Constraining and exploiting the analytic structure of Standard Model scattering amplitudes

[**PB** 2403.18047]
[**PB**, Tong-Zhi Yang 2408.06325]
[**PB**, Tong-Zhi Yang 2503.16299]
[**PB** 2504.xxxxx]

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17.04.2025
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European Research Council
Established by the European Commission

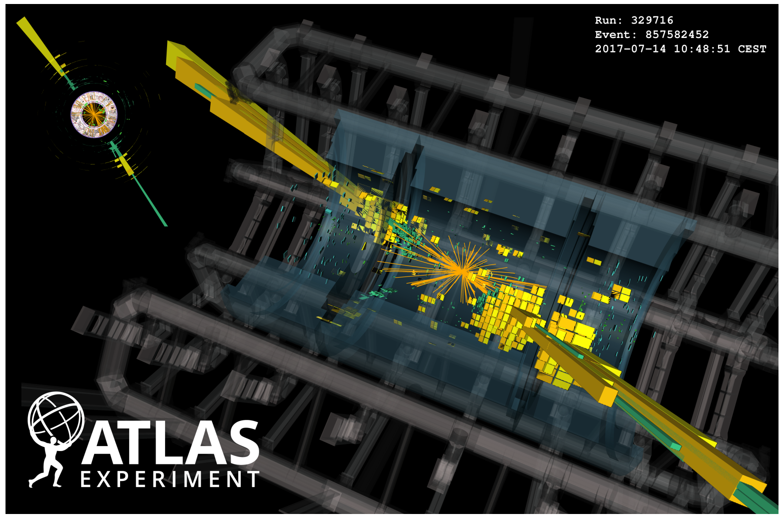
Presentation plan

- Introduction to the scattering amplitudes frontier
- Exploiting **dispersion relations** and unitarity at the integrated level of amplitudes
- Constraining Feynman integrals to a functionally distinct **finite basis**
- Exploiting singularity cancellation to make amplitudes **locally finite**
- Outlook

