{ab} + \hat{\sigma}^{\mathrm{RV}}_{ab} + \hat{\sigma}^{\mathrm{RR}}_{ab} + \hat{\sigma}^{\mathrm{C2}}_{ab} + \hat{\sigma}^{\mathrm{C1}}_{ab}$$



Technically substantially more complicated!

Main bottlenecks:

- Real - real → overlapping singularities
  Many possible limits → good organization principle needed
- Real - virtual → stable matrix elements
- Virtual - virtual → complicated case-by-case analytic treatment

Slicing and Subtraction

Slicing

- Conceptually simple
- Recycling of lower computations
- Non-local cancellations/power-corrections
  → computationally expensive

NNLO QCD schemes

qT-slicing [Catani'07],
N-jettiness slicing [Gaunt'15/Boughezal'15]

Subtraction

- Conceptually more difficult
- Local subtraction → efficient
- Better numerical stability
- Choices:
  - Momentum mapping
  - Subtraction terms
  - Numerics vs. analytic

Antenna [Gehrmann'05-'08],
Colorful [DelDuca'05-'15],
Sector-improved residue subtraction [Czakon'10-'14'19]
Projection [Cacciari'15],
Nested collinear [Caola'17],
Torino [Magnea'21]
Geometric [Herzog'18],
Unsubtraction [Aguilera-Verdugo'19],
…

Slicing and Subtraction

Level of generality
→ complexity of colour structure

Color-singlet, massive quarks

Color-singlet+single jet

General jet processes with arbitrary final state and collinear factorised initial states

NNLO QCD schemes

qT-slicing [Catani'07],
N-jettiness slicing [Gaunt'15/Boughezal'15]

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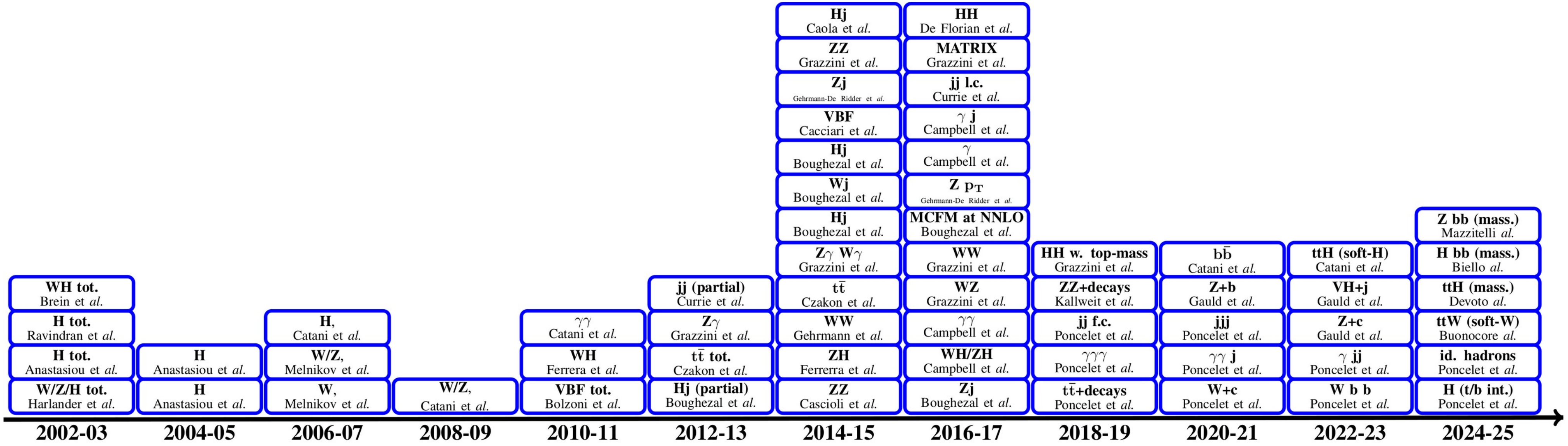
...

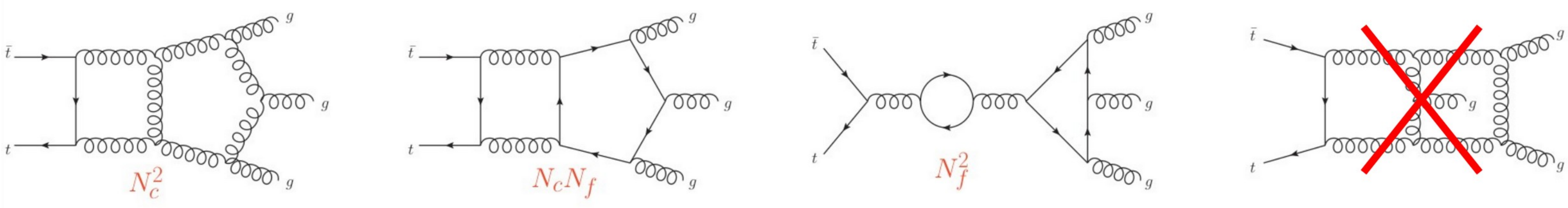
NNLO QCD phenomenology



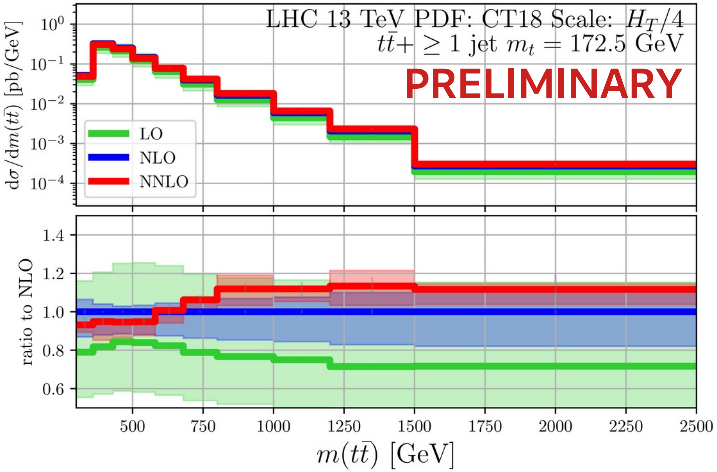
Top-quark pair production in association with a jet

Two-loop amplitudes in leading colour approximation

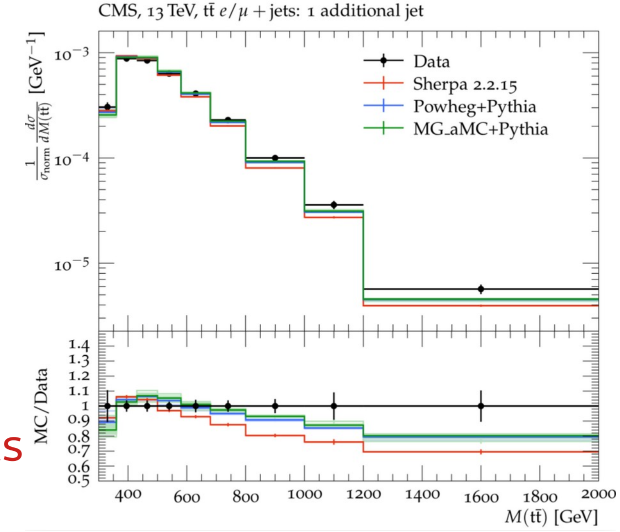
[Badger, Becchetti, Brancaccio, Hartanto, Zoia, 2412.13876] + [Czakon, Poncelet]



First phenomenological studies:



Expected to lift various tensions In tt+j measurements



[1803.08856]

