Scale dependence of PDFs and GPDs: Anomalous dimensions from consistency relations

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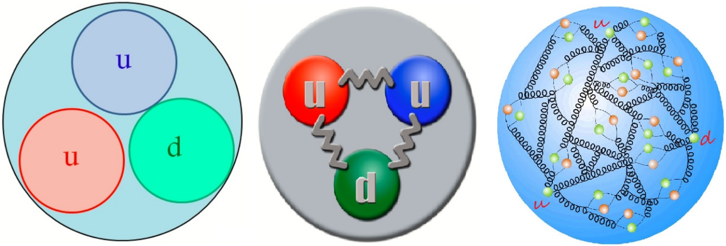
Joint ECFA-NuPECC-APPEC Activity Workshop on Synergies between the EIC and the LHC

23 September 2025

How to gain insight into the structure of hadrons

- Hadrons such as the proton are a mess of many interacting quarks/gluons!



- Nevertheless, they have well-defined physical properties such as mass, spin etc.
  ⇒ How can we explain these in terms of the properties of the constituent partons?
- Experimentally: Perform high-energy **scattering experiments** that can resolve the inner hadron structure (DIS, DVCS)

Description of scattering experiments

- Hard scale $\Rightarrow$ Factorization between short-range and long-range physics

$$\hat{\sigma}(p_A) = \sum_a \int_0^1 \mathrm{d}x_a \, f_{a/A}(x_a, \mu_F^2) \, \sigma_a(x_a p_A; \mu_F^2)$$

- Short-range physics characterized by the perturbative partonic cross section σ_a
- Long-range physics described by non-perturbative parton distributions like **PDFs** and **GPDs**
- Through application of the **OPE**, these distributions are related to hadronic matrix elements of composite QCD operators

Leading-twist operators

The OPE is dominated by leading-twist operators. Based on the representations of the QCD flavour group, we can distinguish two sets of such operators

$$\mathcal{O}^{(N)}_{q\,\mathrm{NS};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi'}\gamma_{\mu_1} D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$
$$\mathcal{O}^{(N)}_{g\,\mathrm{S};\mu_1\ldots\mu_N}(x) = \frac{1}{2}\mathcal{S}\left[\,F^{a_1}{}_{\mu\mu_1}\, D^{a_1 a_2}_{\mu_2}\ldots D^{a_{N-2}a_{N-1}}_{\mu_{N-1}} F^{a_{N-1};\mu}{}_{\mu_N}\,\right]$$
$$\mathcal{O}^{(N)}_{q\,\mathrm{S};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\gamma_{\mu_1} D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$$
$$D_\mu = \partial_\mu - i g_s T^a A^a_\mu$$
$$D^{ac}_\mu = \partial_\mu \delta^{ac} + g_s f^{abc} A^b_\mu$$

f^{abc} are the standard QCD structure constants.

Scale dependence of parton distributions

Contrary to the distributions themselves, their energy scale dependence can be calculated perturbatively!

Forward case (DGLAP [Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]):

$$\frac{\mathrm{d}f_i(x,\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} P_{ij}(y) f_j\left(\frac{x}{y},\mu^2\right)$$

Non-forward case ([Müller et al., 1994],[Radyushkin, 1996],[Ji, 1997]):

$$\frac{\mathrm{d}\mathcal{G}(x,\xi,t;\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} \mathcal{P}\left(\frac{x}{y},\frac{\xi}{y}\right) \mathcal{G}(y,\xi,t;\mu^2)$$

Scale dependence of parton distributions

Because of the direct relation between the distributions and QCD operators, the scale dependence of the distributions is determined by the scale dependence of the operators, characterized by their anomalous dimension

$$\frac{\mathrm{d}[\mathcal{O}]}{\mathrm{d}\ln\mu^2} = \gamma[\mathcal{O}].$$

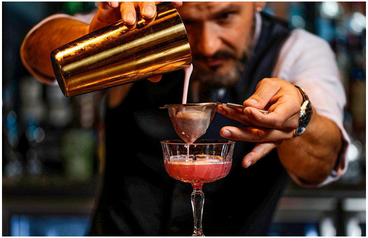
These anomalous dimensions can be computed perturbatively in QCD by renormalizing the (off-shell) partonic matrix elements of the operators. They can be related to the evolution kernels through a **Mellin transform**

$$\gamma^{ij}_{N,N} = -\int_0^1 \mathrm{d}x\, x^N P_{ij}(x)\,,$$

$$\sum_{k=0}^{N} \gamma_{N,k} y^k = -\int_0^1 \mathrm{d}x\, x^N\, \mathcal{P}(x,y).$$