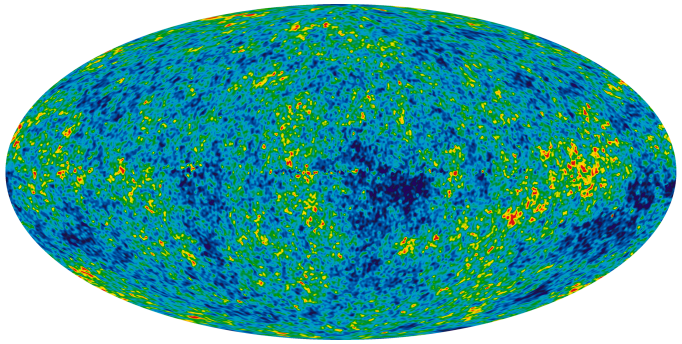
Overview

Scattering amplitudes

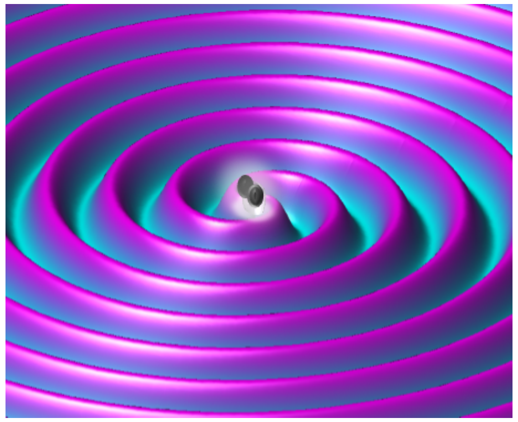
$\mathcal{N} = 4$ super Yang Mills



collider phenomenology



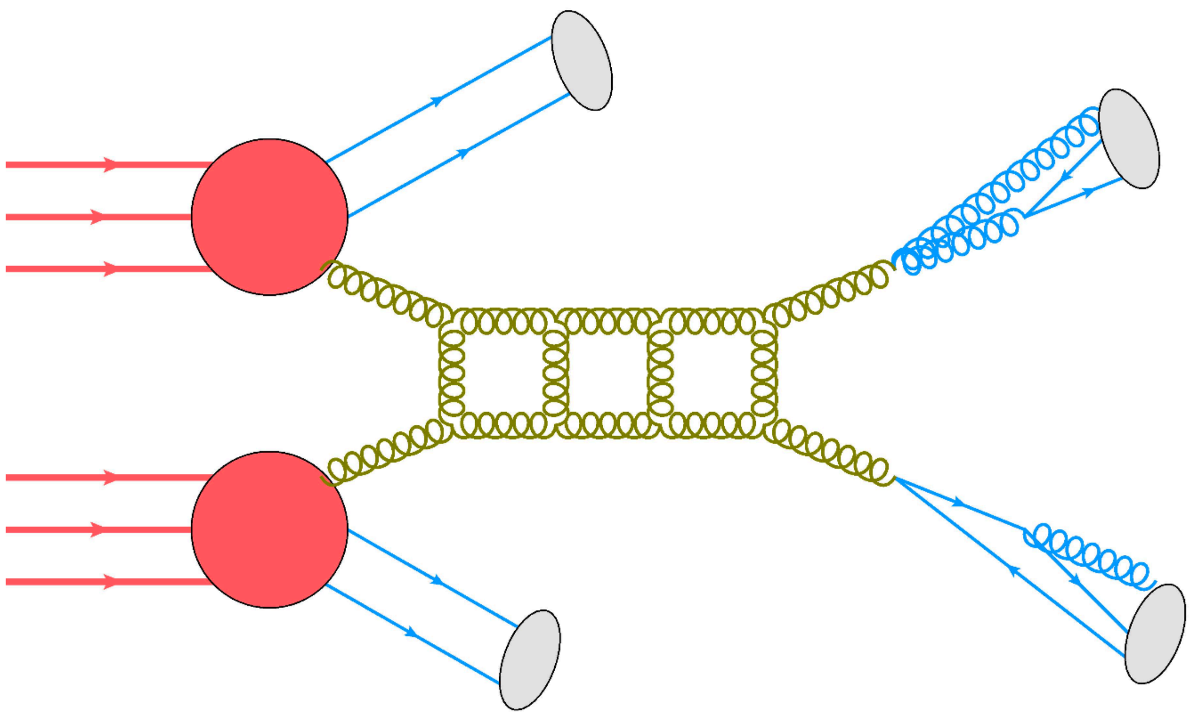
cosmological correlators

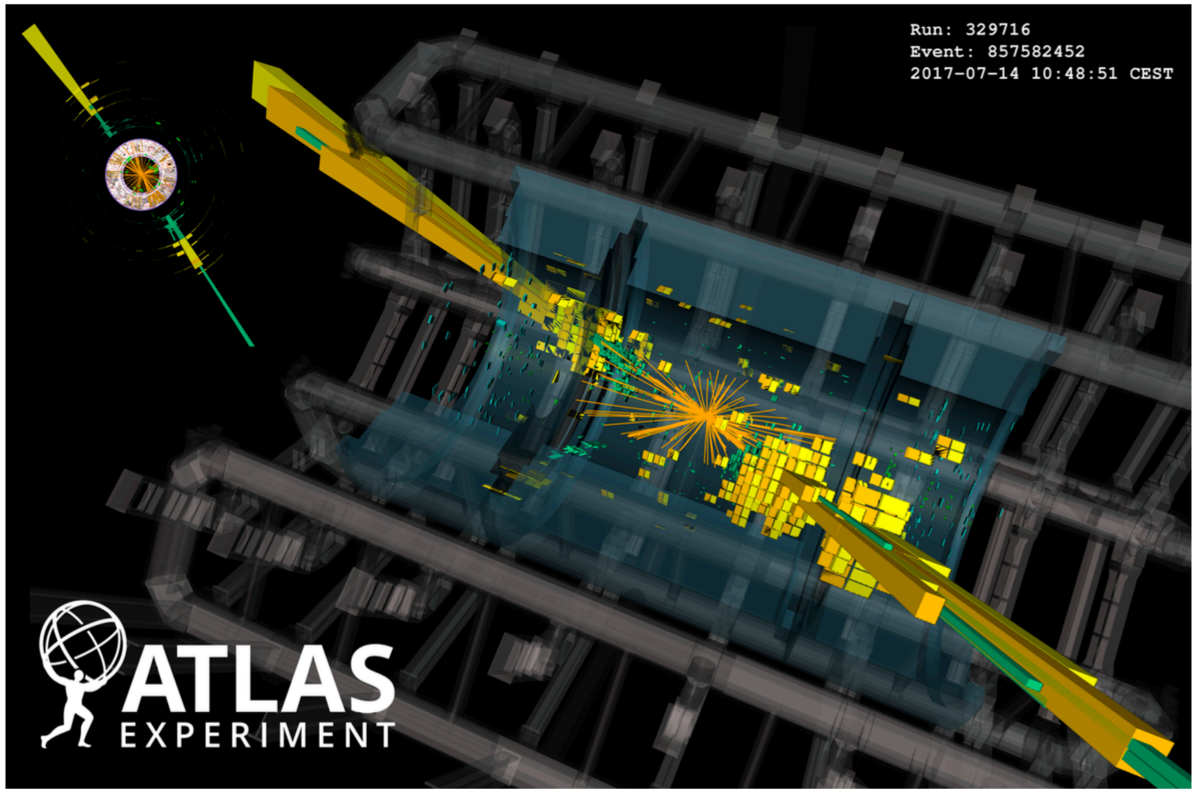


post-Minkowskian gravity

and more ...

Amplitudes for LHC phenomenology

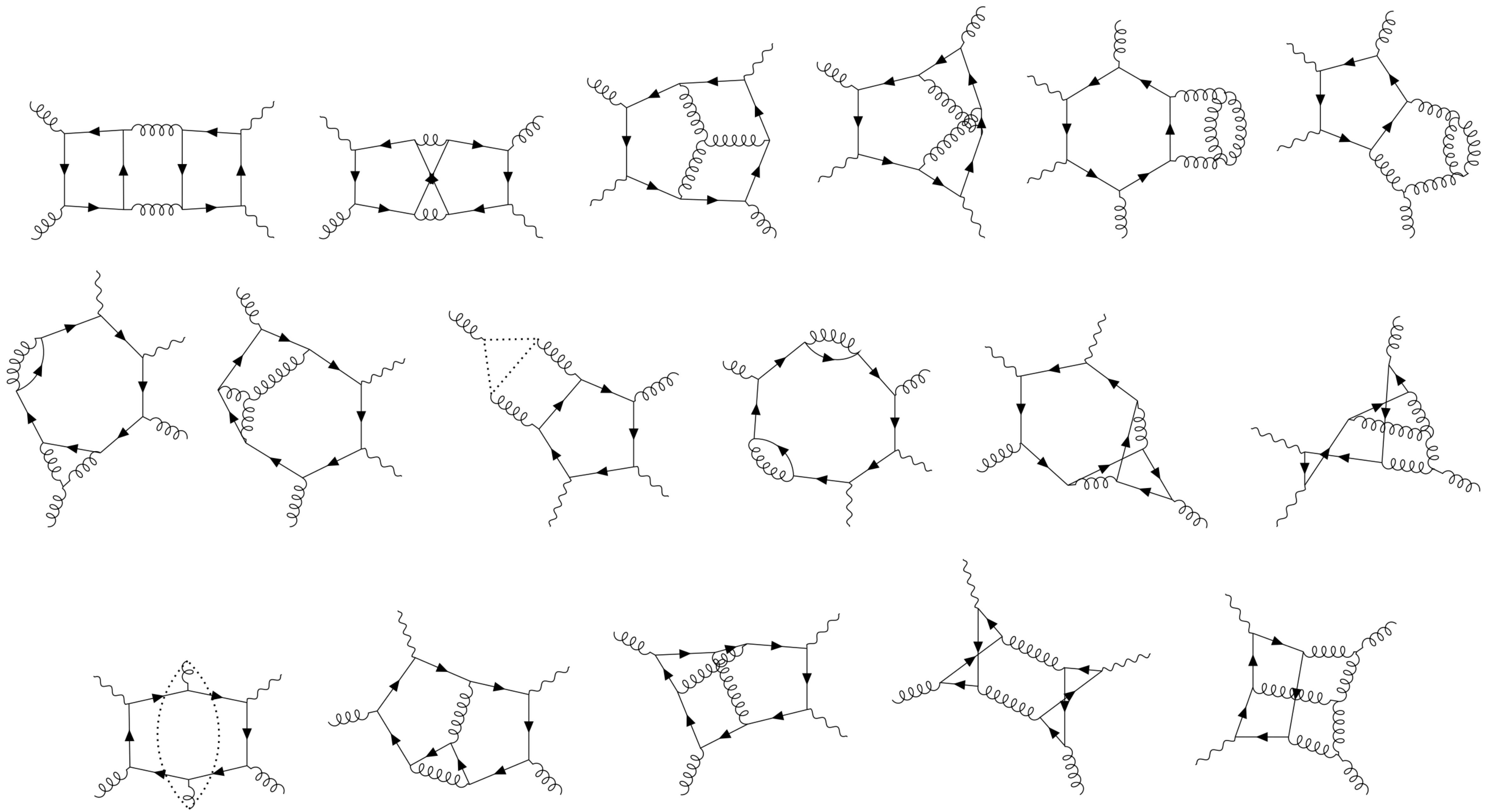




cross section = PDFs ⊗ hard scattering ⊗ real radiation ⊗ hadronization

amplitude contribution →

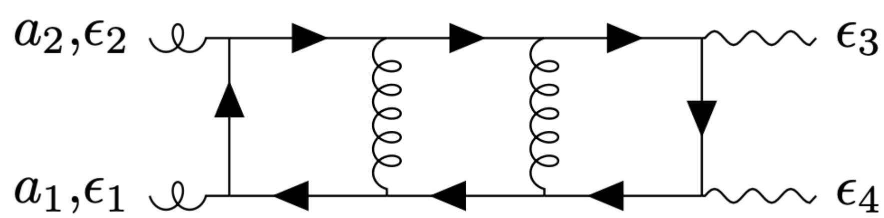
Amplitude = Σ Feynman diagrams



…

often **thousands** in modern applications

Amplitude structures

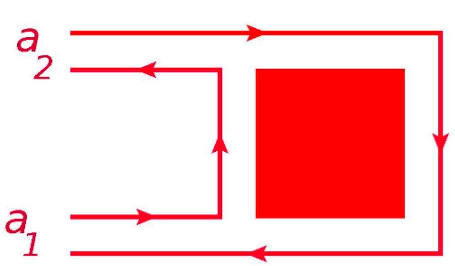


[**PB**, Caola, Manteuffel, Tancredi 2111.13595]

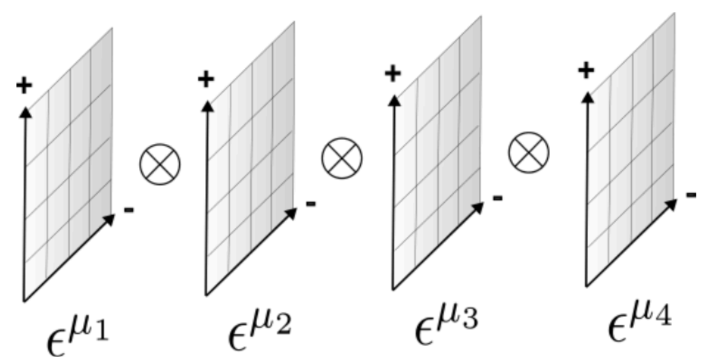
$$= \frac{1}{2}\, g_s^6\, e^2\, n_f^{(V_2)}\, C_f^2\, \delta^{a_1,a_2} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d}$$

$$\times \frac{\mathrm{tr}\left[\not\epsilon_1 (\not k_1) \not\epsilon_2 (\not k_1 + \not p_2) \gamma^\mu (\not k_{13} + \not p_2) \gamma^\nu (\not k_2 + \not p_4) \not\epsilon_4 (\not k_2) \not\epsilon_3 (\not k_2 - \not p_3) \gamma_\nu (\not k_{13} - \not p_1) \gamma_\mu (\not k_1 - \not p_1)\right]}{(k_1)^2 (k_1 + p_2)^2 (k_{13} + p_2)^2 (k_2 + p_4)^2 (k_2)^2 (k_2 - p_3)^2 (k_{13} - p_1)^2 (k_1 - p_1)^2 (k_3)^2 (k_{123} - p_{13})^2}$$