Beyond the parton level

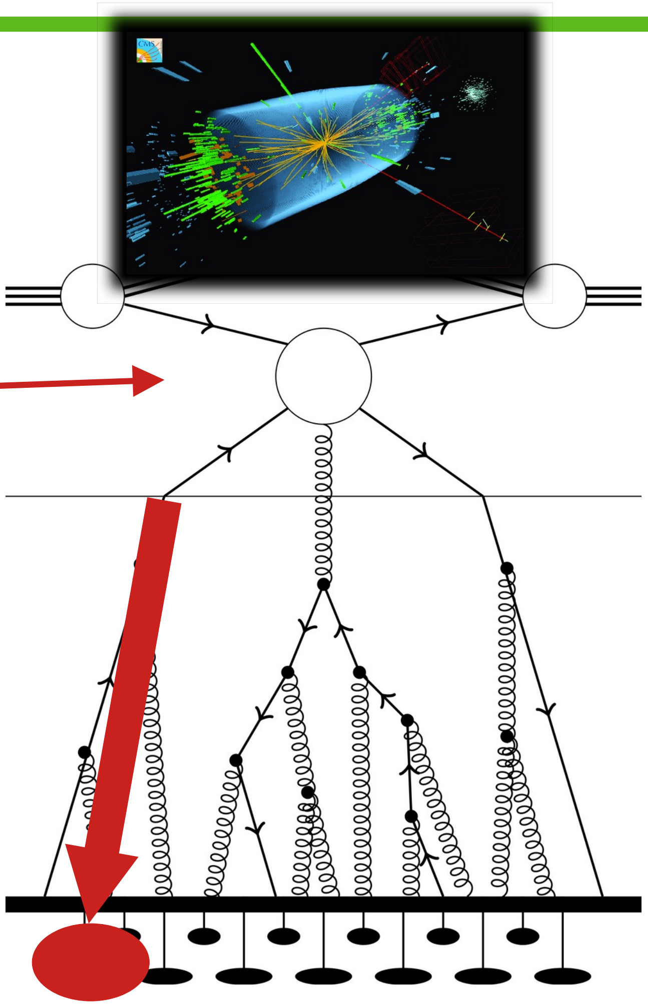
Guiding principle: factorization
"What you see depends on the energy scale"

In Quantum Chromodynamics (QCD):

$Q \gg \Lambda_{\mathrm{QCD}}$ **Fixed-order perturbation theory**
scattering of individual partons

Parton to identified object transition "**Fragmentation**"
→ Resummation of collinear logs through 'DGLAP'
→ Non perturbative fragmentation functions
Example: B-hadrons in e+e-

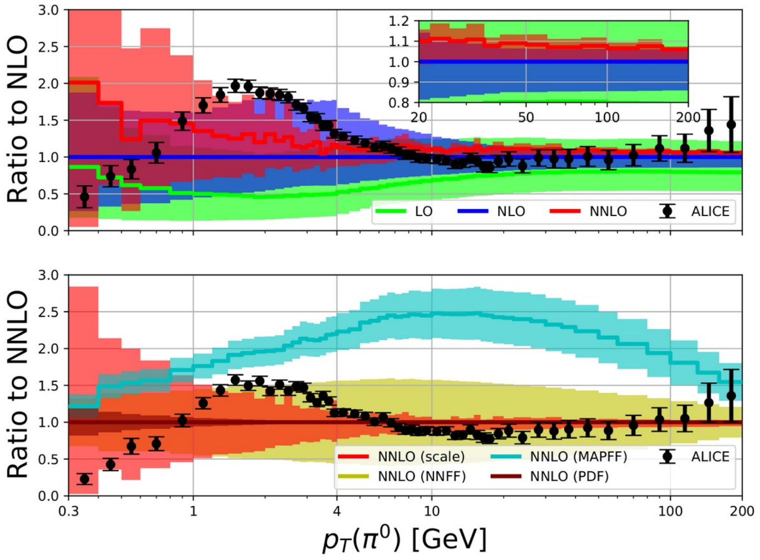
$$\frac{d\sigma_B(m_b,z)}{dz} = \sum_i \left\{ \frac{d\sigma_i(\mu_{Fr},z)}{dz} \otimes D_{i\to B}(\mu_{Fr},m_b,z) \right\}(z) + \mathcal{O}(m_b^2)$$



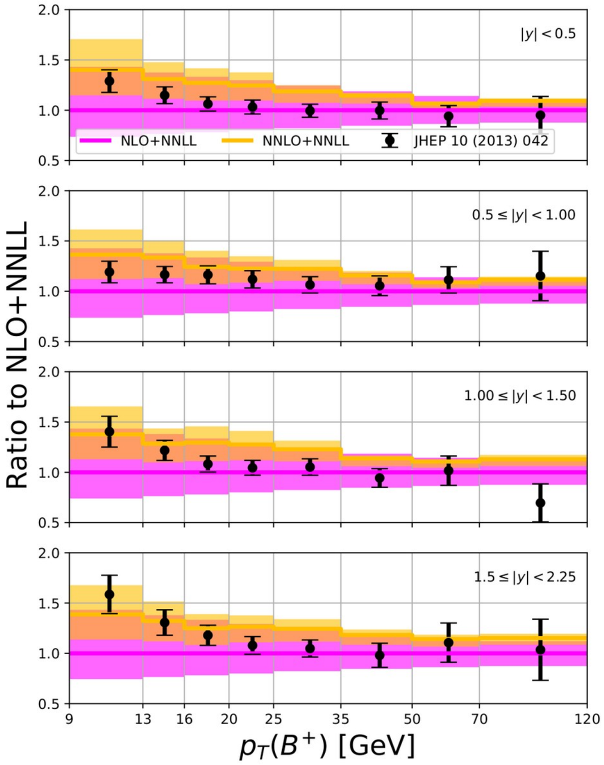
Identified hadrons

Inclusion of fragmentation through NNLO QCD:
[Czakon, Generet, Mitov, Poncelet]
- B-hadrons in top-decays [2210.06078,2102.08267]
- Open-bottom [2411.09684]
- Identified hadrons [2503.11489]

$$d\sigma_{pp\to h}(p) = \sum_i \int dz\ d\hat{\sigma}_{pp\to i}\left(\frac{p}{z}\right) D_{i\to h}(z)$$



Pion production



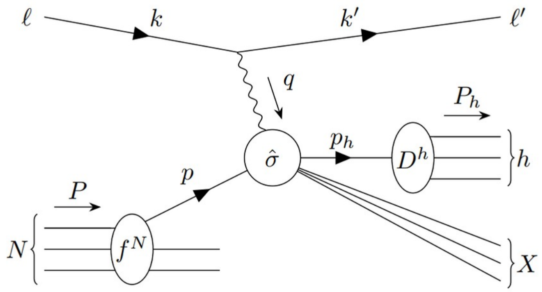
Open-bottom @FONNLL:

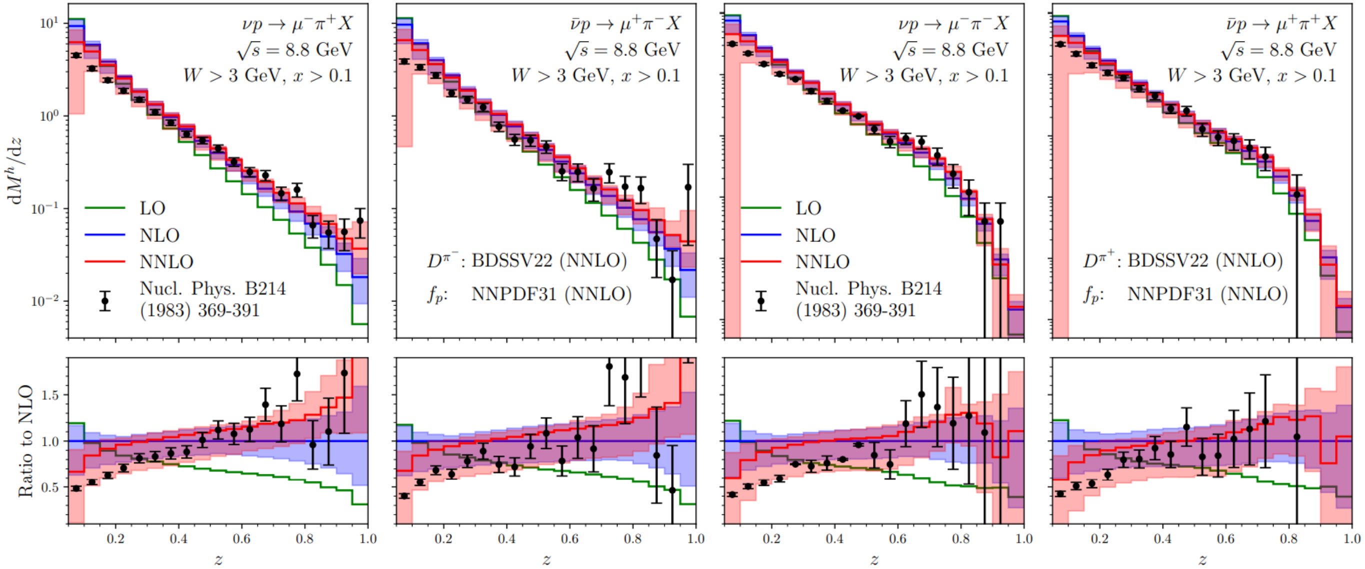
$$\sigma(p_T) = \sigma(m, p_T) + G(p_T)(\sigma(0, p_T) - \sigma(0, p_T)|_{\rm FO})$$

SIDIS

Series of works on SIDIS through NNLO QCD:
[Bonino, Gehrmann, Loechner, Schoenwald, Stagnitto]

- Polarised initial states [2404.08597]
- Neutrino-Nucleon Scattering [2504.05376]
- CC and NC [2506.19926]





[2504.05376]

Jet substructure

Semi-inclusive jet function [1606.06732, 2410.01902]

$$\frac{d\sigma_{\rm LP}}{dp_T d\eta} = \sum_{i,j,k} \int_{x_{i,\min}}^1 \frac{dx_i}{x_i} f_{i/P}(x_i,\mu) \int_{x_{j,\min}}^1 \frac{dx_j}{x_j} f_{j/P}(x_j,\mu) \int_{z_{\min}}^1 \frac{dz}{z} \mathcal{H}_{ij}^k(x_i, x_j, p_T/z, \eta, \mu) \times J_k\left(z, \ln\frac{p_T^2 R^2}{z^2\mu^2}, \mu\right),$$

The same hard function as for identified hadrons!

Modified RGE:

[2402.05170,2410.01902]

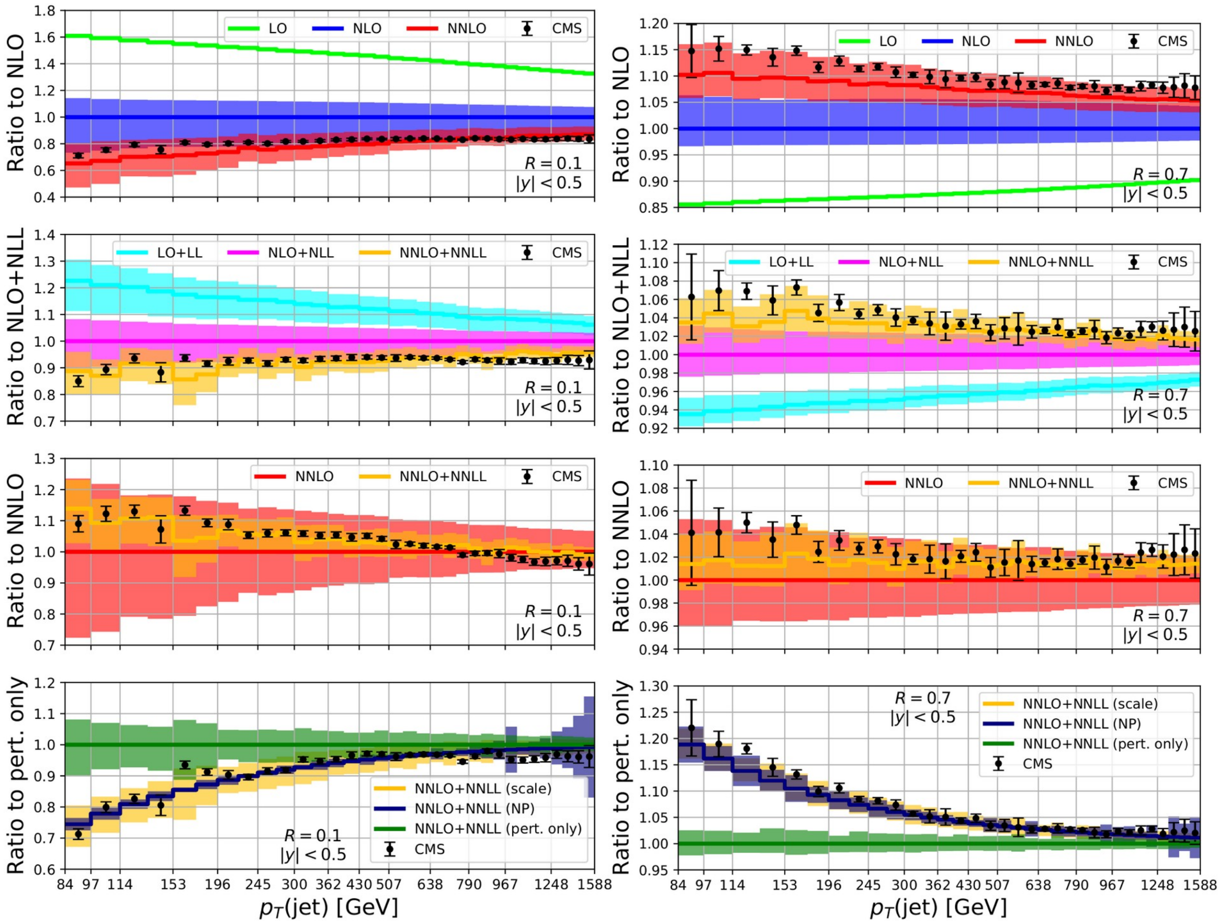
$$\frac{d\vec{J}\left(z, \ln\frac{p_T^2R^2}{z^2\mu^2}, \mu\right)}{d\ln\mu^2} = \int_z^1 \frac{dy}{y} \vec{J}\left(\frac{z}{y}, \ln\frac{y^2 p_T^2 R^2}{z^2\mu^2}, \mu\right) \cdot \widehat{P}_T(y)$$

Energy-Energy correlators obey similar factorization!