Operator mixing: A cocktail of anomalous dimensions

- Without mixing: $1/\varepsilon$-pole of matrix element $\Rightarrow$ anomalous dimension
- With mixing: $1/\varepsilon$-pole gets multiple contributions $\Rightarrow$ how to disentangle?



- Non-forward kinematics: Mixing with total-derivative operators
- In flavour singlet case: Mixing with alien operators

For specific choices of operator bases, both sources of mixing can be analyzed using conjugation relations

Conjugations

Suppose we have a function f of some discrete variable N. A conjugation is then a specific sum over f that, when applied twice, gives back the **original** function [Vermaseren, 1999]. For example, if we have some function $f(N)$, then its **binomial transform**,

$$[f(N)]^C = \sum_{i=0}^{N} (-1)^i \binom{N}{i} f(i)$$

is a conjugation since

$$\Rightarrow \left([f(N)]^C\right)^C = \sum_{j=0}^{N} (-1)^j \binom{N}{j} \sum_{i=0}^{j} (-1)^i \binom{j}{i} f(i) = f(N)\,.$$

Conjugations are very helpful to significantly **restrict the function space**!

Solving conjugation relations

- To take full advantage of conjugation relations, one needs to be able to evaluate them analytically
- Use principles of **symbolic summation**!
- **Creative telescoping** [Zeilberger, 1991] + **Gosper's algorithm** [Gosper, 1978] : evaluate the sum of interest by rewriting it as a recursion relation
- The closed-form expression of the sum then corresponds to the linear combination of the solutions of the recursion that has the same initial values as the sum.

→ For single sums: `Sigma` [Schneider, 2004, Schneider, 2007]

→ For multiple sums: `EvaluateMultiSums` [Schneider, 2013, Schneider, 2014]

Non-forward anomalous dimensions

To treat the mixing of operators with total-derivative ones in non-forward kinematics, we select the following basis (focus on flavour-non-singlet case)

$$\mathcal{O}^{\mathcal{D}}_{k,N-k} = (\Delta\cdot\partial)^k\{\overline{\psi'}(\Delta\cdot\Gamma)(\Delta\cdot D)^{N-k}\psi\} \qquad [\Delta^2=0]$$

By also considering operators in which the covariant derivative acts on $\overline{\psi'}$, one can construct recursion relations between the operators which lead to consistency relations between the anomalous dimensions

$$\forall k: \qquad \sum_{j=k}^{N}\left\{(-1)^k\binom{j}{k}\gamma^{qq,\mathrm{NS}}_{N,j} - (-1)^j\binom{N}{j}\gamma^{qq,\mathrm{NS}}_{j,k}\right\} = 0\,.$$

$\rightarrow$ Valid to all orders in perturbation theory!
Note that, for $k=0$, this reduces to a conjugation as defined above

$$\left[\gamma^{qq,\mathrm{NS}}_{N,0}\right]^C = \sum_{j=0}^{N}\gamma^{qq,\mathrm{NS}}_{N,j}$$

Non-forward anomalous dimensions

- These relations were used in [Moch and Van Thurenhout, 2021] to determine the vector ($\Gamma = \gamma_\mu$) anomalous dimensions in the leading-n_f limit to 5-loop accuracy and in the planar limit to 2 loops
- Relations **independent** of the Dirac structure
  $\rightarrow$ 4-loop transversity ($\Gamma \sim [\gamma_\mu, \gamma_\nu]$) anomalous dimensions in leading-n_f limit [Van Thurenhout, 2022]
- Similar relations can be derived for **different types** of operators; e.g. in the flavour-singlet sector [Van Thurenhout, 2025]

$$\forall k > 0 : \sum_{j=k}^{N} \left\{ (-1)^k \binom{j-1}{k-1} \gamma_{N,j}^{gg} - (-1)^j \binom{N-1}{j-1} \gamma_{j,k}^{gg} \right\} = 0$$

$\rightarrow$ Derived at 1-loop level in which mixing with aliens can be ignored
$\rightarrow$ Hints that it nevertheless stays valid beyond 1-loop accuracy [needs further investigation!]