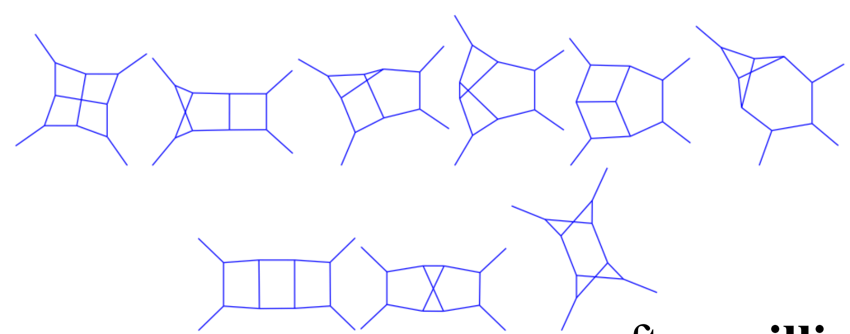
colour



tensors



integrals

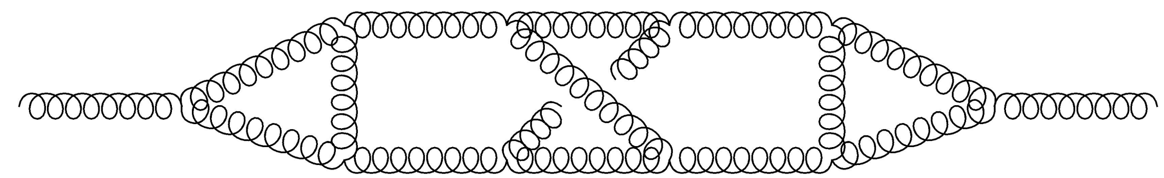


often **millions**
in modern applications

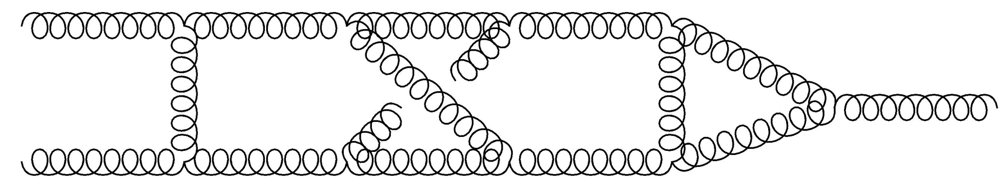
$$\mathcal{A}^{\vec{a},\vec{\lambda}} = \sum_{c,t,i} \mathcal{C}_c^{\vec{a}}\, T_t^{\vec{\lambda}}\, \mathcal{I}_i\, r_{c,t,i}$$

QCD amplitudes frontier

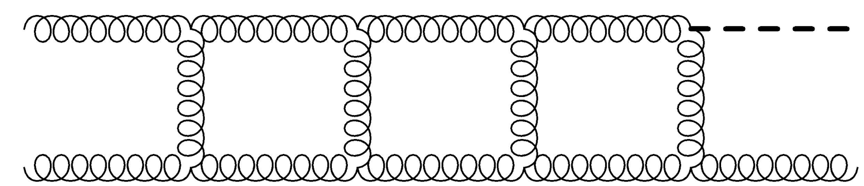
- 2-point : 5-loop (β function)
  [Herzog, Ruijl, Ueda, Vermaseren, Vogt 1701.01404]



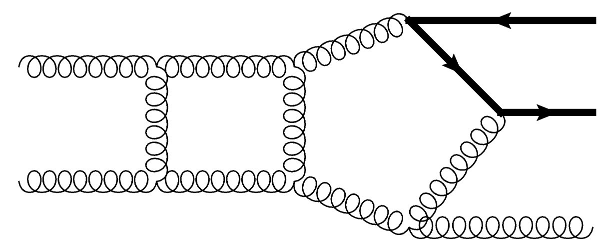
- 3-point : 4-loop (form factors)
  [Lee, Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser 2202.04660]



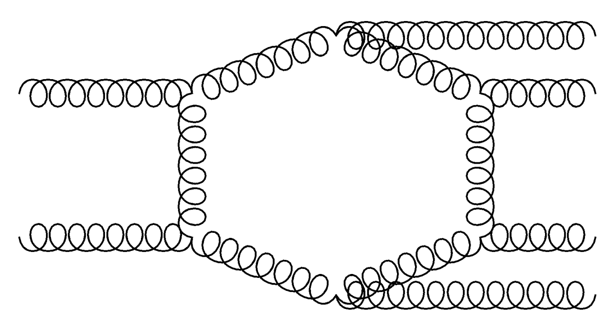
- 4-point : 3-loop ($\leq$1-off shell)
  [**PB**, Caola, Chakraborty, Gambuti, Manteuffel, Tancredi]
  [Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi]
  [Chen, Guan, Mistlberger]



- 5-point : 2-loop (multiscale)
  [Abreu, Badger, Brancaccio, Buccioni, Chawdhry, Chicherin, Cordero, Czakon, Devoto, Gambuti, Ita, Manteuffel, Mitov, Page, Peraro, Poncelet, Sotnikov, Tancredi, Zoia, ...]

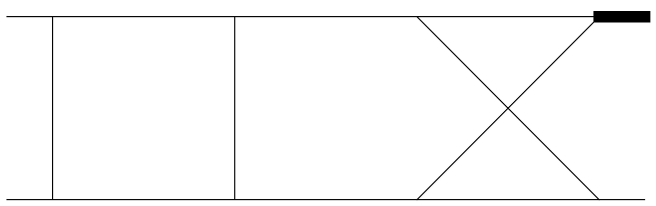


- $\geq$6-point : 1-loop (numerical, automated)
  [Blackhat, GoSam, MadLoops, OpenLoops, Rocket, ...]

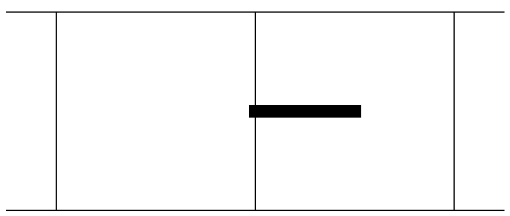


Feynman integrals frontier

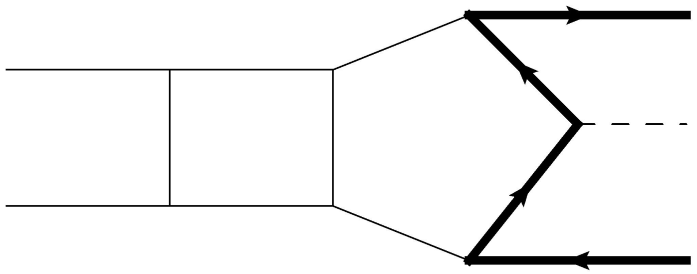
[Gehrmann, Henn, Jakubčík, Lim, Mella, Syrrakos, Tancredi, Bobadilla 2410.19088]



[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2306.15431]



[Cordero, Figueiredo, Kraus, Page, Reina 2312.08131]



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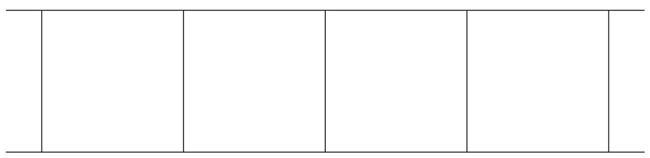
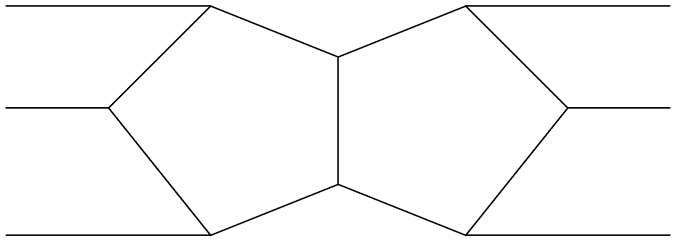
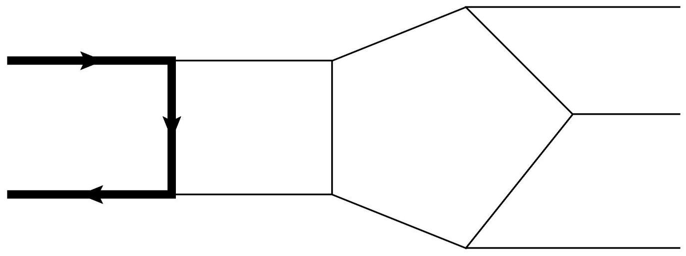


