Top-down approach to improving the BK equation

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Synergies between the EIC and the LHC

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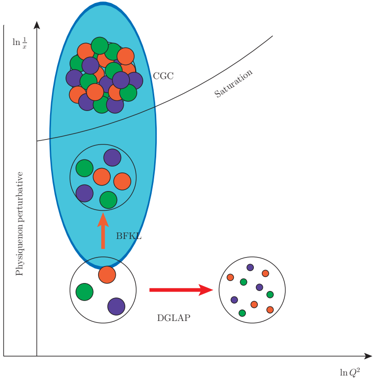
In collaboration with P, Caucal and Y. Mehtar-Tani

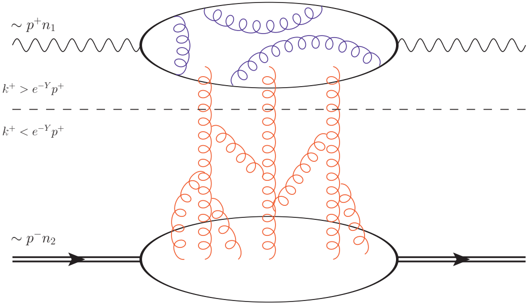
QCD at small $x_{\mathrm{Bj}} \sim Q^2/s$

Regge limit: $Q^2 \ll s$



Rapidity separation

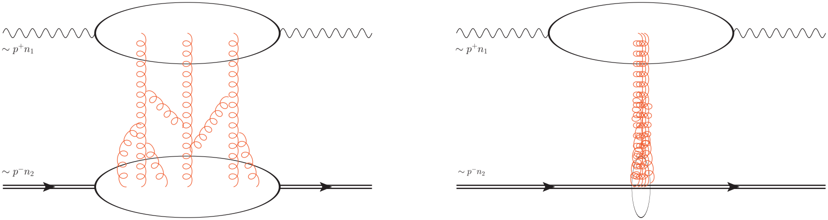
[McLerran, Venugopalan, 1994] [Balitsky, 1995]



Let us split the gluonic field between "fast" and "slow" gluons

$$\mathcal{A}^{\mu a}(k^+, k^-, \boldsymbol{k}) = a^{\mu a}(|k^+| > e^{-Y} p^+, k^-, \boldsymbol{k}) + A_{\rm cl}^{\mu a}(|k^+| < e^{-Y} p^+, k^-, \boldsymbol{k})$$

Large longitudinal boost to the projectile frame



$$A^+_{\rm cl}(x^+, x^-, \boldsymbol{x}) \qquad \frac{1}{\Lambda} A^+_{\rm cl}(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x})$$

$$A^-_{\rm cl}(x^+, x^-, \boldsymbol{x}) \quad \longrightarrow \quad \Lambda A^-_{\rm cl}(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x})$$

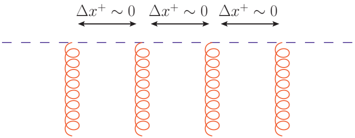
$$A^i_{\rm cl}(x^+, x^-, \boldsymbol{x}) \qquad \Lambda \sim \sqrt{\frac{s}{m_t^2}} \qquad A^i_{\rm cl}(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x})$$

$$A^\mu_{\rm cl}(x) \to A^-_{\rm cl}(x)\, n_2^\mu = \delta(x^+)\, \mathbf{A}(\boldsymbol{x})\, n_2^\mu + O(\sqrt{\frac{m_t^2}{s}})$$

Shock wave approximation

Effective Feynman rules in the slow background field

The interactions with the background field can be exponentiated

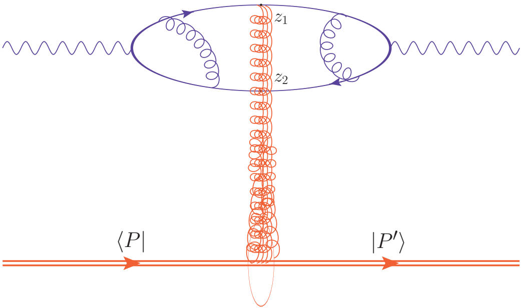


$$D_F(x_2, x_0)|_{x_2^+ > 0, x_0^+ < 0} = \int \mathrm{d}^D x_1 \, \delta(x_1^+) \, D_0(x_2, x_1) \, \gamma^+ \, U_{\mathbf{x}_1} \, D_0(x_1, x_0)$$