Small-R jets

Application to small-R jets
[Generet, Lee, Moult, Poncelet, Zhang]
[2503.21866]

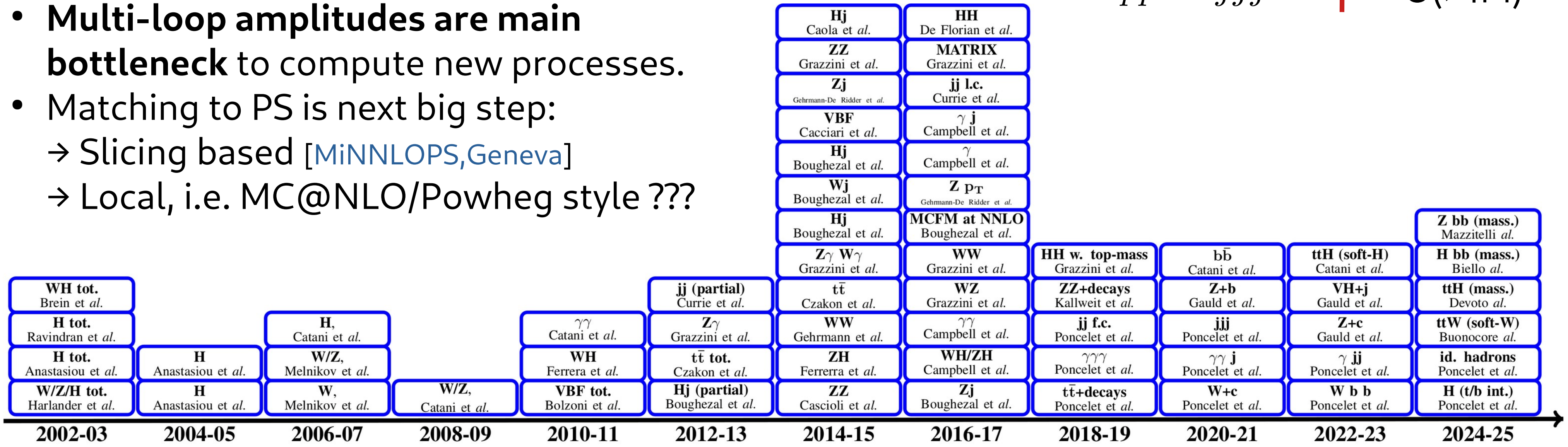
‘Triple’ differential measurement by CMS:
Y, pT, R [2005.05159]



Summary/Outlook

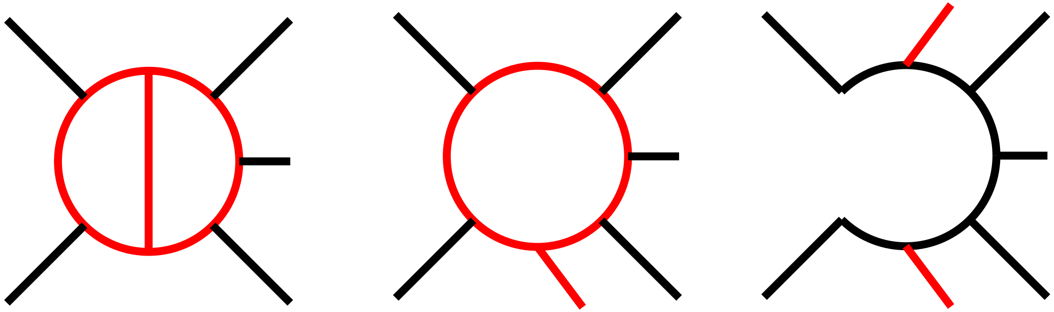
- Formally NNLO IR subtraction is done
  → technically **efficiency remains a crucial question**!
- Many phenomenological applications:
  - Done: 2 → 2, 2 → 3 massless
  - State-of-the-art: 2 → 3 with masses
  - Fragmentation processes start to appear
- **Multi-loop amplitudes are main bottleneck** to compute new processes.
- Matching to PS is next big step:
  → Slicing based [MiNNLOPS,Geneva]
  → Local, i.e. MC@NLO/Powheg style ???

Process class	Core-hours
$pp \to V$	~hours
$pp \to VV$	O(1k)
$pp \to V + j$	O(>10k)
$pp \to jj$	O(>100k)
$pp \to jjj$	O(>1M)



Backup

Sector-improved residue subtraction



Sector decomposition I

Considering working in CDR:

→ Virtuals are usually done in this regularization: $\hat{\sigma}_{ab}^{VV} = \sum_{i=-4}^{0} c_i \epsilon^i + \mathcal{O}(\epsilon)$

→ Can we write the real radiation as such expansion?

→ Difficult integrals, analytical impractical (except very simple observables)!

→ Numerics not possible, integrals are divergent → ε-poles!

How to extract these poles? → Sector decomposition!

Divide and conquer the phase space:

$$1 = \sum_{i,j} \left[\sum_k \mathcal{S}_{ij,k} + \sum_{k,l} \mathcal{S}_{i,k;j,l} \right] \quad \rightarrow \quad \hat{\sigma}_{ab}^{\mathrm{RR}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+2} \sum_{i,j} \left[\sum_k \mathcal{S}_{ij,k} + \sum_{k,l} \mathcal{S}_{i,k;j,l} \right] \left\langle \mathcal{M}_{n+2}^{(0)} \middle| \mathcal{M}_{n+2}^{(0)} \right\rangle \mathrm{F}_{n+2}$$

Sector decomposition II

Divide and conquer the phase space

- Each $\mathcal{S}_{i,k}$ (NLO), $\mathcal{S}_{ij,k}/\mathcal{S}_{i,k;j,l}$ (NNLO) has simpler divergences:
  - Soft limits of partons i and j
  - Collinear w.r.t partons k (and l) of partons i and j

$$\mathcal{S}_{i,k} = \frac{1}{D_1 d_{i,k}} \quad D_1 = \sum_{ik} \frac{1}{d_{i,k}} \qquad d_{i,k} = \frac{E_i}{\sqrt{s}}(1 - \cos\theta_{ik})$$

- Parametrization w.r.t. reference parton makes divergences explicit

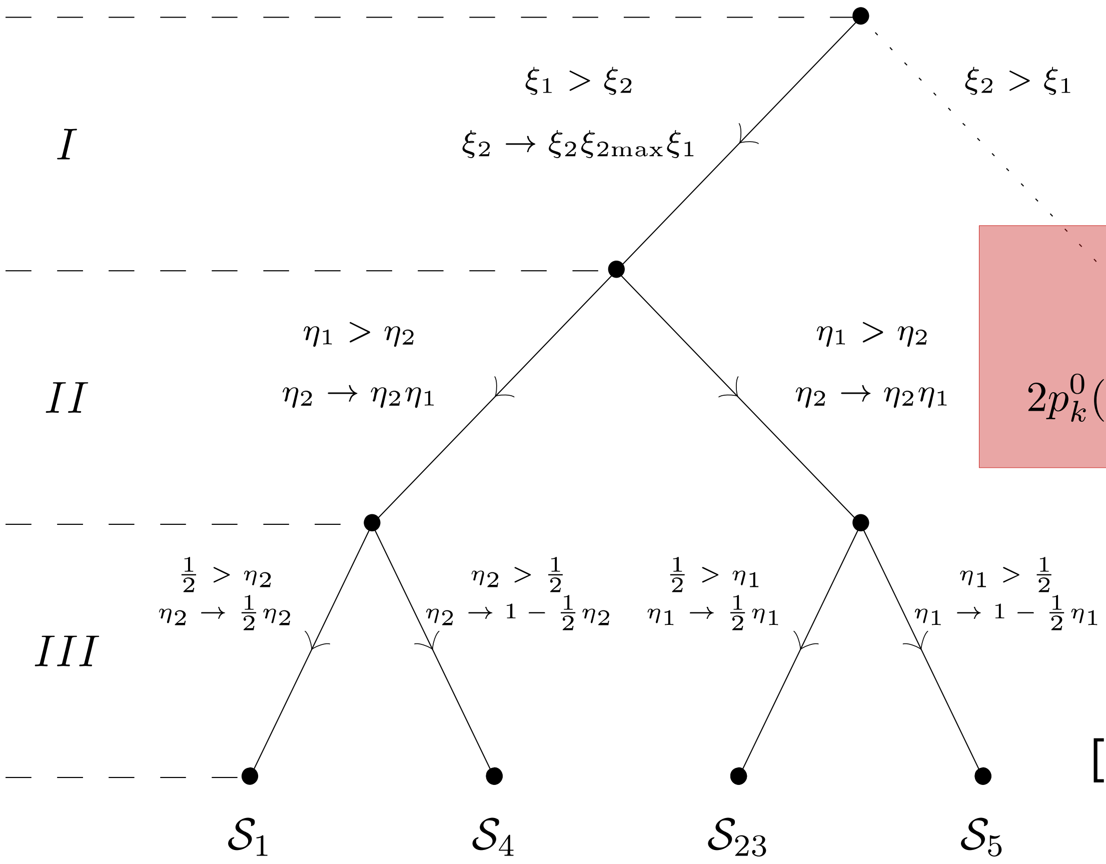
$$\hat{\eta}_i = \tfrac{1}{2}(1 - \cos\theta_{ik}) \in [0,1] \qquad \hat{\xi}_i = \frac{u_i^0}{u_{\max}^0} \in [0,1]$$