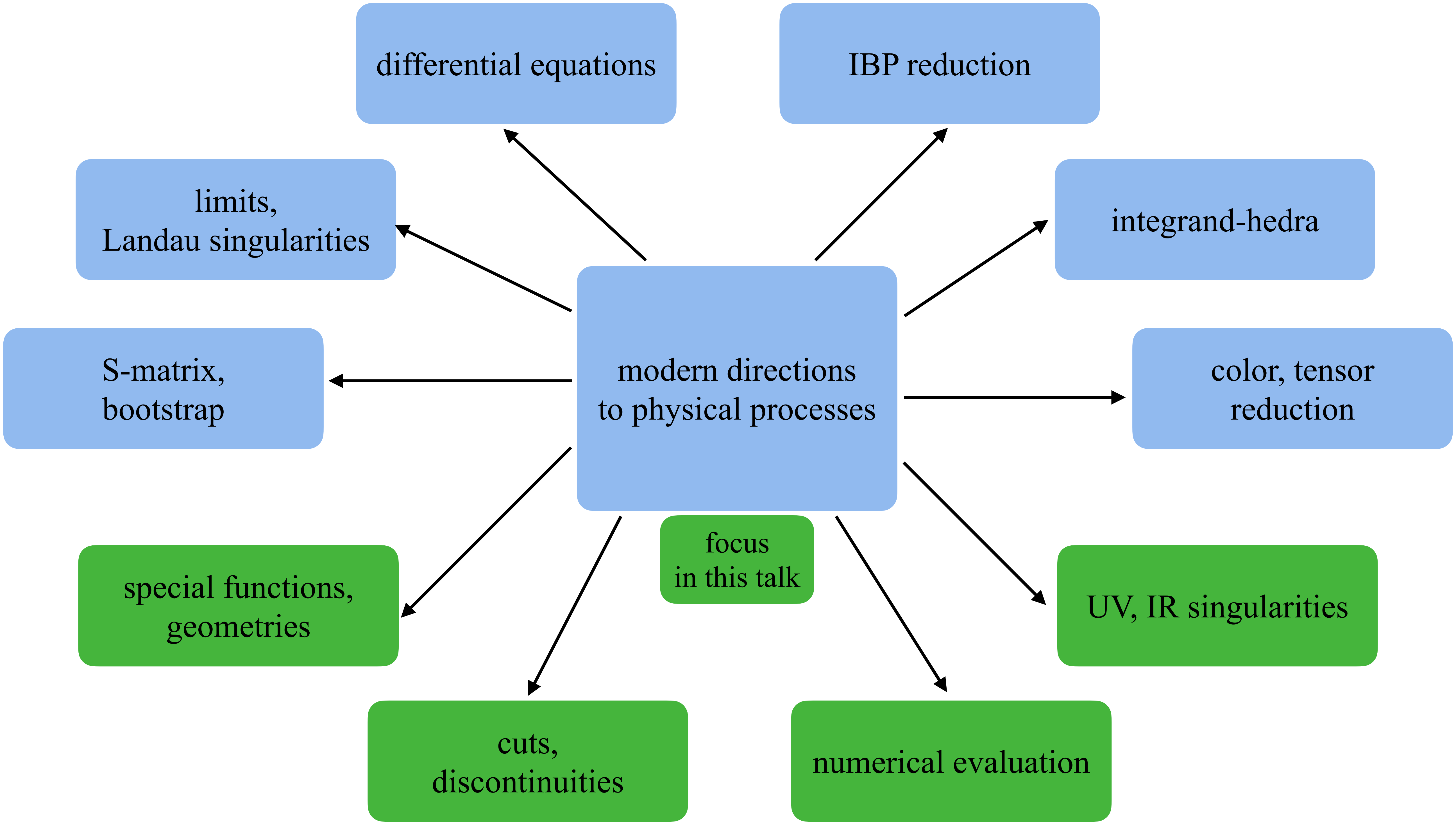
[Abreu, Monni, Page, Usovitsch 2412.19884]
[Henn, Matijašić, Miczajka, Peraro, Xu, Zhang 2501.01847]



[Badger, Becchetti, Giraudo, Zoia 2404.12325]



Computational complexity



Exploiting the analytic amplitude structure

[**PB** 2403.18047]

Integrated Unitarity for Scattering Amplitudes

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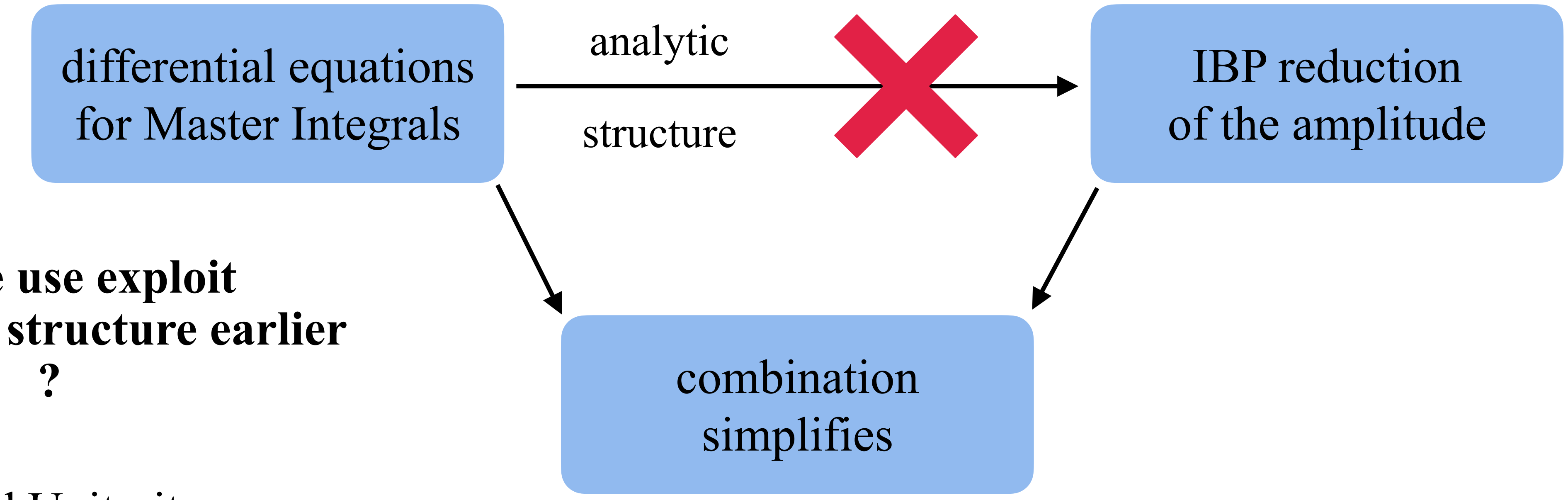
We present a new method for computing multi-loop scattering amplitudes in Quantum Field Theory. It extends the Generalized Unitarity method by constraining not only the integrand of the amplitude but also its full integrated form. Our approach exploits the relation between cuts and discontinuities of the amplitude. Explicitly, by the virtue of analyticity and unitarity of the S-matrix, the amplitude can be expressed in terms of lower-loop on-shell amplitudes dispersively integrated along cuts. As both cuts and discontinuities can be computed systematically in dimensional regularization, we validated our method by reproducing the four-gluon amplitude in two-loop massless Quantum Chromodynamics. Moreover, since our approach improves the performance of the calculation, we provide a new result for the four-loop four-point massless planar ladder Feynman integral. It is expressed in terms of Harmonic Polylogarithms with letters 0 and 1.