Each fast parton is dressed by an infinite Wilson line

$$U_{\mathbf{x}} \equiv \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} \mathrm{d}x \cdot A_{\mathrm{cl}}(x) \right]$$

Factorized picture



Factorized amplitude

$$\mathcal{S} = \int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2\, \Phi^Y(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \langle P'|[\mathrm{Tr}(U^Y_{x_1} U^{Y\dagger}_{x_2}) - N_c]|P\rangle$$

Written similarly for any number of Wilson lines in any color representation!

Y independence: B-JIMWLK, BK equations. Resums logarithms of s

The seemingly incompatible nature of the distributions

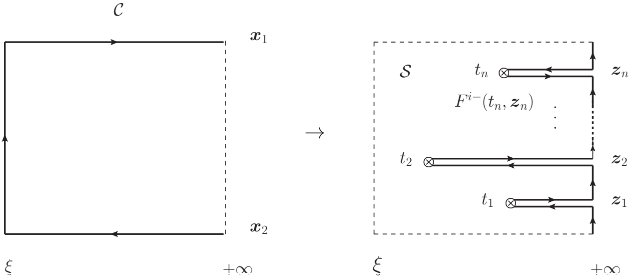
Two different kinds of gluon distributions

Moderate x distributions	Low x distributions
TMD, PDF...	Dipole scattering amplitude
$\langle P \| F^{-i} W F^{-j} W \| P \rangle$	$\langle P \| \mathrm{tr}(U_1 U_2^\dagger) \| P \rangle$

The Wilson line ↔ parton distribution equivalence

Most general equivalence: use the Non-Abelian Stokes theorem

[RB, Mehtar-Tani]



$$\mathcal{P}\exp\left[\oint_{\mathcal{C}} dx_\mu A^\mu(x)\right] = \mathcal{P}\exp\left[\int_{\mathcal{S}} d\sigma_{\mu\nu}\, W F^{\mu\nu} W^\dagger\right]$$

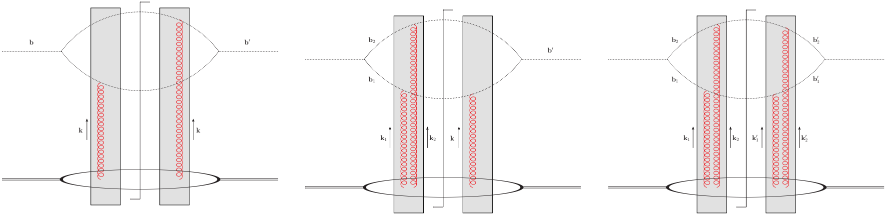
$$U_{x_{1\perp}} U^\dagger_{x_{2\perp}} = [\hat{x}_{1\perp}, \hat{x}_{2\perp}]$$

Inclusive low x cross section

Non-exclusive low x cross section = TMD cross section

[Altinoluk, RB, Kotko], [Altinoluk, RB]

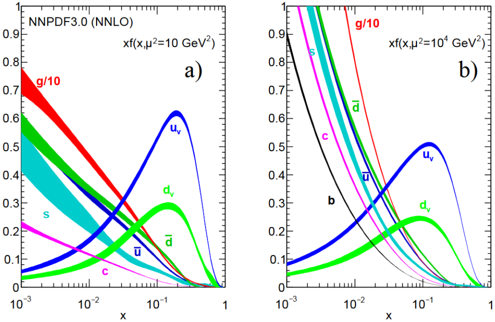
Generalizes [Dominguez, Marquet, Xiao, Yuan]



$$\sigma = \mathcal{H}_2^{ij}(\boldsymbol{k}) \otimes f_2^{ij}(x=0, \boldsymbol{k})$$
$$+\mathcal{H}_3^{ijk}(\boldsymbol{k}, \boldsymbol{k}_1) \otimes f_3^{ijk}(x=0, x_1=0, \boldsymbol{k}, \boldsymbol{k}_1)$$
$$+\mathcal{H}_4^{ijkl}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_1') \otimes f_4^{ijkl}(x=0, x_1=0, x_1'=0, \boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_1')$$