Forward singlet anomalous dimensions

In the flavour-singlet sector, one needs to take into account alien operators (ghosts + EOM)

→ Recently, G. Falcioni and F. Herzog derived a method to consistently construct the aliens to any loop-order [Falcioni and Herzog, 2022].

- In their approach, the aliens are derived using **generalized gauge symmetry** of the QCD Lagrangian, which is then promoted to a **generalized (anti-)BRST symmetry**
- Each alien operator features a coupling which can be interpreted as the renormalization constant that characterizes the mixing of the physical operators into the alien.
- The couplings obey certain **N-dependent constraints**. These were solved for fixed $N \leq 20$ in [Falcioni and Herzog, 2022, Falcioni et al., 2024a].
- Recently, we were able to compute the couplings for **arbitrary** values of N [Falcioni et al., 2024b]

Forward singlet anomalous dimensions

$$\mathcal{O}_{\text{EOM}}^{(N),II} = g_s \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi \right) f^{abc} \sum_{i+j=N-3} \kappa_{ij} (\partial^i A^b)(\partial^j A^c),$$

$$\mathcal{O}_c^{(N),II} = -g_s f^{abc} \sum_{i+j=N-3} \eta_{ij} (\partial \overline{c}^a)(\partial^i A^b)(\partial^{j+1} c^c)$$

$$\kappa_{ij} + \kappa_{ji} = 0,$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N) \binom{i+j+1}{i},$$

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = 0$$

NOTE: Bottom relation = conjugation!

Forward singlet anomalous dimensions

Another neat feature of the alien relations is that they show a **bootstrap**: Complicated higher-order couplings can be related to simpler lower-order ones

$$\eta^{(1)}_{ijkl} + \eta^{(1)}_{jilk} + \eta^{(1)}_{lkji} + \eta^{(1)}_{klij} = 2[\kappa^{(1)}_{ij(k+l+1)} + \kappa^{(1)}_{(k+l+1)ji}]\binom{k+l+1}{k} + \text{permutations}$$

$$\eta^{(1)}_{ijk} + \eta^{(1)}_{kij} + \eta^{(1)}_{jki} = 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2\kappa_{k(i+j+1)}\binom{i+j+1}{i} + 2\kappa_{j(i+k+1)}\binom{i+k+1}{k}.$$

$$\eta_{ij} + \eta_{ji} = \eta(N)\left[\binom{i+j+1}{i} + \binom{i+j+1}{j}\right]$$

The coupling $\eta(N)$ is known to $O(a_s^3)$

[Dixon and Taylor, 1974, Hamberg and van Neerven, 1992, Gehrmann et al., 2023]

Word of caution: Kernel functions

If an Ansatz is generated using conjugation-type relations, one is in principle free to add non-zero functions that live in the kernel of these relations. For example, a term of the form

$$-\frac{f(N)}{4}\left((-1)^j + \binom{N-2}{i+1} - \binom{N-2}{i}\right)$$

automatically obeys the relation for η_{ij} above [for arbitrary $f(N)$!] and hence can be added without ruining internal consistency. In general, the exclusion of this type of function can only be confirmed by **comparison with fixed-N computations**.

Summary & outlook

- The scale dependence of PDFs and GPDs can be computed perturbatively as the **anomalous dimensions** of the operators that define them
- Such perturbative calculations are complicated due to several sources of operator mixing
- Uniform approach: Consistency relations based on **conjugations**
- Several **extensions** in principle possible but still to be looked at (e.g. properly taking into account aliens for the non-forward flavour-singlet anomalous dimensions)

End [1]

Thank you for your attention!