Introducing Integrated Unitarity

standard amplitude workflow



**can we use exploit
the analytic structure earlier
?**

YES : Integrated Unitarity

- dispersion relations : discontinuities algorithmic
- cut canonical differential equations (DEQ) : algorithmic in dimReg

[see Appendix for technical definitions]

- Generalized Unitarity @ integrated level : constrains both Master Integrals (MIs) and their coefficients using cuts
- less subsectors, less MIs, simpler DEQ and IBP system

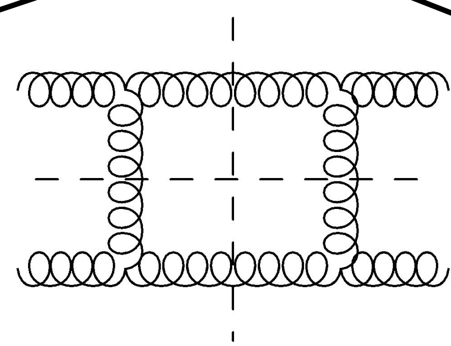
Generalized VS Integrated Unitarity

consider a toy 1-loop amplitude

$$\mathscr{A}^{(1)} \sim \int \frac{d^d k}{(2\pi)^d} \frac{\mathscr{N}(k)}{\mathscr{D}_1(k)\,\mathscr{D}_2(k)\,\mathscr{D}_3(k)\,\mathscr{D}_4(k)} = r_{1234} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\mathscr{D}_1(k)\,\mathscr{D}_2(k)\,\mathscr{D}_3(k)\,\mathscr{D}_4(k)} + \text{subsectors}$$

cut all 4 propagators

$$\text{Cut}_{1234}\,\mathscr{A}^{(1)} \sim \int \frac{d^d k}{(2\pi)^d} \mathscr{A}_1^{(0)}(k)\ \mathscr{A}_2^{(0)}(k)\ \mathscr{A}_3^{(0)}(k)\ \mathscr{A}_4^{(0)}(k)\, \delta^+(\mathscr{D}_1(k))\, \delta^+(\mathscr{D}_2(k))\, \delta^+(\mathscr{D}_3(k))\, \delta^+(\mathscr{D}_4(k)) = r_{1234} \int \frac{d^d k}{(2\pi)^d} \delta^+(\mathscr{D}_1(k))\, \delta^+(\mathscr{D}_2(k))\, \delta^+(\mathscr{D}_3(k))\, \delta^+(\mathscr{D}_4(k))$$



Generalized Unitarity

can compute integral coefficient

$$r_{1234} = \frac{1}{2} \sum_{\text{2 cut solns } k^*} \mathscr{A}_1^{(0)}(k^*)\ \mathscr{A}_2^{(0)}(k^*)\ \mathscr{A}_3^{(0)}(k^*)\ \mathscr{A}_4^{(0)}(k^*)$$

scalar integral remains to be computed

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\mathscr{D}_1(k)\,\mathscr{D}_2(k)\,\mathscr{D}_3(k)\,\mathscr{D}_4(k)}$$

Integrated Unitarity

compute cut amplitude

$$\text{Cut}_{1234}\,\mathscr{A}^{(1)} = r_{1234} \int \frac{d^d k}{(2\pi)^d} \delta^+(\mathscr{D}_1(k))\, \delta^+(\mathscr{D}_2(k))\, \delta^+(\mathscr{D}_3(k))\, \delta^+(\mathscr{D}_4(k))$$

can reconstruct whole amplitude

$$\mathscr{A}^{(1)}(s,u) \sim \int_0^\infty \frac{ds'}{s'-s} \int_0^\infty \frac{du'}{u'-u} \text{Cut}_{1234}\,\mathscr{A}^{(1)}(s',u')$$

Dispersion relation

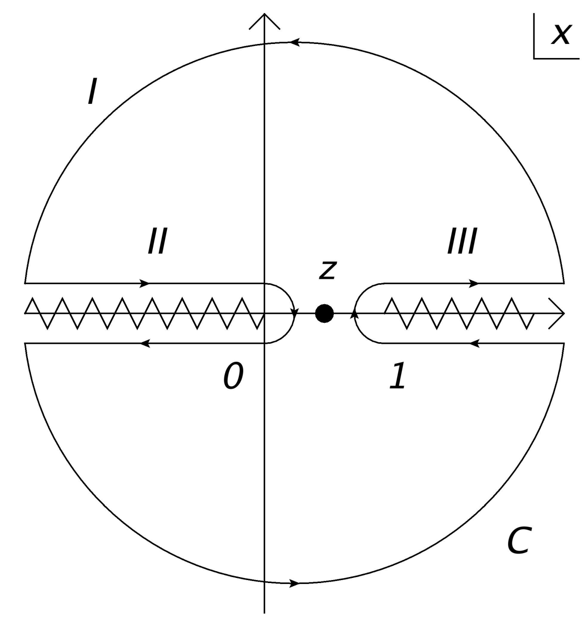
Cauchy's integral formula

$$\mathscr{A}(z) = \frac{1}{2\pi i}\oint_C \frac{\mathscr{A}(x)dx}{x-z}$$

[Cutkosky, Mandelstam, Eden, Landshoff, Olive, Polkinghorne, Remiddi, van Neerven, Kniehl, Sirlin]

$x = -\frac{t}{s}$

4-point massless



piecewise contour

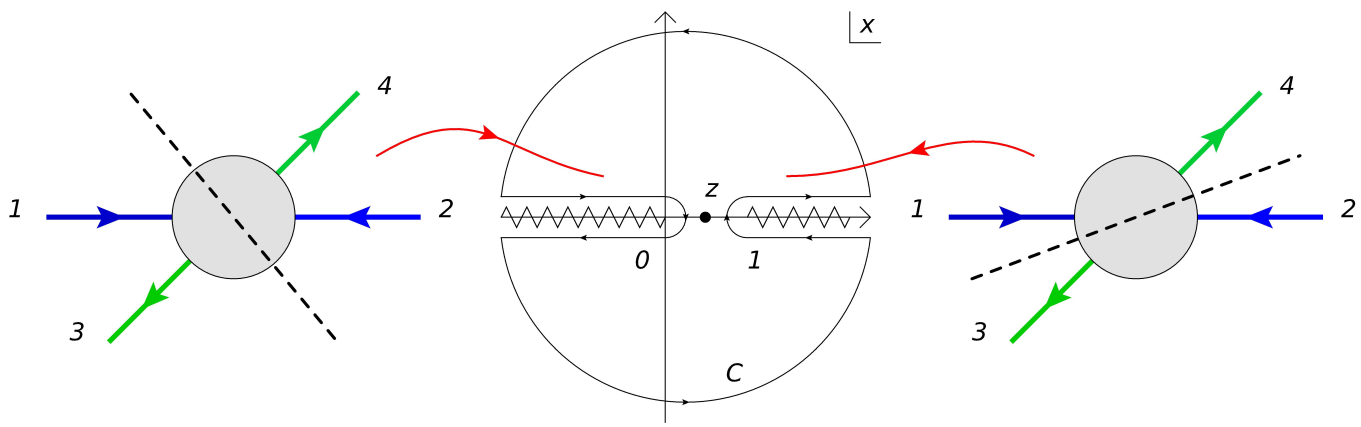
$$\mathscr{A}(z) = c_\infty + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Disc}_0 + \int_1^\infty \mathrm{Disc}_1\right)\frac{\mathscr{A}(x)dx}{x-z}$$

cancel constant from infinite arc

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Disc}_0 + \int_1^\infty \mathrm{Disc}_1\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$

unitarity

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Cut}_t + \int_1^\infty \mathrm{Cut}_u\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$