All distributions are evaluated in the strict $x = 0$ limit

All distributions are evaluated in the strict $x = 0$ limit



[NNLO NNPDF3.0 global analysis, taken from PDG2018]

Instabilities in the collinear corner of the phase space

The double log limit

Collinear logs are a problem for small-x_{Bj} physics

Proposed *ad hoc* solutions to the symptoms:

- Imposed kinematic orderings on – momenta or light cone times
  [Beuf], extends [Ciafaloni, Colferai, Salam, Stasto...]
- Resummation of logarithms
  [Iancu, Madrigal, Mueller, Soyez, Triantafyllopoulos], [Liu, Kang, Liu]
  [Caucal, Salazar, Schenke, Venugopalan], [Taels, Altinoluk, Beuf, Marquet]
- Non-local factorization
  [Iancu, Mueller, Triantafyllopoulos]
- Better choice of evolution variable
  [Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos]

All these schemes mimic a dependence on x in hopes of postponing the issue with the collinear limit.

All distributions are evaluated in the strict $x = 0$ limit

Hard part $\mathcal{H}$ and gluon distribution f for an inclusive observable:

Bjorken limit	Leading twist of the CGC
$s \sim Q^2$	$s \gg Q^2, Q^2 \to \infty$
$\int \mathrm{d}x f(x)\mathcal{H}(x)$	$f(0) \int \mathrm{d}x \mathcal{H}(x)$

Too late to restore a dependence on x via evolution: x is already integrated over

Summary so far

Distributions involved in pQCD observables

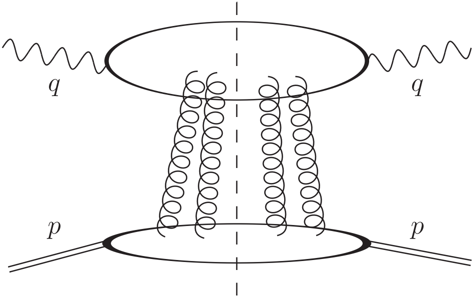
Overarching scheme?

$$f(x_1 \ldots x_n;\, k_{\perp 1} \ldots k_{\perp n})$$

Bjorken limit	Regge limit
$s \sim Q^2$	$s \gg Q^2$
$f(x;\, 0_\perp) + O(Q^{-2})$	$f(0 \ldots 0,\, k_{\perp 1} \ldots k_{\perp n}) + O(x_{\mathrm{Bj}})$

Look for an interpolating scheme for simple observables

An interpolating scheme for collinearly factorizable observables: DIS, DVCS, TCS, DDVCS



[RB, Mehtar-Tani]

Bjorken limit

$$s \sim Q^2$$

$$f(x, k_\perp = 0) + O(Q^{-2})$$

Regge limit

$$s \gg Q^2$$

$$f(x = 0, k_\perp) + O(x_{\mathrm{Bj}})$$

Interpolation?

$$s \gtrsim Q^2$$

$$f(x, k_\perp) + O(x_{\mathrm{Bj}} Q^{-2})$$

Basic observation: in both limits, $k^+ \simeq 0$ for t-channel gluons

Factorization in k^+ space is consistent

[Balitsky, Tarasov]

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the Regge limit:

$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit?

$$A^\mu(x) = A^-(x^+, x^-, \mathbf{x})\, n_2^\mu + A_\perp^\mu(x^+, x^-, \mathbf{x})$$

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Dependence on x^-: sub-sub-leading in twist counting

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$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x})\, n_2^\mu + A_\perp^\mu(x^+, 0^-, \boldsymbol{x})$$

Non-zero $A_\perp$: only two A^i contribute to DDVCS

They can be computed using Ward-Takahashi: only necessary for consistency checks, can be dropped.