[1] Part of this work has been supported by grant K143451 of the National Research, Development and Innovation Fund in Hungary.

Appendices and references

Classical telescoping and Gosper's algorithm

The telescoping algorithm is a well-known method for evaluating finite sums. Suppose we want to evaluate the following sum

$$\sum_{k=a}^{N} f(k)$$

with $a, N \in \mathbb{N}$ and $a \leq N$. Now, if we can find a function $g(N)$ such that

$$f(k) = \Delta g(k) \equiv g(k+1) - g(k)$$

then

$$\begin{aligned}\sum_{k=a}^{N} f(k) &= \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k) \\ &= g(N+1) - g(a).\end{aligned}$$

Here, Δ represents the finite difference operator. The telescoping function $g(N)$ can be found by application of Gosper's algorithm [Gosper, 1978].

Classical telescoping and Gosper's algorithm

The algorithm consists of three main steps. Assume we want to calculate the telescoping function for some sequence $\{a_N\}$

$$a_N = \Delta b(N).$$

It is assumed that $\{a_N\}$ is a hypergeometric sequence, that is

$$\frac{a_{N+1}}{a_N} = q(N)$$

with $q(N)$ a rational function of N. The steps of Gosper's algorithm can then be summarized as follows

Classical telescoping and Gosper's algorithm

1. Determine three functions $f(x), g(x)$ and $h(x)$ such that

$$q(x) = \frac{f(x+1)}{f(x)} \frac{g(x)}{h(x+1)}$$

   and

$$\gcd[g(x), h(x+n)] = 1 \quad (n \in \mathbb{N}_0).$$

2. Solve the so-called Gosper equation,

$$f(x) = g(x)y(x+1) - h(x)y(x),$$

   for the polynomial $y(x)$.

3. If such a polynomial solution does not exist, it means that the sum in question does not have a hypergeometric closed form. Otherwise, the telescoping function is determined by

$$t(x) = \frac{h(x)}{f(x)} y(x) \qquad \text{with } b(N) = t(N)a(N)$$

More details can e.g. be found in [Kauers and Paule, 2011]

Creative telescoping

Classical telescoping works when dealing with sequences that depend on one variable only. When we want to determine a closed form for a summation of a sequence depending on two variables, we can use the creative telescoping algorithm by Zeilberger [Zeilberger, 1991]. The idea is similar to that of classical telescoping. Suppose we want to evaluate

$$\sum_{k=a}^{b} f(N,k) \equiv S(N).$$

The way to go about this is by attempting to find d functions $c_0(N), \ldots, c_d(N)$ and a function $g(N,k)$ such that

$$g(N,k+1) - g(N,k) = c_0(N) f(N,k) + ... + c_d(N) f(N+d,k).$$

Summing both sides, and applying classical telescoping to the left-hand side then gives

$$g(N,b+1) - g(N,a) = c_0(N) \sum_{k=a}^{b} f(N,k) + ... + c_d(N) \sum_{k=a}^{b} f(N+d,k).$$

Creative telescoping

This leads to an inhomogeneous recursion relation for the original sum of the form

$$q(N) = c_0(N)S(N) + ... + c_d(N)S(N + d).$$

Typically, one starts this procedure at $d = 0$, which is equivalent to classical telescoping. The value of d is then increased stepwise until a solution is found. The creative telescoping algorithm can be applied when the sequence under consideration is holonomic. A sequence $\{a_N\}$ is said to be holonomic if there exist polynomials $p_0(x), \ldots, p_r(x)$ such that the following recursion relation is obeyed [Kauers and Paule, 2011]

$$p_0(N)a_N + p_1(N)a_{N+1} + \cdots + p_r(N)a_{N+r} = 0 \quad (N \in \mathbb{N}, p_r(N) \neq 0).$$

For example, the harmonic numbers $\{S_1(N)\}$ form a holonomic sequence as they obey

$$(N + 1)S_1(N) - (2N + 3)S_1(N + 1) + (N + 2)S_1(N + 2) = 0.$$

More details on the summation algorithms reviewed here can e.g. be found in the excellent books [Graham et al., 1989, Petkovšek et al., 1996].