- Example: Splitting function

$$\sim \frac{1}{s_{ik}} P(z) \qquad s_{ik} = (p_i + p_k)^2 = 2 p_k^0 u_{\max}^0 \xi_i \eta_i \qquad \sim \frac{1}{\eta_i \xi_i}$$

Sector decomposition II – triple collinear factorization

Three particle invariant does not factorize trivially:



Double soft factorization:

$$\theta(u_i^0 - u_j^0) + \theta(u_i^0 - u_j^0)$$

$$(p_k + u_i + u_j)^2 =$$

$$2p_k^0(\xi_i \eta_i u_{\max}^{0,i} + \xi_j \eta_j u_{\max}^{0,j} + \xi_i \xi_j \frac{u_{max}^{0,i} u_{max}^{0,j}}{p_k^0} \angle(u_i, u_j))$$

[Czakon'10,Caola'17]

Sector decomposition III

Factorized singular limits in each sector:

$$\frac{1}{2\hat{s}}\int \mathrm{d}\Phi_{n+2}\,\mathcal{S}_{kl,m}\left\langle \mathcal{M}^{(0)}_{n+2}\middle|\mathcal{M}^{(0)}_{n+2}\right\rangle \mathrm{F}_{n+2} = \sum_{\text{sub-sec.}}\int \mathrm{d}\Phi_n \prod \mathrm{d}x_i\,\underbrace{x_i^{-1-b_i\epsilon}}_{\text{singular}}\,\mathrm{d}\tilde{\mu}(\{x_i\})\underbrace{\prod x_i^{a_i+1}\langle \mathcal{M}_{n+2}|\mathcal{M}_{n+2}\rangle}_{\text{regular}}\mathrm{F}_{n+2}$$

$x_i \in \{\eta_1, \xi_1, \eta_2, \xi_2\}$

Regularization of divergences:

$$x^{-1-b\epsilon} = \underbrace{\frac{-1}{b\epsilon}}_{\text{pole term}} + \underbrace{\left[x^{-1-b\epsilon}\right]_+}_{\text{reg. + sub.}}$$

$$\int_0^1 \mathrm{d}x\,\left[x^{-1-b\epsilon}\right]_+ f(x) = \int_0^1 \frac{f(x)-f(0)}{x^{1+b\epsilon}}$$

Finite NNLO cross section

$$\hat{\sigma}_{ab}^{\mathrm{RR}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+2} \left\langle \mathcal{M}_{n+2}^{(0)} \middle| \mathcal{M}_{n+2}^{(0)} \right\rangle \mathrm{F}_{n+2}$$

$$\hat{\sigma}_{ab}^{\mathrm{RV}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+1}\, 2\mathrm{Re} \left\langle \mathcal{M}_{n+1}^{(0)} \middle| \mathcal{M}_{n+1}^{(1)} \right\rangle \mathrm{F}_{n+1}$$

$$\hat{\sigma}_{ab}^{\mathrm{VV}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n} \left(2\mathrm{Re} \left\langle \mathcal{M}_{n}^{(0)} \middle| \mathcal{M}_{n}^{(2)} \right\rangle + \left\langle \mathcal{M}_{n}^{(1)} \middle| \mathcal{M}_{n}^{(1)} \right\rangle \right) \mathrm{F}_{n}$$

$$\hat{\sigma}_{ab}^{\mathrm{C1}} = (\text{single convolution})\, \mathrm{F}_{n+1}$$

$$\hat{\sigma}_{ab}^{\mathrm{C2}} = (\text{double convolution})\, \mathrm{F}_{n}$$

sector decomposition and master formula

$$x^{-1-b\epsilon} = \underbrace{\frac{-1}{b\epsilon}}_{\text{pole term}} + \underbrace{\left[x^{-1-b\epsilon}\right]_+}_{\text{reg. + sub.}}$$

$$\left(\sigma_F^{RR}, \sigma_{SU}^{RR}, \sigma_{DU}^{RR}\right) \quad \left(\sigma_F^{RV}, \sigma_{SU}^{RV}, \sigma_{DU}^{RV}\right) \quad \left(\sigma_F^{VV}, \sigma_{DU}^{VV}, \sigma_{FR}^{VV}\right) \quad \left(\sigma_{SU}^{C1}, \sigma_{DU}^{C1}\right) \quad \left(\sigma_{DU}^{C2}, \sigma_{FR}^{C2}\right)$$

re-arrangement of terms → 4-dim. formulation [Czakon'14,Czakon'19]

$$\underline{\left(\sigma_F^{RR}\right)} \quad \underline{\left(\sigma_F^{RV}\right)} \quad \underline{\left(\sigma_F^{VV}\right)} \quad \underline{\left(\sigma_{SU}^{RR}, \sigma_{SU}^{RV}, \sigma_{SU}^{C1}\right)} \quad \underline{\left(\sigma_{DU}^{RR}, \sigma_{DU}^{RV}, \sigma_{DU}^{VV}, \sigma_{DU}^{C1}, \sigma_{DU}^{C2}\right)} \quad \underline{\left(\sigma_{FR}^{RV}, \sigma_{FR}^{VV}, \sigma_{FR}^{C2}\right)}$$

separately finite: ε poles cancel

Improved phase space generation

Phase space cut and differential observable introduce

mis-binning : mismatch between kinematics in subtraction terms

→ leads to increased variance of the integrand

→ slow Monte Carlo convergence

New phase space parametrization [Czakon'19]:

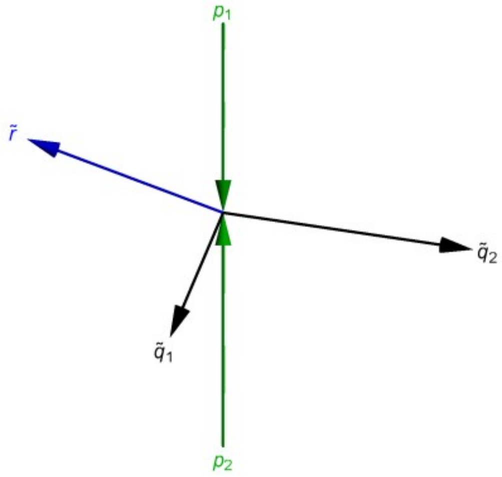
Minimization of # of different subtraction kinematics in each sector

Improved phase space generation

New phase space parametrization:

Minimization of # of different subtraction kinematics in each sector

Mapping from n+2 to n particle phase space: $\{P, r_j, u_k\} \to \left\{\tilde{P}, \tilde{r}_j\right\}$



Requirements:

- Keep direction of reference r fixed
- Invertible for fixed u_i : $\left\{\tilde{P}, \tilde{r}_j, u_k\right\} \to \{P, r_j, u_k\}$
- Preserve Born invariant mass: $q^2 = \tilde{q}^2\,, \;\; \tilde{q} = \tilde{P} - \sum_{j=1}^{n_{fr}} \tilde{r}_j$

Main steps:

- Generate Born configuration
- Generate unresolved partons u_i
- Rescale reference momentum $r = x\tilde{r}$
- Boost non-reference momenta of the Born configuration

Improved phase space generation

New phase space parametrization:

Minimization of # of different subtraction kinematics in each sector

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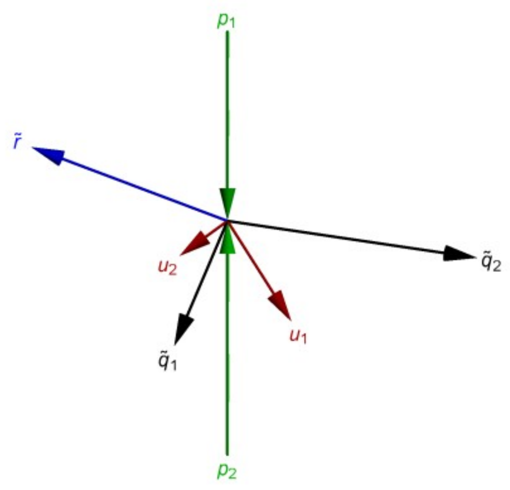

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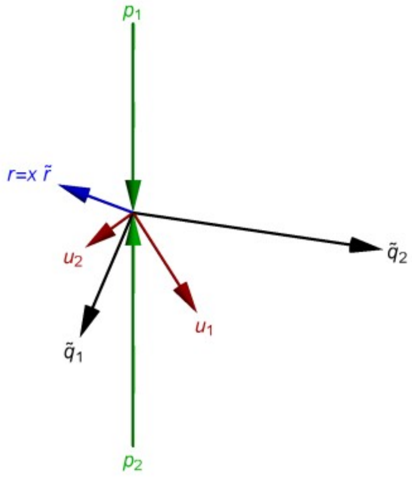

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Improved phase space generation

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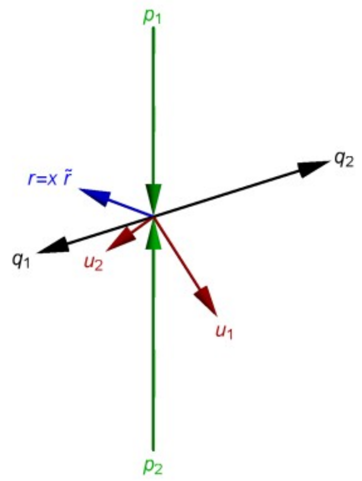


Requirements:

- Keep direction of reference r fixed
- Invertible for fixed u_i : $\left\{\tilde{P}, \tilde{r}_j, u_k\right\} \to \{P, r_j, u_k\}$
- Preserve Born invariant mass: $q^2 = \tilde{q}^2\,, \;\; \tilde{q} = \tilde{P} - \sum_{j=1}^{n_{fr}} \tilde{r}_j$

Main steps:

- Generate Born configuration
- Generate unresolved partons u_i
- Rescale reference momentum $r = x\tilde{r}$
- Boost non-reference momenta of the Born configuration

t'HV corrections

Observables: Implemented by infrared safe measurement function (MF) F_m

Infrared property in STRIPPER context:

- $\{x_i\} \to 0 \leftrightarrow$ single unresolved limit
  $\Rightarrow \mathrm{F}_{n+2} \to \mathrm{F}_{n+1}$
- $\{x_i\} \to 0 \leftrightarrow$ double unresolved limit
  $\Rightarrow \mathrm{F}_{n+2} \to \mathrm{F}_n$
  $\Rightarrow \mathrm{F}_{n+1} \to \mathrm{F}_n$

Tool for new formulation in the 't Hooft Veltman scheme:

Parameterized MF $\mathrm{F}^{\alpha}_{n+1}$

- $\mathrm{F}^{\alpha}_n \equiv 0$ for $\alpha \neq 0$
  (NLO MF)
- 'arbitrary' F^0_n
  (NNLO MF)
- $\alpha \neq 0 \Rightarrow \mathrm{DU} = 0$ and SU separately finite

Example: $F^{\alpha}_{n+1} = F_{n+1}\Theta_\alpha(\{\alpha_i\})$
with $\Theta_\alpha = 0$ if some $\alpha_i < \alpha$

t'HV corrections

$$\sigma_{SU} = \sigma_{SU}^{RR} + \sigma_{SU}^{RV} + \sigma_{SU}^{C1} \qquad \text{where} \qquad \sigma_{SU}^{c} = \int \mathrm{d}\Phi_{n+1} \left(I_{n+1}^{\mathrm{c}} F_{n+1} + I_{n}^{\mathrm{c}} F_{n} \right)$$

NLO measurement function ($\alpha \neq 0$):

$$\int \mathrm{d}\Phi_{n+1} \left(I_{n+1}^{\mathrm{RR}} + I_{n+1}^{\mathrm{RV}} + I_{n+1}^{\mathrm{C1}} \right) F_{n+1}^{\alpha} = \text{finite in 4 dim.}$$

All divergences cancel in d-dimensions:

$$\sum_{c} \int \mathrm{d}\Phi_{n+1} \left[\frac{I_{n+1}^{c,(-2)}}{\epsilon^2} + \frac{I_{n+1}^{c,(-1)}}{\epsilon} \right] F_{n+1}^{\alpha} \equiv \sum_{c} \mathcal{I}^{c} = 0$$

t'HV corrections

$$\sigma_{SU} = \sigma_{SU}^{RR} + \sigma_{SU}^{RV} + \sigma_{SU}^{C1} \underbrace{- \mathcal{I}^{RR} - \mathcal{I}^{RV} - \mathcal{I}^{C1}}_{=0}$$

$$\begin{aligned}
\sigma_{SU}^{c} - \mathcal{I}^{c} &= \int \mathrm{d}\Phi_{n+1} \left\{ \left[\frac{I_{n+1}^{c,(-2)}}{\epsilon^2} + \frac{I_{n+1}^{c,(-1)}}{\epsilon} + I_{n+1}^{c,(0)} \right] F_{n+1} + \left[\frac{I_{n}^{c,(-2)}}{\epsilon^2} + \frac{I_{n}^{c,(-1)}}{\epsilon} + I_{n}^{c,(0)} \right] F_{n} \right\} \\
&\quad - \int \mathrm{d}\Phi_{n+1} \left[\frac{I_{n+1}^{c,(-2)}}{\epsilon^2} + \frac{I_{n+1}^{c,(-1)}}{\epsilon} \right] F_{n+1} \Theta_{\alpha}(\{\alpha_i\}) \\
&= \int \mathrm{d}\Phi_{n+1} \left[\frac{I_{n+1}^{c,(-2)} F_{n+1} + I_{n}^{c,(-2)} F_{n}}{\epsilon^2} + \frac{I_{n+1}^{c,(-1)} F_{n+1} + I_{n}^{c,(-1)} F_{n}}{\epsilon} \right] (1 - \Theta_{\alpha}(\{\alpha_i\})) \\
&\quad + \int \mathrm{d}\Phi_{n+1} \left[I_{n+1}^{c,(0)} F_{n+1} + I_{n}^{c,(0)} F_{n} \right] + \int \mathrm{d}\Phi_{n+1} \left[\frac{I_{n}^{c,(-2)}}{\epsilon^2} + \frac{I_{n}^{c,(-1)}}{\epsilon} \right] F_{n} \Theta_{\alpha}(\{\alpha_i\})
\end{aligned}$$

$$=: \underbrace{Z^{c}(\alpha)}_{\text{integrable, zero volume for } \alpha \to 0} + \underbrace{C^{c}}_{\text{no divergencies}} + \underbrace{N^{c}(\alpha)}_{\text{only } F_n \to \text{DU}}$$

t'HV corrections

Looks like slicing, but it is slicing *only* for divergences
$\rightarrow$ no actual slicing parameter in result