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the Regge limit:

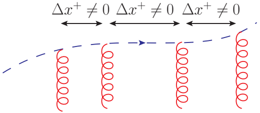
$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit:

$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x})\, n_2^\mu$$

Effective Feynman rules in the slow background field

Effective fermion propagator in the external classical field

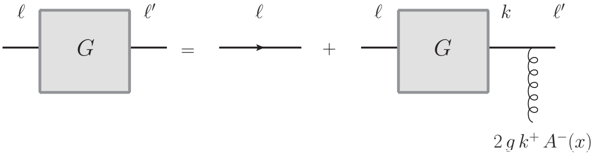


- $A^i_{\rm cl} = 0,\ A^+_{\rm cl} = 0$: the Dirac structure factorizes
- $A_{\rm cl}$ does not depend on x^-: conservation of + momentum

$$D_F(\ell', \ell) = i\frac{\gamma^+}{2\ell^+}(2\pi)^D \delta^D(\ell' - \ell) + i\frac{\not{\ell}'\gamma^+\not{\ell}}{2\ell^+} G_{\rm scal}(\ell', \ell)$$

Effective Feynman rules in the slow background field

Effective scalar propagator in the external classical field

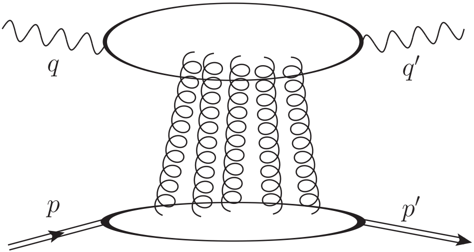


$$G_{\rm scal}(\ell',\ell) - G_0(\ell')(2\pi)^D\delta^D(\ell'-\ell)$$
$$= 2g\int \mathrm{d}^D z \int \frac{\mathrm{d}^D k}{(2\pi)^D} \mathrm{e}^{i(\ell'-k)\cdot z} G_0(\ell')\,(k\cdot A)(z)\,G_{\rm scal}(k,\ell).$$

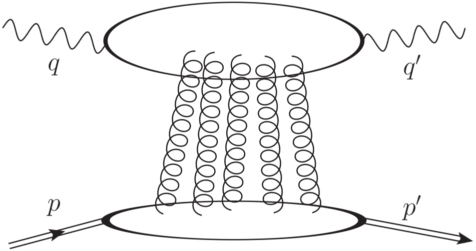
In coordinate space, it satisfies the Klein-Gordon equation in a potential

$$[-\Box_z + 2igA(z)\cdot\partial_z]\,G_{\rm scal}(z,z_0) = \delta^D(z-z_0)$$

Application to the $\gamma^{(*)}(q)P(p) \to \gamma^{(*)}(q')P(p')$ amplitude



Computing the $\gamma^{(*)}(q)P(p) \to \gamma^{(*)}(q')P(p')$ amplitude



$$\mathcal{A} = \frac{e^2}{\mu^{d-2}} \varepsilon_q^{\mu} \varepsilon_{q'}^{\nu *} \sum_f q_f^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \int \frac{\mathrm{d}^D k}{(2\pi)^D}$$

$$\times \langle p' | \mathrm{tr} \left[\gamma_\nu D_F(k, \ell) \gamma_\mu D_F(-q + \ell, -q' + \ell + k) \right] | p \rangle$$

Final result for DDVCS

Final expression for the DDVCS amplitude

$$\mathcal{A} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \int \mathrm{d}^d \boldsymbol{k}$$
$$\times (\partial^i \Phi)(z, \boldsymbol{\ell} - \boldsymbol{k}/2)(\partial^j \Phi^*)(z, \boldsymbol{\ell} + \boldsymbol{k}/2)$$
$$\times \int \mathrm{d}x \frac{\mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})}{x - x_{\mathrm{Bj}} - \frac{\boldsymbol{\ell}^2}{2z\bar{z}q^+P^-} + i0}$$

Standard wave functions Φ

x-dependent unintegrated GPD operator $\mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})$ (includes polarized terms)

Final result

Final expression for the DIS amplitude

$$\mathcal{A} = -i\pi g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \int \mathrm{d}^d \boldsymbol{k}$$
$$\times (\partial^i \Phi)(z, \boldsymbol{\ell} - \boldsymbol{k}/2)(\partial^j \Phi^*)(z, \boldsymbol{\ell} + \boldsymbol{k}/2)$$
$$\times \int \mathrm{d}x \, \mathcal{G}^{ij}(x, \boldsymbol{k}) \delta\left(x - x_{\mathrm{Bj}} - \frac{\boldsymbol{\ell}^2}{2z\bar{z}q^+P^-}\right)$$