Relation between non-forward anomalous dimensions

In practical computations we use a different representation of the consistency relations

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j}.$$

- ✓ Order-independent consistency check
- ✓ Can be used to construct the full ADM from the knowledge of the forward anomalous dimensions $\gamma_{N,N}$ + boundary condition to ensure uniqueness of the solution ($\gamma^{\mathcal{D}}_{N,0}$, from Feynman diagrams)

Construction of the alien operators

The complete gauge-fixed QCD action is written as

$$S = \int \mathrm{d}^D x \, (\mathcal{L}_0 + \mathcal{L}_{\mathrm{GF+G}}) \, .$$

Here $\mathcal{L}_0$ represents the classical part of the QCD Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4} F_a^{\mu\nu} \, F_{\mu\nu}^a + \sum_{f=1}^{n_f} \overline{\psi}^f (i\not{D} - m_f)\psi^f \, ,$$

with

$$\mathcal{L}_{\mathrm{GF+G}} = -\frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 - \overline{c}^a \, \partial^\mu D_\mu^{ab} \, c^b$$

and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c$$
$$D_\mu = \partial_\mu - i g_s T^a A_\mu^a$$
$$D_\mu^{ac} = \partial_\mu \delta^{ac} + g_s f^{abc} A_\mu^b$$

f^{abc} are the standard QCD structure constants.

Construction of the alien operators

The QCD Lagrangian can be extended to also include the leading-twist spin-N gauge-invariant operators, which we define as

$$\mathcal{O}_{\rm g}^{(N)}(x) = \frac{1}{2} F_\nu(x)\, D^{N-2} F^\nu(x)\,,$$
$$\mathcal{O}_{\rm q}^{(N)}(x) = \overline{\psi}(x) \not{\Delta}\, D^{N-1} \psi(x)\,.$$

Here Δ_μ is a lightlike vector and we introduced the notation

$$F^{\mu;a} = \Delta_\nu\, F^{\mu\nu;a}, \qquad A^a = \Delta_\mu A^{\mu;a}, \qquad D = \Delta_\mu\, D^\mu, \qquad \partial = \Delta_\mu \partial^\mu\,.$$

These physical operators now mix under renormalization with aliens, which are (a) proportional to the field EOMs and (b) contain ghosts. Schematically the complete Lagrangian is then

$$\widetilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_{\mathsf{GF+G}} + w_{\rm i}\, \mathcal{O}_{\rm i} + \mathcal{O}_{\mathsf{EOM}}^{(N)} + \mathcal{O}_c^{(N)}$$

Construction of the alien operators

The most general form of the EOM operator is [Falcioni and Herzog, 2022]

$$\mathcal{O}_{\text{EOM}}^{(N)} = \left(D \cdot F^a + g_s \overline{\psi} T^a \not{\Delta} \psi\right) \mathcal{G}^a(A^a, \partial A^a, \partial^2 A^a, \ldots)$$

with $\mathcal{G}^a$ a generic local function of the gauge field and its derivatives. Expanding $\mathcal{G}^a$ in a series of contributions with an increasing number of gauge fields then leads to

$$\mathcal{O}_{\text{EOM}}^{(N)} = \mathcal{O}_{\text{EOM}}^{(N),I} + \mathcal{O}_{\text{EOM}}^{(N),II} + \mathcal{O}_{\text{EOM}}^{(N),III} + \mathcal{O}_{\text{EOM}}^{(N),IV} + \ldots$$

Construction of the alien operators

$$\mathcal{O}_{\mathrm{EOM}}^{(N),I} = \eta(N)\ \left(D\cdot F^a + g_s\overline{\psi}\,\not{A}\,T^a\psi\right)\ \left(\partial^{\,N-2}A^a\right),$$

$$\mathcal{O}_{\mathrm{EOM}}^{(N),II} = g_s\left(D\cdot F^a + g_s\overline{\psi}\,\not{A}\,T^a\psi\right)\sum_{\substack{i+j\\=N-3}} C_{ij}^{abc}(\partial^i A^b)(\partial^j A^c),$$

$$\mathcal{O}_{\mathrm{EOM}}^{(N),III} = g_s^2\ \left(D\cdot F^a + g_s\overline{\psi}\not{A}T^a\psi\right)\sum_{\substack{i+j+k\\=N-4}} C_{ijk}^{abcd}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d),$$

$$\mathcal{O}_{\mathrm{EOM}}^{(N),IV} = g_s^3\ \left(D\cdot F^a + g_s\overline{\psi}\not{A}T^a\psi\right)\sum_{\substack{i+j+k+l\\=N-5}} C_{ijkl}^{abcde}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e).$$