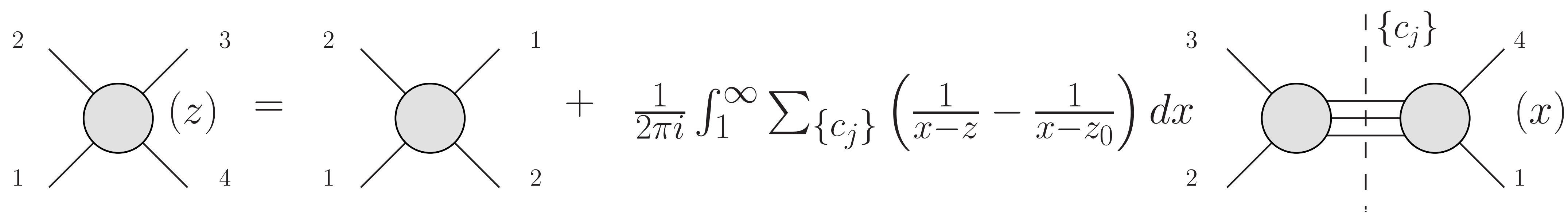
Dispersion relation

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Cut}_t + \int_1^\infty \mathrm{Cut}_u\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$

expressing cuts as phase space integrals

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\Bigg(\int_0^\infty \sum_{\{c_i\}\in\mathscr{C}_t} \int d\mathrm{PS}_{t,\{c_i\}}\, \mathscr{A}_{t,\{c_i\},L}\, \mathscr{A}^*_{t,\{c_i\},R}$$
$$+ \int_1^\infty \sum_{\{c_j\}\in\mathscr{C}_u} \int d\mathrm{PS}_{u,\{c_j\}}\, \mathscr{A}_{u,\{c_j\},L}\, \mathscr{A}^*_{u,\{c_j\},R}\Bigg)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)dx$$

diagrammatically, in the planar case



Integrated Unitarity : 3 methods

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Cut}_t + \int_1^\infty \mathrm{Cut}_u\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$

A. explicit integration : convergent e.g. for canonical MIs

subtraction terms needed for full amplitude

$$\mathscr{A}(x) \to S(x)\ \mathscr{A}(x) \qquad S(x) = \frac{(1-x)^p x^q}{(x-z_1)^r} \qquad -\mathrm{Res}_{x\to z_1}\mathscr{A}(x)\,S(x)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)$$

B. ansatz reconstruction with 2 discontinuities + # evaluations :

$$\begin{cases} \mathrm{Disc}_0\,\mathscr{A} &= \mathrm{Cut}_t\,\mathscr{A} \\ \mathrm{Disc}_1\,\mathscr{A} &= \mathrm{Cut}_u\,\mathscr{A} \\ \mathscr{A}(z_i) &= \mathscr{A}_i \end{cases} \qquad \mathscr{A}(z) = \epsilon^{\#}\sum_{n\geq 0}\epsilon^n \sum_{\vec{\alpha}\,:\,|\vec{\alpha}|\leq n} r_{n,\vec{\alpha}}(z)\,G(\vec{\alpha}, z)$$

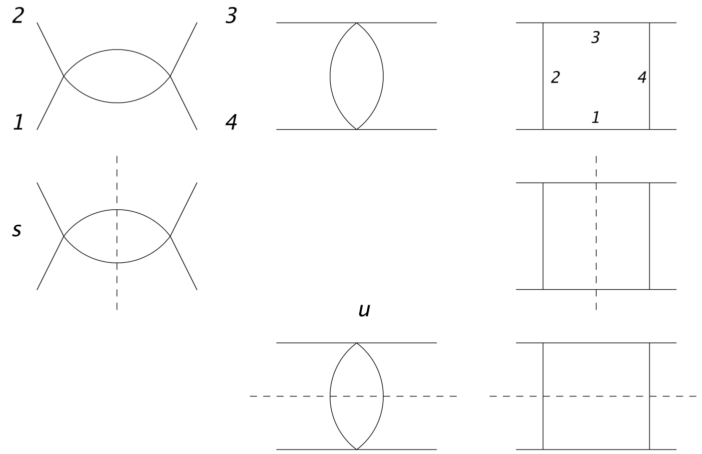
unknowns

C. ansatz reconstruction with 3 discontinuities :

$$\begin{cases} \mathrm{Disc}_0\,\mathscr{A} &= \mathrm{Cut}_t\,\mathscr{A} \\ \mathrm{Disc}_1\,\mathscr{A} &= \mathrm{Cut}_u\,\mathscr{A} \\ \mathrm{Disc}_\infty\,\mathscr{A} &= \mathrm{Cut}_s\,\mathscr{A} \end{cases} \qquad \mathscr{A}(z) = \epsilon^{\#}\sum_{n\geq 0}\epsilon^n \sum_{\vec{\alpha}\,:\,|\vec{\alpha}|\leq n} r_{n,\vec{\alpha}}(z)\,G(\vec{\alpha}, z)$$

[Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu 2007.13747]

Integrated Unitarity : example



cut MIs

$$M_i^c \in \{\epsilon(2\epsilon-1)\, I_{1,0,1,0}\,,\, \epsilon(2\epsilon-1)\, I_{0,1,0,1}\,,\, \epsilon^2(x-1)\, I_{1,1,1,1}\}$$

$$\mathrm{Cut}_s M_i^c \in \{\epsilon(2\epsilon-1)\, I_{1,0,1,0;1,3},\ 0,\ \epsilon^2(x-1)\, I_{1,1,1,1;1,3}\}$$

$$\mathrm{Cut}_u M_i^c \in \{0,\ \epsilon(2\epsilon-1)\, I_{0,1,0,1;2,4},\, \epsilon^2(x-1)\, I_{1,1,1,1;2,4}\}$$

$$\mathrm{Cut}_t M_i^c \in \{0,\ 0,\ 0\}$$

cut DEQ

$$\partial_x M_i^c(x,\epsilon) = \epsilon\, A_{ij}^c(x)\, M_j^c(x,\epsilon)$$

$$\partial_x \mathrm{Cut}_s M_i^c(x,\epsilon) = \epsilon\, A_{ij}^{c,s}(x)\, \mathrm{Cut}_s M_j^c(x,\epsilon)$$

$$\partial_x \mathrm{Cut}_u M_i^c(x,\epsilon) = \epsilon\, A_{ij}^{c,u}(x)\, \mathrm{Cut}_u M_j^c(x,\epsilon)$$

$$\partial_x \mathrm{Cut}_t M_i^c(x,\epsilon) = 0$$

$$A^c(x) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\frac{1}{1-x}} & \mathbf{0} \\ \mathbf{\frac{2}{x}} & \mathbf{\frac{2}{x}} + \mathbf{\frac{2}{1-x}} & \frac{1}{x} + \frac{1}{1-x} \end{pmatrix}$$

unknowns

3 methods

$$M_i^c(z) = \epsilon^{-2} \sum_{n\geq 0} \epsilon^n \sum_{\vec{\alpha}\,:\,|\vec{\alpha}|\leq n} c_{n,\vec{\alpha}}\, G(\vec{\alpha}, z) \qquad \alpha_k \in \{0,1\}$$

A. explicit integration

$$M_i^c(z) = M_{i,0}^c + \frac{1}{2\pi i}\int_1^\infty \left(\frac{dx}{x-z} - \frac{dx}{x}\right) \mathrm{Cut}_u M_i^c(x)$$

B. ansatz + 2cuts + 1pt

ansatz computed

$$\begin{cases} \mathrm{Disc}_0 M_i^c & = \mathrm{Cut}_t M_i^c \\ \mathrm{Disc}_1 M_i^c & = \mathrm{Cut}_u M_i^c \\ M_i^c(0) & = M_{i,0}^c \end{cases}$$

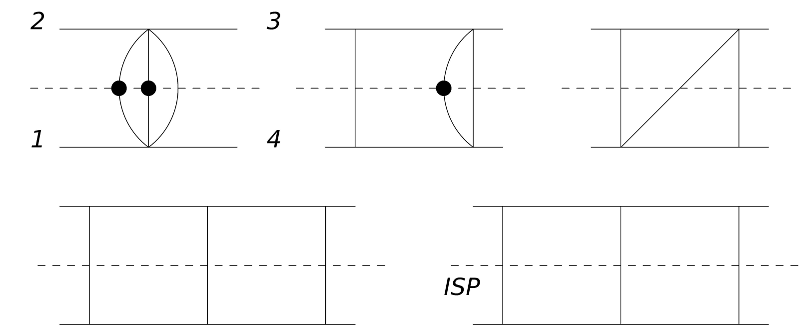
C. ansatz + 3cuts

$$\begin{cases} \mathrm{Disc}_0 M_i^c & = \mathrm{Cut}_t M_i^c \\ \mathrm{Disc}_1 M_i^c & = \mathrm{Cut}_u M_i^c \\ \mathrm{Disc}_\infty M_i^c & = \mathrm{Cut}_s M_i^c \end{cases}$$