Standard wave functions Φ

x-dependent unintegrated PDF operator $\mathcal{G}^{ij}(x, \boldsymbol{k})$

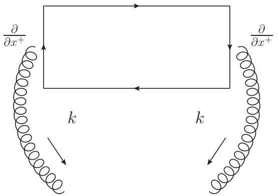
The unintegrated PDF

uGPD as a finite Wilson loop

$$\int \mathrm{d}^2\boldsymbol{k}\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\boldsymbol{r}^i\boldsymbol{r}^j\mathcal{G}^{ij}(x,\xi,\boldsymbol{k},\boldsymbol{\Delta})$$

$$= \frac{1}{\alpha_s}\int\frac{\mathrm{d}^4v_1\mathrm{d}^4v_2}{(2\pi)^4}\delta(v_1^-)\delta(v_2^-)\mathrm{e}^{-i(k-\frac{\Delta}{2})\cdot v_1+i(k+\frac{\Delta}{2})\cdot v_2}$$

$$\times\frac{\partial}{\partial v_1^+}\frac{\partial}{\partial v_2^+}\frac{\langle p'|\mathrm{tr}[v_1^+,v_2^+]_{\boldsymbol{v}_1}[\boldsymbol{v}_1,\boldsymbol{v}_2]_{v_2^+}[v_2^+,v_1^+]_{\boldsymbol{v}_2}[\boldsymbol{v}_2,\boldsymbol{v}_1]_{v_1^+}|p\rangle}{\langle p|p\rangle}$$



x-dependent unintegrated GPD $\Leftrightarrow$ FT of a finite Wilson loop

Summary

Interpolating scheme for exclusive Compton scattering

Overarching scheme

$$\int \mathrm{d}x \int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta}) H^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})$$

Bjorken limit

$$\int \mathrm{d}x H^{ij}(x, \xi, \boldsymbol{0}, \boldsymbol{\Delta}) \times \left[\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})\right]$$

Regge limit

$$\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(0, \xi, \boldsymbol{k}, \boldsymbol{\Delta}) \times \left[\int \mathrm{d}x H^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})\right]$$

We found an interpolating scheme

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Bjorken limit

$$\int \mathrm{d}x H^{ij}(x, \xi, \mathbf{0}, \boldsymbol{\Delta}) \times [\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})]$$

Regge limit

$$\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(0, \xi, \boldsymbol{k}, \boldsymbol{\Delta}) \times [\int \mathrm{d}x H^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})]$$

We found an interpolating scheme

Can we find an interpolating evolution equation?

Does it correct the collinear structure of BK?

Deriving the evolution equation

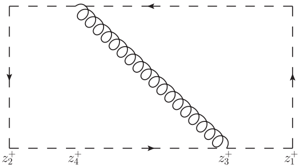
Evolution equation for the x-dependent dipole operator

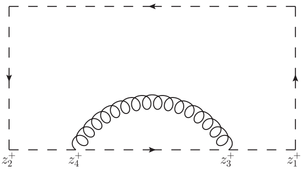
$$S^{(2)}(x, \boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \frac{1}{N_c} \int \mathrm{d}z_1^+ \int_{-\infty}^{z_1^+} \mathrm{d}z_2^+ \, \mathrm{e}^{ixP^- z_{12}^+} \frac{\partial^2}{\partial z_1^+ \partial z_2^+} \mathrm{tr}[z_1^+, z_2^+]_{\boldsymbol{x}_1} [z_2^+, z_1^+]_{\boldsymbol{x}_2}$$