Construction of the alien operators

The coefficients $C^{a_1 \dots a_n}_{i_1 \dots i_{n-1}}$ can be written in terms of a set of independent colour tensors, each of them multiplying an associated coupling constant, as follows

$$
\begin{aligned}
C^{abc}_{ij} &= f^{abc}\kappa_{ij}, \\
C^{abcd}_{ijk} &= (f\,f)^{abcd}\kappa^{(1)}_{ijk} + d_4^{abcd}\kappa^{(2)}_{ijk} + d^{abcd}_{\widehat{4ff}}\kappa^{(3)}_{ijk}, \\
C^{abcde}_{ijkl} &= (f\,f\,f)^{abcde}\kappa^{(1)}_{ijkl} + d^{abcde}_{4f}\kappa^{(2)}_{ijkl}
\end{aligned}
$$

To avoid overcounting: κ-couplings inherit properties of the colour structures they multiply, e.g. $\kappa_{ij} = -\kappa_{ji}$

The standard gauge transformations leave $\mathcal{L}_0$ and $\mathcal{O}_i$ invariant, but not $\mathcal{O}^{(N)}_{\text{EOM}}$

$\Rightarrow$ generalized gauge transformation

$$A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \delta^{\Delta}_\omega A^a_\mu$$

Construction of the alien operators

$$A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \delta^\Delta_\omega A^a_\mu$$

$$\delta_\omega A^a_\mu = D^{ab}_\mu \omega^b(x),$$

$$\delta^\Delta_\omega A^a_\mu = -\Delta_\mu \Bigg[\eta(N)\, \partial^{N-1}\omega^a + g_s \sum_{\substack{i+j\\=N-3}} \widetilde{C}^{aa_1a_2}_{ij} \left(\partial^i A^{a_1}\right) \left(\partial^{j+1}\omega^{a_2}\right)$$

$$+ g_s^2 \sum_{\substack{i+j+k\\=N-4}} \widetilde{C}^{aa_1a_2a_3}_{ijk} \left(\partial^i A^{a_1}\right) \left(\partial^j A^{a_2}\right) \left(\partial^{k+1}\omega^{a_3}\right)$$

$$+ g_s^3 \sum_{\substack{i+j+k+l\\=N-5}} \widetilde{C}^{aa_1a_2a_3a_4}_{ijkl} \left(\partial^i A^{a_1}\right) \left(\partial^j A^{a_2}\right) \left(\partial^k A^{a_3}\right) \left(\partial^{l+1}\omega^{a_4}\right) + \mathcal{O}(g_s^4) \Bigg]$$

Construction of the alien operators

$$\widetilde{C}_{ij}^{abc} = f^{abc}\eta_{ij},$$
$$\widetilde{C}_{ijk}^{abcd} = (f\,f)^{abcd}\eta_{ijk}^{(1)} + d_4^{abcd}\eta_{ijk}^{(2)} + d_{\widehat{4ff}}^{abcd}\eta_{ijk}^{(3)},$$
$$\widetilde{C}_{ijkl}^{abcde} = (f\,f\,f)^{abcde}\eta_{ijkl}^{(1)} + d_{4f}^{abcde}\eta_{ijkl}^{(2a)} + d_{4f}^{aebcd}\eta_{ijkl}^{(2b)}.$$

The generalized gauge symmetry implies that the couplings $\eta_{n_1\ldots n_j}^{(k)}$ are related to $\kappa_{n_1\ldots n_j}^{(k)}$