Powerlog-expansion:

$$N^c(\alpha) = \sum_{k=0}^{\ell_{\max}} \ln^k(\alpha) N_k^c(\alpha)$$

- all $N_k^c(\alpha)$ regular in α
- start expression independent of $\alpha \Rightarrow$ all logs cancel
- only $N_0^c(0)$ relevant

Putting parts together:

$$\sigma_{SU} - \sum_c N_0^c(0) \text{ and } \sigma_{DU} + \sum_c N_0^c(0)$$

are finite in 4 dimension

$\Downarrow$

SU contribution: $\sigma_{SU} - \sum_c N_0^c = \sum_c C^c$
original expression σ_{SU} in 4-dim without poles, no further ϵ pole cancellation

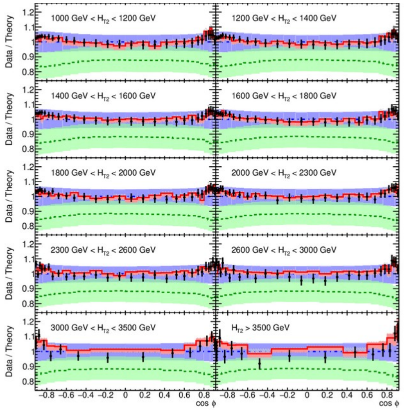
C++ framework

- Formulation allows efficient algorithmic implementation → STRIPPER
- **High degree of automation**:
  - Partonic processes (taking into account all symmetries)
  - Sectors and subtraction terms
  - Interfaces to Matrix-element providers + O(100) hardcoded: AvH, OpenLoops, Recola, NJET, HardCoded

    → In practice: Only two-loop matrix elements required
- **Broad range of applications** through additional facilities:
  - Narrow-Width & Double-Pole Approximation
  - Fragmentation
  - Polarised intermediate massive bosons
  - (Partial) Unweighting → Event generation for **HighTEA**
  - Interfaces: FastNLO, FastJet

HighTEA

HighTEA



= ~100 MCPUh

How to make this more efficient/environment-friendly/accessible/faster?

high tea

for your freshly brewed analysis

https://www.precision.hep.phy.cam.ac.uk/hightea

HighTEA: High energy Theory Event Analyser
[2304.05993]

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Basic idea

- Database of precomputed "Theory Events"
  - **Equivalent to a full fledged computation**
  - Currently this means partonic fixed order events
  - Extensions to included showered/resummed/hadronized events is feasible
  - (Partially) Unweighting to increase efficiency
- Analysis of the data through an user interface
  - Easy-to-use
  - Fast
  - Flexible:
    - Observables from basic 4-momenta
    - Free specification of bins
    - Renormalization/Factorization Scale variation
    - PDF (member) variation
    - Specify phase space cuts

Not so new idea:
LHE [Alwall et al '06],
Ntuple [BlackHat '08'13],

Factorizations

Factorizing renormalization and factorization scale dependence:

$$w_s^{i,j} = w_{\mathrm{PDF}}(\mu_F, x_1, x_2) w_{\alpha_s}(\mu_R) \left(\sum_{i,j} c_{i,j} \ln\left(\mu_R^2\right)^i \ln\left(\mu_F^2\right)^j \right)$$

PDF dependence:

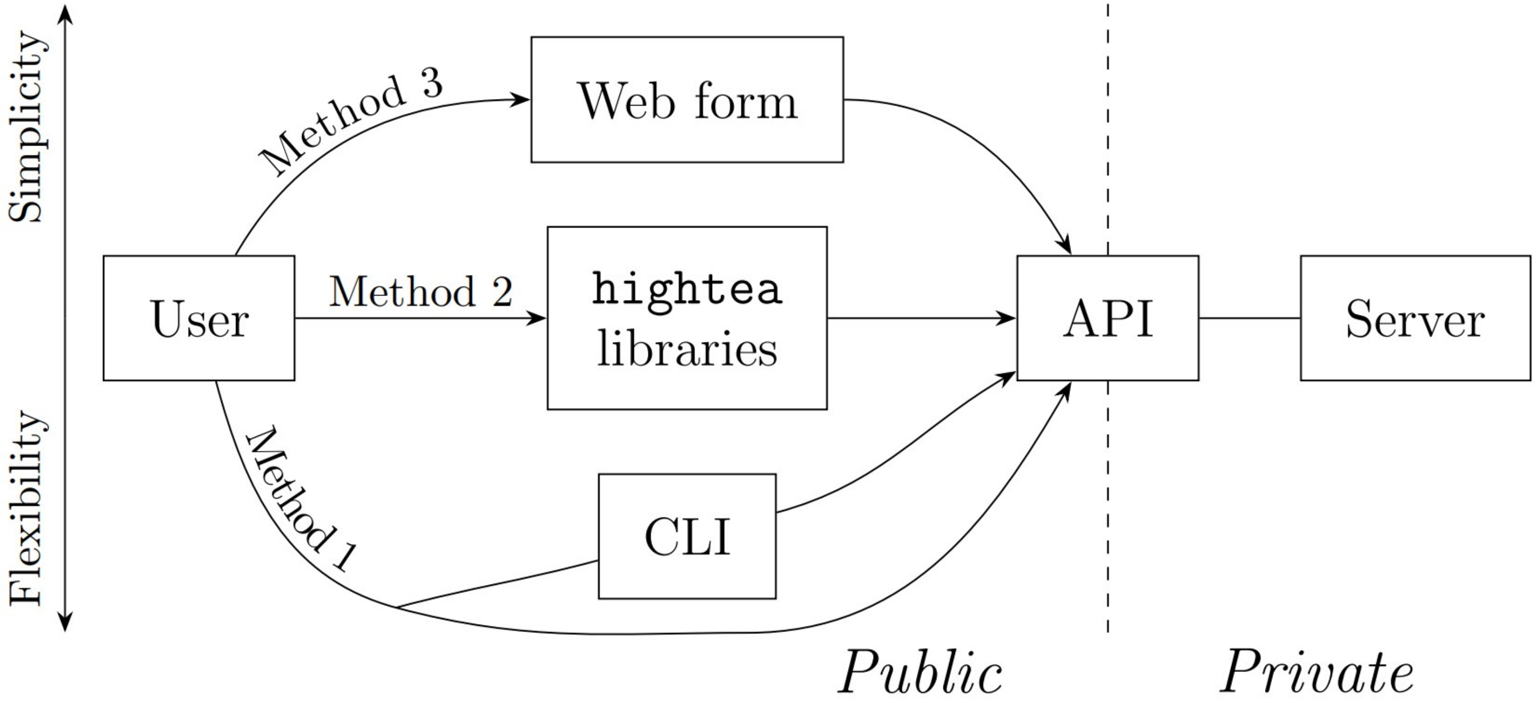
$$w_{\mathrm{PDF}}(\mu, x_1, x_2) = \sum_{ab \in \mathrm{channel}} f_a(x_1, \mu) f_b(x_2, \mu)$$