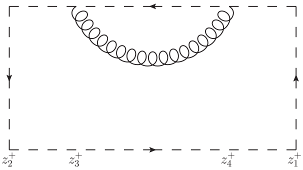
Deriving the evolution equation

Quantum corrections

$$A^\mu = A^\mu_{\text{cl}} + a^\mu$$



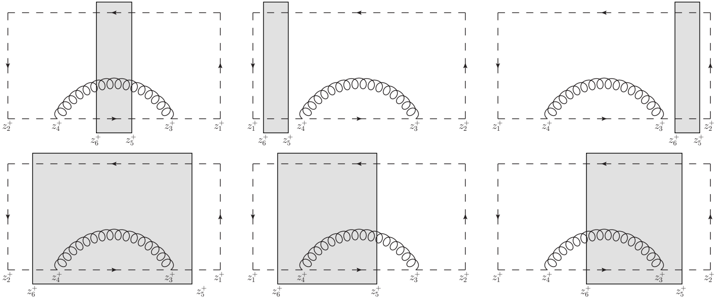




Every line is dressed with classical gluon fields

Deriving the evolution equation

Operator algebra: each diagram can be decomposed



Only lines inside grey blobs are dressed with classical gluon fields

The second line is power suppressed in the Regge limit

Deriving the evolution equation

Result, up to subeikonal terms

$$\Delta S^{(2)}(x, \boldsymbol{x}_1, \boldsymbol{x}_2)$$
$$= -\frac{2\alpha_s}{N_c}\int \frac{\mathrm{d}k^+}{k^+}\theta(k^+)\int \frac{\mathrm{d}^d\boldsymbol{k}_1}{(2\pi)^d}\frac{\mathrm{d}^d\boldsymbol{k}_2}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{x}_3$$
$$\times \frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\left(\mathrm{e}^{i(\boldsymbol{k}_1\cdot\boldsymbol{x}_{13})}-\mathrm{e}^{i(\boldsymbol{k}_1\cdot\boldsymbol{x}_{23})}\right)\left(\mathrm{e}^{-i(\boldsymbol{k}_2\cdot\boldsymbol{x}_{13})}-\mathrm{e}^{-i(\boldsymbol{k}_2\cdot\boldsymbol{x}_{23})}\right)}{(\boldsymbol{k}_1^2-2xP^-k^+-i0)(\boldsymbol{k}_2^2-2xP^-k^+-i0)}$$
$$\times \int \mathrm{d}z_1^+\int_{-\infty}^{z_1^+}\mathrm{d}z_2^+\mathrm{e}^{i\left(xP^- - \frac{\boldsymbol{k}_1^2-i0}{2k^+}\right)z_{12}^+}$$
$$\times\, \partial_{z_1^+}\partial_{z_2^+}\left\{\mathrm{tr}[z_1^+,z_2^+]_{\boldsymbol{x}_1}[z_2^+,z_1^+]_{\boldsymbol{x}_3}\mathrm{tr}[z_1^+,z_2^+]_{\boldsymbol{x}_3}[z_2^+,z_1^+]_{\boldsymbol{x}_2}-N_c\mathrm{tr}[z_1^+,z_2^+]_{\boldsymbol{x}_1}[z_2^+,z_1^+]_{\boldsymbol{x}_2}\right\}$$

x-dependent BK kernel, x-dependent double dipole

Deriving the evolution equation

Result, at $x = 0$

$$
\begin{aligned}
&\Delta S^{(2)}(x=0,\boldsymbol{x}_1,\boldsymbol{x}_2)\\
&= -\frac{2\alpha_s}{N_c}\int \frac{\mathrm{d}k^+}{k^+}\theta(k^+)\int \frac{\mathrm{d}^d\boldsymbol{k}_1}{(2\pi)^d}\frac{\mathrm{d}^d\boldsymbol{k}_2}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{x}_3\\
&\times \frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\left(\mathrm{e}^{i(\boldsymbol{k}_1\cdot\boldsymbol{x}_{13})}-\mathrm{e}^{i(\boldsymbol{k}_1\cdot\boldsymbol{x}_{23})}\right)\left(\mathrm{e}^{-i(\boldsymbol{k}_2\cdot\boldsymbol{x}_{13})}-\mathrm{e}^{-i(\boldsymbol{k}_2\cdot\boldsymbol{x}_{23})}\right)}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}\\
&\times \int \mathrm{d}z_1^+\int_{-\infty}^{z_1^+}\mathrm{d}z_2^+\,\mathrm{e}^{-i\frac{\boldsymbol{k}_1^2-i0}{2k^+}z_{12}^+}\\
&\times \partial_{z_1^+}\partial_{z_2^+}\left\{\mathrm{tr}[z_1^+,z_2^+]_{\boldsymbol{x}_1}[z_2^+,z_1^+]_{\boldsymbol{x}_3}\mathrm{tr}[z_1^+,z_2^+]_{\boldsymbol{x}_3}[z_2^+,z_1^+]_{\boldsymbol{x}_2}-N_c\mathrm{tr}[z_1^+,z_2^+]_{\boldsymbol{x}_1}[z_2^+,z_1^+]_{\boldsymbol{x}_2}\right\}
\end{aligned}
$$