Construction of the alien operators

$$\eta_{ij} = 2\kappa_{ij} + \eta(N)\binom{i+j+1}{i},$$

$$\eta^{(1)}_{ijk} = 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2[\kappa^{(1)}_{ijk} + \kappa^{(1)}_{kji}],$$

$$\eta^{(2)}_{ijk} = 3\kappa^{(2)}_{ijk},$$

$$\eta^{(3)}_{ijk} = 2[\kappa^{(3)}_{ijk} - \kappa^{(3)}_{kji}],$$

$$\eta^{(1)}_{ijkl} = 2[\kappa^{(1)}_{ij(l+k+1)} + \kappa^{(1)}_{(l+k+1)ji}]\binom{l+k+1}{k} + 2[\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{ilkj} + \kappa^{(1)}_{likj} + \kappa^{(1)}_{lkij}],$$

$$\eta^{(2a)}_{ijkl} = 3\kappa^{(2)}_{ij(k+l+1)}\binom{k+l+1}{k} + 2\kappa^{(2)}_{ijkl},$$

$$\eta^{(2b)}_{ijkl} = 2\kappa^{(2)}_{lijk}.$$

Construction of the alien operators

The generalized gauge transformation can now be promoted to a generalized BRST (gBRST) transformation

$$A_\mu^a \to A_\mu^a + \delta_c A_\mu^a + \delta_c^\Delta A_\mu^a$$

The ghost operator is now generated by the action of gBRST on a suitable ancestor operator [Falcioni and Herzog, 2022], giving

$$\mathcal{O}_c^{(N)} = \mathcal{O}_c^{(N),I} + \mathcal{O}_c^{(N),II} + \mathcal{O}_c^{(N),III} + \mathcal{O}_c^{(N),IV} + \ \ldots$$

Construction of the alien operators

$$\mathcal{O}_c^{(N),I} = -\eta(N)(\partial\overline{c}^a)(\partial^{N-1}c^a),$$
$$\mathcal{O}_c^{(N),II} = -g_s \sum_{\substack{i+j\\=N-3}} \widetilde{C}_{ij}^{abc}(\partial\overline{c}^a)(\partial^i A^b)(\partial^{j+1}c^c),$$
$$\mathcal{O}_c^{(N),III} = -g_s^2 \sum_{\substack{i+j+k\\=N-4}} \widetilde{C}_{ijk}^{astu}(\partial\overline{c}^a)(\partial^i A^s)(\partial^j A^t)(\partial^{k+1}c^u),$$
$$\mathcal{O}_c^{(N),IV} = -g_s^3 \sum_{\substack{i+j+k+l\\=N-5}} \widetilde{C}_{ijkl}^{abcde}(\partial\overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1}c^e).$$

Construction of the alien operators

In fact, there is another, and equivalent, approach to generate the ghost operators. Namely, we could also start from anti-gBRST, for which $\omega^a(x)$ in the generalized gauge transformation should be replaced by the anti-ghost field $\overline{c}^a(x)$

$$A^a_\mu \to A^a_\mu + \delta_{\overline{c}} A^a_\mu + \delta^{\Delta}_{\overline{c}} A^a_\mu$$

$\rightarrow$ This should lead to the same operators!
$\rightarrow$ Nevertheless, the functional form of the resulting operators is different from those derived from gBRST

$\Rightarrow$ Non-trivial identities for the η-couplings!

Construction of the alien operators

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = 0,$$

$$\eta^{(1)}_{ijk} = \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{(m+n+k)!}{m!\, n!\, k!} (-1)^{m+n+k} \eta^{(1)}_{(j-n)(i-m)(k+m+n)},$$

$$\eta^{(1)}_{ijkl} = -\sum_{s_1=0}^{i} \sum_{s_2=0}^{j} \sum_{s_3=0}^{k} \frac{(s_1+s_2+s_3+l)!}{s_1!\, s_2!\, s_3!\, l!} (-1)^{s_1+s_2+s_3+l} \eta^{(1)}_{(k-s_3)(j-s_2)(i-s_1)(s_1+s_2+s_3+l)}.$$