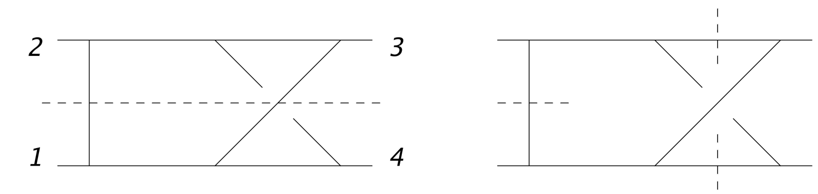
Integrated Unitarity : further checks

Master Integrals (from DEQ)

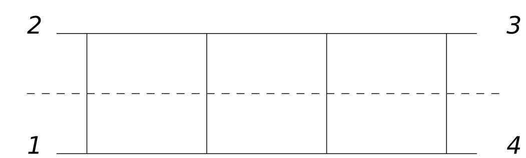
- 2-loop planar (#MIs : 8 → 5) ✓



- 2-loop nonplanar (#MIs : 12 → 2+6) ✓



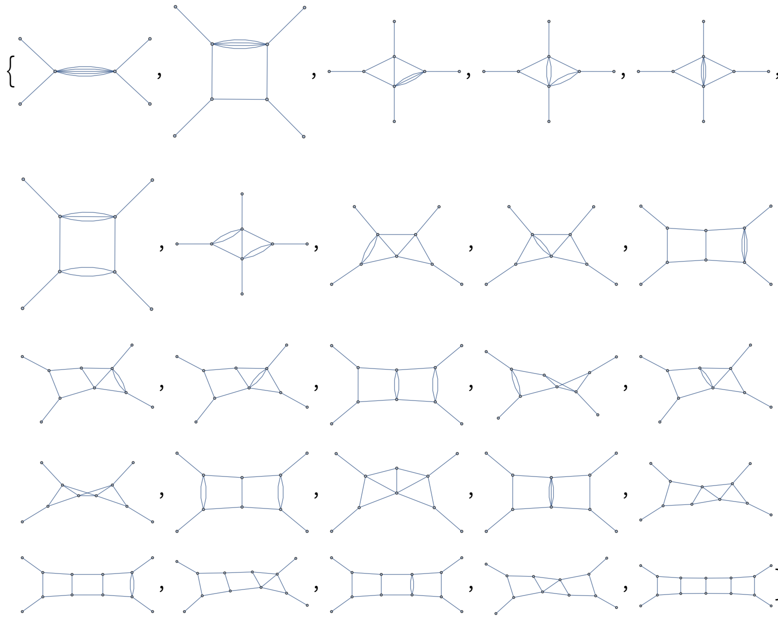
- 3-loop planar ladder (#MIs : 26 → 17) ✓



Amplitudes (from form factor method)

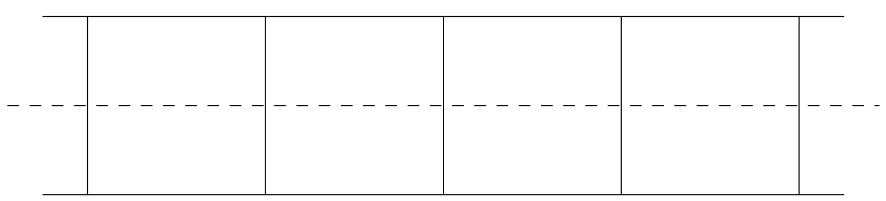
- 1-loop gg → gg (#INTs : /~2 per cut) ✓
- 2-loop gg → gg planar (#INTs : /~8 per cut) ✓
- 2-loop gg → gg nonplanar (#INTs : /~4 per cut) ✓

New application : four-loop ladder



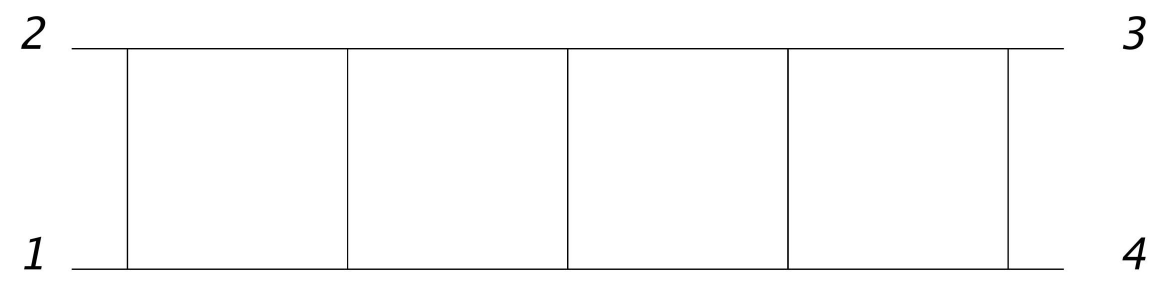
procedure

- 22 generalized propagators = 13 denominators + 9 ISPs
- cut u : 5 propagators
- IBP : 59 MIs $\mathrm{Cut}_u M_i^c$ ($\texttt{LiteRed} + \texttt{Kira}$)
- canonical DEQ : $A_{ij}^c(x) = \dfrac{a_{ij}}{x} + \dfrac{b_{ij}}{1-x}$ ($\texttt{CANONICA} + \texttt{MultivariateApart} + \texttt{FiniteFlow}$)
- canonical general solution : $M_u^c(x,\epsilon) = \mathbb{P}e^{\epsilon\int A^c(x)dx} M_{u,0}^c(\epsilon)$ ($\texttt{PolyLogTools}$ + in-house)
- regularity constraints on BCs $M_{u,0}^c(\epsilon)$: $59 \to 5$ (in-house)
- 5 remaining BCs : weight 7 ($\texttt{AMFlow}$)
- method B : $\mathrm{Disc}_0 M_i^c = \mathrm{Cut}_t M_i^c = 0 \quad \& \quad \mathrm{Disc}_1 M_i^c = \mathrm{Cut}_u M_i^c$ (in-house)



fixed all HPL coefficients to weight 8

New application : four-loop ladder



Future directions

kinematic limits

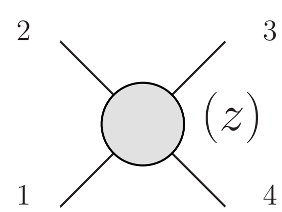
e.g. $\text{Disc}_1 \mathscr{A} = \text{Cut}_u \mathscr{A}$ alone gives u→0 limit at fixed Log accuracy to any subleading power

$$\lim_{x\to 1} \mathscr{A}(x) \sim c_{1,1}(x)\, G(1,1,x) + c_{0,1}(x)\, G(0,1,x) + c_{\dots,0}(x)\, G(\dots,0,x)$$

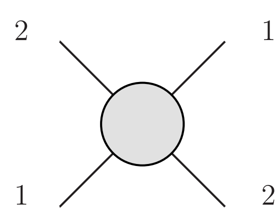
Leading Log → $G(1,1,x)$; Next-to-Leading Log → $G(0,1,x)$; suppressed → $G(\dots,0,x)$

recursive approach

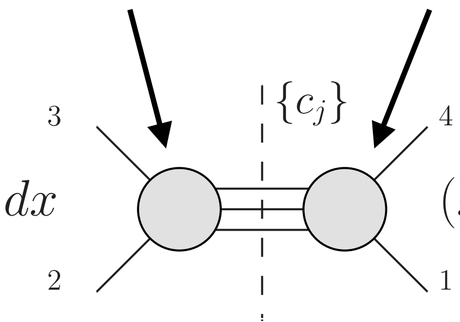
lower-loop amplitudes



$(z) =$

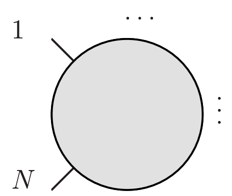


$$+ \frac{1}{2\pi i}\int_1^\infty \sum_{\{c_j\}} \left(\frac{1}{x-z} - \frac{1}{x-z_0}\right) dx$$

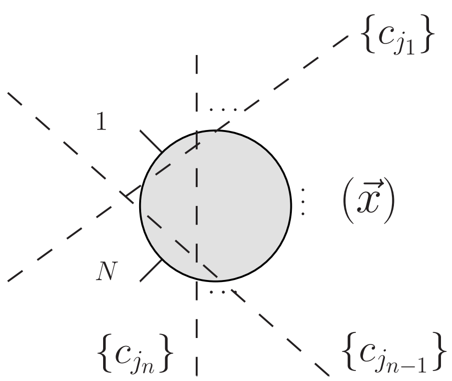


(x)