Non-zero phase: not BK!

Deriving the evolution equation

Result, at $x = 0$

Standard BK:

$$\int \frac{\mathrm{d}k^+}{k^+}\theta(k^+)\, \mathrm{e}^{-i\frac{\boldsymbol{k}_1^2 - i0}{2k^+}z_{12}^+} \to \int_{\rho_0^+}^{\rho_0^+ + \delta\rho_0^+} \frac{\mathrm{d}k^+}{k^+}\theta(k^+) = \delta \ln \rho_0^+$$

Here:

$$\int \frac{\mathrm{d}k^+}{k^+}\theta(k^+)\, \mathrm{e}^{-i\frac{\boldsymbol{k}_1^2 - i0}{2k^+}z_{12}^+} \to (\rho^+)^\eta \int \frac{\mathrm{d}k^+}{(k^+)^{1+\eta}} \mathrm{e}^{-i\frac{\boldsymbol{k}_1^2 - i0}{2k^+}z_{12}^+}\, \theta(k^+)$$

Then,

$$\Delta S^{(2)}(x = 0, \boldsymbol{x}_1, \boldsymbol{x}_2) = \Delta S^{(2)}_{\mathrm{BK}}(x = 0, \boldsymbol{x}_1, \boldsymbol{x}_2) - \ln \frac{\boldsymbol{k}_1^2}{2\rho_0^+ P^-} \otimes S^{(3)}(x = 0) + \mathrm{const.}$$

Evolution equation

Result, at $x = 0$

$$\Delta S_{12} - \bar{\alpha}_s K_{\rm BK} \otimes S_{12}$$
$$= -\bar{\alpha}_s \int_3 \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2} \left(\ln \frac{|\mathbf{x}_{13}||\mathbf{x}_{32}|\mu}{|\mathbf{x}_{12}|} + \ln |\mathbf{x}_{12}|\mu + \frac{\mathbf{x}_{23}^2 - \mathbf{x}_{13}^2}{\mathbf{x}_{12}^2} \ln \frac{|\mathbf{x}_{13}|}{|\mathbf{x}_{32}|} \right) (S_{13} S_{32} - S_{12})$$

where $\mu^2 = 2\rho_0^+ P^-$

Balitsky and Chirilli's conformal dipole evolution, and an extra term

Evolution equation

Result, at $x = 0$

$$\Delta S_{12} - \bar{\alpha}_s K_{\mathrm{BK}} \otimes S_{12}$$
$$= -\bar{\alpha}_s \int_3 \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{32}^2} \left(\ln \frac{|\boldsymbol{x}_{13}||\boldsymbol{x}_{32}|\mu}{|\boldsymbol{x}_{12}|} + \ln |\boldsymbol{x}_{12}|\mu + \frac{\boldsymbol{x}_{23}^2 - \boldsymbol{x}_{13}^2}{\boldsymbol{x}_{12}^2} \ln \frac{|\boldsymbol{x}_{13}|}{|\boldsymbol{x}_{32}|} \right) (S_{13} S_{32} - S_{12})$$

where $\mu^2 = 2\rho_0^+ P^-$

Conformal dipole term: cancels double logs, but generates instabilities

Extra term: compensates those instabilities

How to fix BK: generalities

Fixing BK by a change of variables

Let us introduce a vector S of Wilson line operators in the Balitsky hierarchy: $S^{(2)}$=dipole, $S^{(3)}$=double dipole... and the BK operator so that for $\zeta = 2\rho^+ P^-$, the BK hierarchy of equation reads