α_s dependence:

$$w_{\alpha_s}(\mu) = (\alpha_s(\mu))^m$$

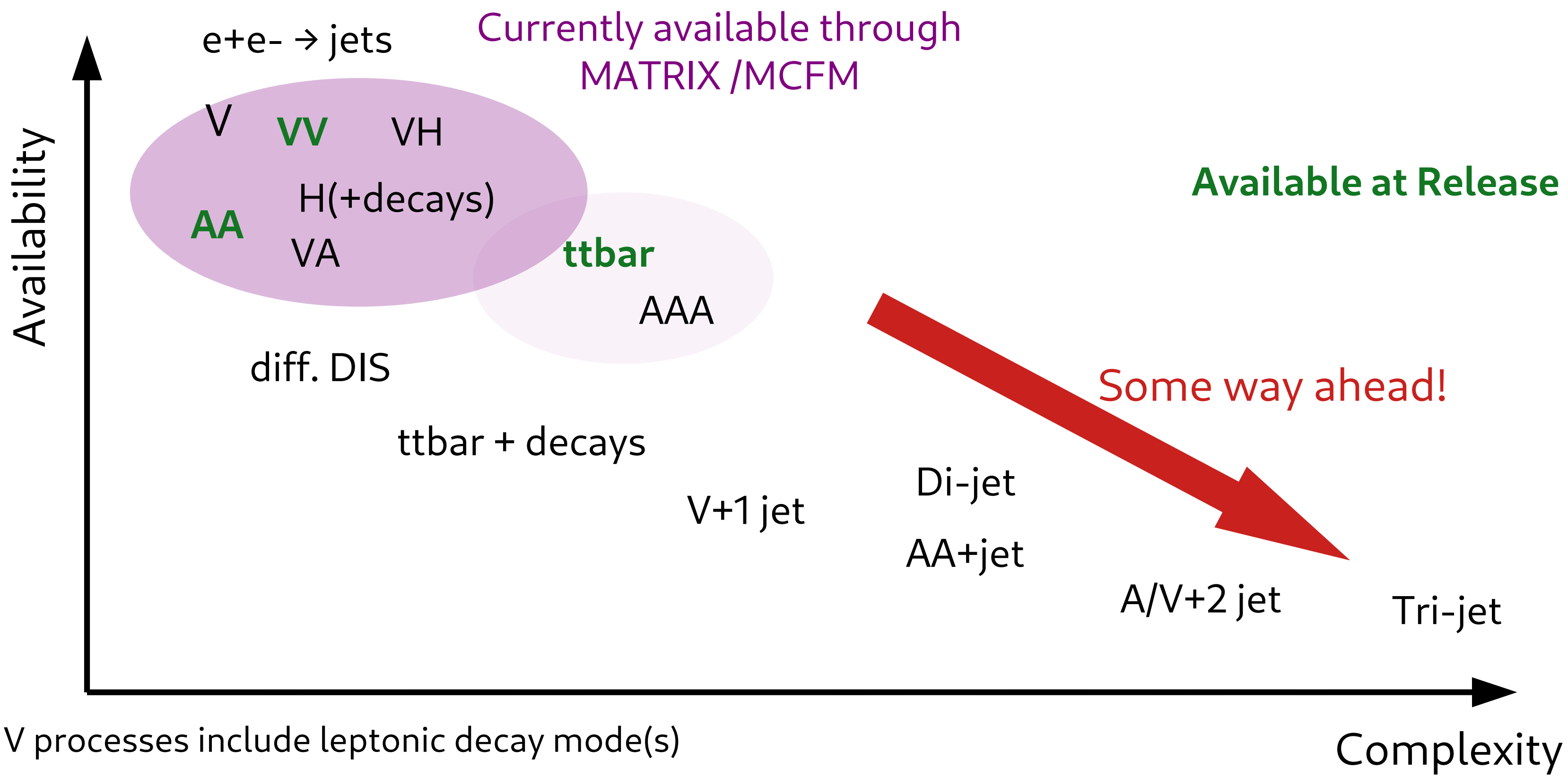
Allows **full control over scales and PDF**

HighTEA interface



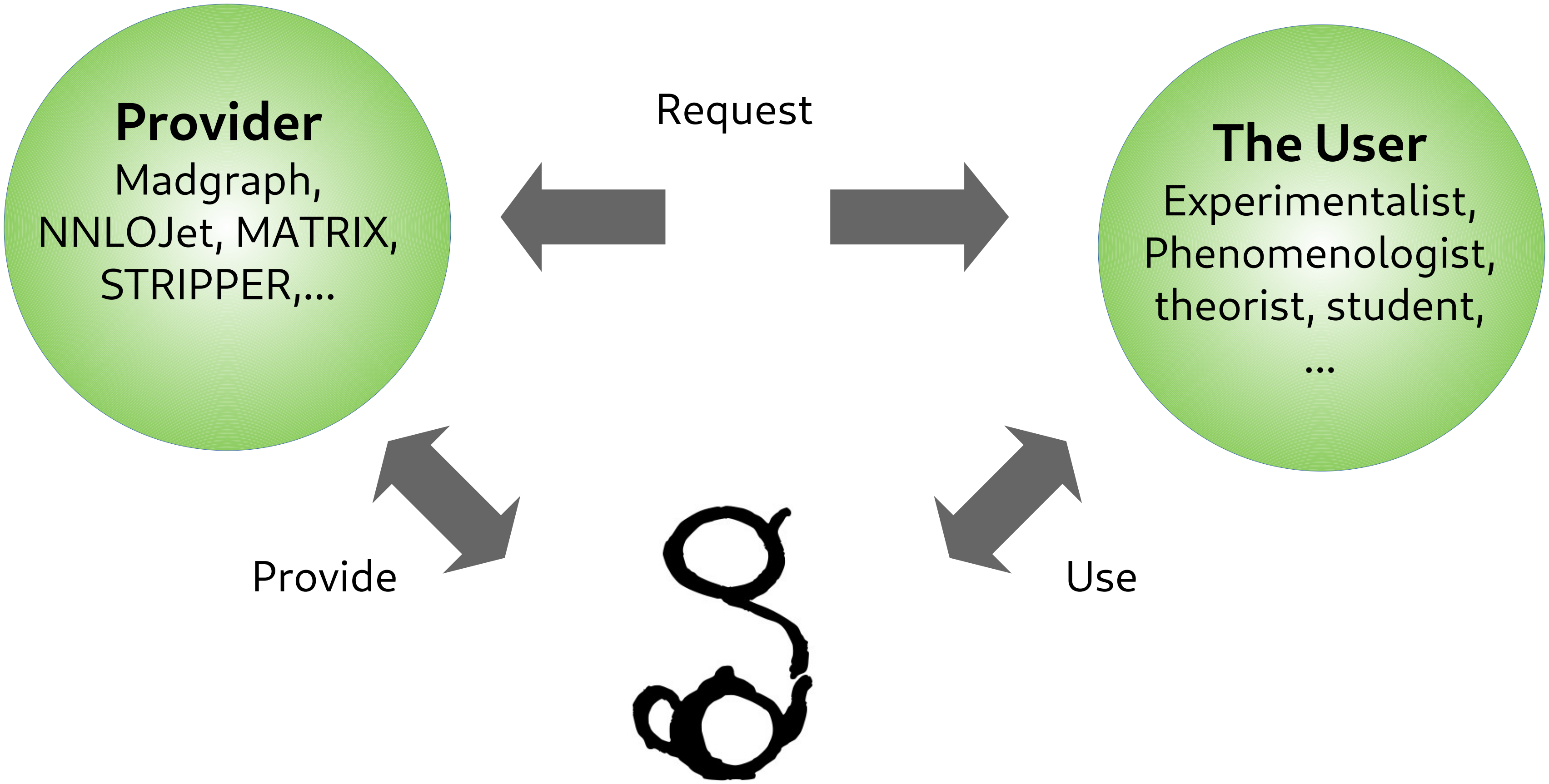
Available Processes

Processes currently implemented in our STRIPPER framework through NNLO QCD



* V processes include leptonic decay mode(s)

The Vision


Provider
Madgraph,
NNLOJet, MATRIX,
STRIPPER,...
Request
The User
Experimentalist,
Phenomenologist,
theorist, student,
...
Provide
Use

Transverse Thrust @ NNLO QCD

NNLO QCD corrections to event shapes at the LHC

Alvarez, Cantero, Czakon, Llorente, Mitov, Poncelet [2301.01086]

ATLAS [2007.12600]