The complete Lagrangian is now

$$\widetilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_{\mathsf{GF+G}} + w_{\rm i}\,\mathcal{O}_{\rm i} + \mathcal{O}^{(N)}_{\mathsf{EOM}} + \mathcal{O}^{(N)}_c$$
$$= \mathcal{L}_0(A^a_\mu, g_s) + \mathcal{L}_{\mathsf{GF+G}}(A^a_\mu, c^a, \bar{c}^a, g_s, \xi) + \sum_{\rm k} \mathcal{C}_{\rm k}\,\mathcal{O}_{\rm k},$$

where $\mathcal{C}_{\rm k}$ labels all the distinct couplings of the operators, $\mathcal{C}_{\rm k} = \{w_{\rm i}, \eta(N), \kappa^{(i)}_{n_1\ldots n_j}, \eta^{(k)}_{n_1\ldots n_j}\}$. The UV singularities associated with the QCD Lagrangian are absorbed by introducing the bare fields/parameters

$$A^{a;\mathsf{bare}}_\mu(x) = \sqrt{Z_3} A^a_\mu(x)$$
$$c^{a;\mathsf{bare}}(x) = \sqrt{Z_c} c^a(x)$$
$$\bar{c}^{a;\mathsf{bare}}(x) = \sqrt{Z_c} \bar{c}^a(x)$$
$$g_s^{\mathsf{bare}} = \mu^\epsilon Z_g g_s$$
$$\xi^{\mathsf{bare}} = \sqrt{Z_3}\xi$$

Flavour-singlet renormalization

This is not enough to make the OMEs finite. Instead they need an additional renormalization

$$\mathcal{O}_{\mathrm{i}}^{\mathsf{ren}}(x) = Z_{\mathrm{ij}}\,\mathcal{O}_{\mathrm{j}}^{\mathsf{bare}}(x),$$

The renormalized Lagrangian becomes

$$\begin{aligned}\widetilde{\mathcal{L}} &= \mathcal{L}_0(A_\mu^{a;\mathsf{bare}}, g_s^{\mathsf{bare}}) + \mathcal{L}_{\mathsf{GF+G}}(A_\mu^{a;\mathsf{bare}}, c^{a;\mathsf{bare}}, \bar{c}^{a;\mathsf{bare}}, g_s^{\mathsf{bare}}, \xi^{\mathsf{bare}}) \\ &\quad + \sum_{\mathrm{k}} \mathcal{C}_{\mathrm{k}}^{\mathsf{bare}}\,\mathcal{O}_{\mathrm{k}}^{\mathsf{bare}}, \\ \mathcal{C}_{\mathrm{i}}^{\mathsf{bare}} &= \sum_{\mathrm{k}} \mathcal{C}_{\mathrm{k}}\,Z_{\mathrm{k\,i}},\end{aligned}$$

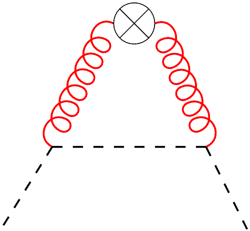
where $\mathcal{C}_{\mathrm{k}}$ is the (finite) renormalized coupling of the operator $\mathcal{O}_{\mathrm{k}}$. The UV-finite OMEs featuring a single insertion of $\mathcal{O}_{\mathrm{g/q}}^{\mathsf{ren}}$ are computed by setting the renormalized couplings $\mathcal{C}_{\mathrm{i}} = \delta_{\mathrm{i\,g/q}}$, which gives

$$\mathcal{C}_{\mathrm{i}}^{\mathsf{bare}} = Z_{\mathrm{g/q\,i}}.$$

Flavour-singlet renormalization

$\Rightarrow$ The couplings of the bare operators $\eta^{\text{bare}}(N), \ldots$ are interpreted as the renormalization constants that mix the physical operators into the aliens

$\rightarrow$ Extracted from the direct calculation of the singularities of the OMEs, e.g.



$$\eta^{\text{bare}}(N) = Z_{\mathrm{g}\,c} = -\frac{a_s}{\epsilon}\frac{C_A}{N(N-1)} + O(a_s^2)$$

We note that this quantity is known to $O(a_s^3)$

[Dixon and Taylor, 1974, Hamberg and van Neerven, 1992, Gehrmann et al., 2023]

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