$$\frac{\partial S(\zeta)}{\partial \zeta} = \bar{\alpha}_s K \cdot S(\zeta)$$

Let us introduce a new scale μ^2 and the composite vector $\bar{S}$ as

$$\bar{S}(\zeta, \mu^2) = \mathrm{e}^{-\bar{\alpha}_s L(\mu^2)} S(\zeta)$$

Then $\bar{S}$ evolves as

$$\frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \zeta} = \bar{\alpha}_s \bar{K} \cdot \bar{S}(\zeta, \mu^2)$$

with

$$\bar{K}(\mu^2) = \mathrm{e}^{-\bar{\alpha}_s L(\mu^2)} K \mathrm{e}^{\bar{\alpha}_s L(\mu^2)}$$

Fixing BK by a change of variables

$L(\mu^2)$ is arbitrary: we can build it so that ([Balitsky, Chirilli]: factor 1/2)

$$\frac{\partial \bar{S}(\zeta,\mu^2)}{\partial \ln \mu^2} = -\frac{\partial \bar{S}(\zeta,\mu^2)}{\partial \zeta}$$

Then

$$\frac{\partial \bar{K}(\zeta,\mu^2)}{\partial \ln \mu^2} = 0$$

and $\bar{S}(\zeta,\mu^2) = \bar{S}(\zeta/\mu^2)$.

For $L(\mu^2)$ polynomial in $\ln \mu^2$ of degree < number of loops, we find at NNLL:

$$L_{\rm LL} = L_{10} + K_{\rm LL} \ln \mu^2$$

$$L_{\rm NLL} = L_{20} + \left\{ K_{\rm NLL} + \frac{1}{2}[K_{\rm LL}, L_{10}] \right\} \ln \mu^2$$

$$L_{\rm NNLL} = K_{\rm NNLL} + \left\{ K_{\rm NNLL} + \frac{1}{2}[K_{\rm LL}, L_{20}] + \frac{1}{2}[K_{\rm NLL}, L_{10}] + \frac{1}{12}[[K_{\rm LL}, L_{10}], L_{10}] \right\} \ln \mu^2 + \frac{1}{12}[K_{\rm LL}, [K_{\rm LL}, L_{10}]] \ln^2 \mu^2$$

Fixing BK by a change of variables

Modified kernel:

$$\bar{K}_{\mathrm{NNLL}} = K_{\mathrm{NNLL}} + \bar{\alpha}_s[K_{\mathrm{LL}}, L_{10}] + \bar{\alpha}_s^2[K_{\mathrm{NLL}}, L_{10}] + \bar{\alpha}_s^2[K_{\mathrm{LL}}, L_{20}] + \frac{\bar{\alpha}_s^2}{2}[[K_{\mathrm{LL}}, L_{10}], L_{10}]$$

Educated choice for the $L(\mu^2)$ constant terms L_{10}, L_{20}: convolution of the BK kernel with transverse logs of daughter dipole sizes

$$L_{n0} \cdot S = (K \otimes \ln \hat{\boldsymbol{r}}^2) \cdot S \Rightarrow L(\mu^2) = K_{\mathrm{LL}} \otimes \ln(\mu^2 \hat{\boldsymbol{r}}^2) + \ldots$$

Commutators: Logarithms of ratios of daughter dipole sizes to parent dipole sizes.

Good choices of transverse logs cancel double logs

Summary: change of variables

We can define a composite dipole $\bar{S}$ which:

- Evolves with rapidity $Y = \zeta/\mu^2 \to 1/x_{\mathrm{Bj}}$ instead of projectile momentum
- Evolves with a collinearly stable evolution equation
- Is still compatible with impact factors computed with regularization in $k+$

Correction to standard impact factors, for cross section independence on the choice of L:

$$\bar{H} = H\,\mathrm{e}^{\bar{\alpha}_s \overleftarrow{L}(\mu^2)}$$

Conclusion

Summary of our findings

Bad news

- Semi-classical small x physics has, at its core, issues with collinear logarithms
- The problem can be traced down to the very starting point

Good news

- We now have a minimal correction of semi-classical small x which solves the problem from first principles
- Known impact factors are compatible with our scheme