Multivariate Integrated Unitarity



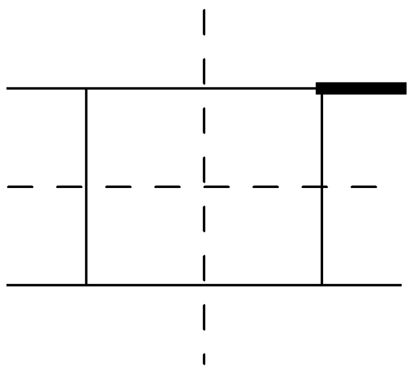
$$(\vec{z}) = \frac{1}{(2\pi i)^n}\left(\prod_{m=1}^{n} \int_{x_{\text{thres}}^{(m)}}^{\infty} \frac{dx_m}{x_m - z_m} \sum_{\{c_{j_m}\}}\right)$$



$$(\vec{x}) + \dots$$

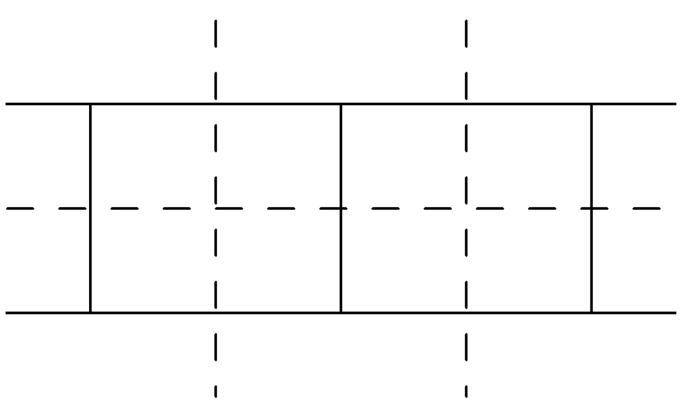
Beyond a single cut

is it always possible to reconstruct an integral from its **maximal** cut ?



YES

[**PB**, Otaifi, Tancredi]



ongoing

trade-off in properties

- many cut propagators → simpler IBPs & DEQs
- genealogical constraints → possible simplifications

  [Caron-Huot, Dixon, McLeod, Hippel 1609.00669]

  [Hannesdottir, Lippstreu, McLeod, Polackova 2406.05943]
- iterative Cauchy formula → multivariate complex analysis
- Landau singularities → nontrivial analytic structure

  [Helmer, Papathanasiou, Tellander 2402.14787]

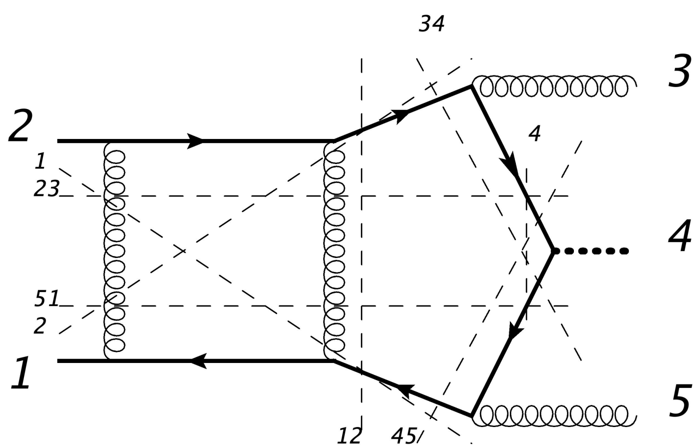
  [Correia 2212.06157]

  [Correia, Giroux, Mizera 2503.16601]
- how do amplitudes factorize on anomalous thresholds ?

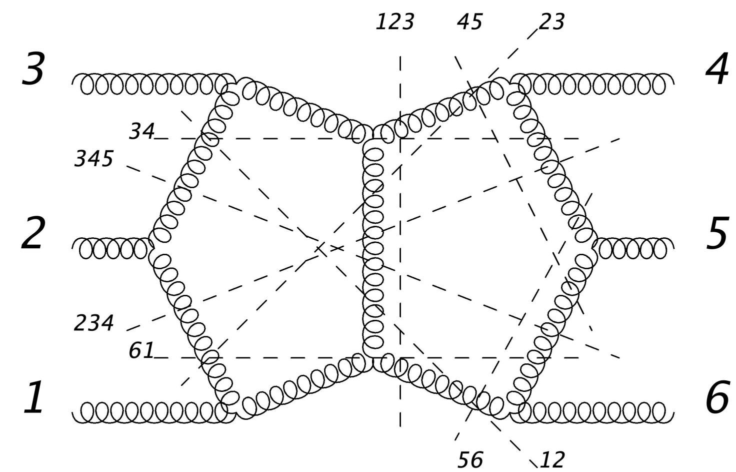
Analytic structure at 2 loops

with each new scale,
more complicated analytic structure

$pp \to t\bar{t}H$ @ 2 loops

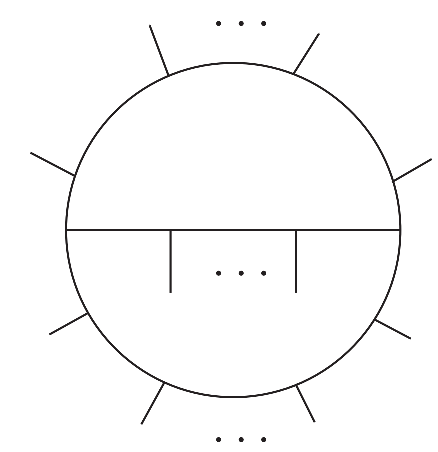


$gg \to gggg$ @ 2 loops



...

n-point



where does it **end** ?

a.k.a. the finite functional basis problem

Constraining the structure of Feynman integrals

[**PB**, Tong-Zhi Yang 2408.06325]

On the finite basis topologies for multi-loop high-multiplicity Feynman integrals

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[1] *Physik-Institut, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland*

In this work, we systematically analyse Feynman integrals in the 't Hooft-Veltman scheme. We write an explicit reduction resulting from partial fractioning the high-multiplicity integrands to a finite basis of topologies at any given loop order. We find all of these finite basis topologies at two loops in four external dimensions. Their maximal cut and the leading singularity are expressed in terms of the Gram determinant and Baikov polynomial. By performing an Integration-By-Parts reduction without any cut constraint on a numerical probe for one of these topologies, we show that the computational complexity drops significantly compared to the Conventional Dimensional Regularization scheme. Formally, our work implies an upper bound on the rigidity of special functions appearing in the iterated integral solutions at each loop order in perturbative Quantum Field Theory. Phenomenologically, the integrand-level reduction we present will substantially simplify the task of providing high-precision predictions for future high-multiplicity collider observables.

[**PB**, Tong-Zhi Yang 2503.16299]
+ ongoing with Frellesvig, Marzucca, Morales, Seefeld, Wilhelm

On the finite basis of two-loop 't Hooft-Veltman Feynman integrals

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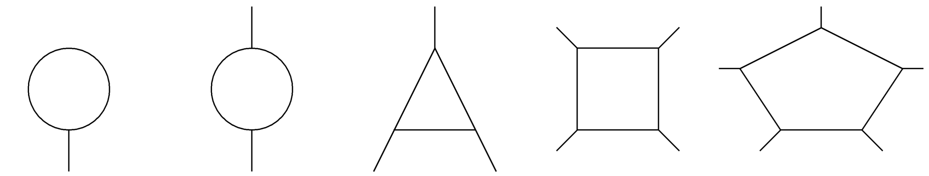
Abstract: In this work, we investigate the finite basis topologies of two-loop dimensionally regularized Feynman integrals in the 't Hooft-Veltman scheme in the Standard Model. We present a functionally distinct finite basis of Master Integrals which spans the whole transcendental space of all two-loop Feynman integrals with external momenta in four dimensions. We also indicate that all the two-loop Master Integrals, in an appropriate basis, with more than 8 denominators do not contribute to the finite part of any two-loop scattering amplitude. In addition, we elaborate on the application of the 't Hooft-Veltman decomposition to improve the performance of numerical evaluation of Feynman integrals using AMFlow and DCT packages. Moreover, we analyze the spectrum of special functions and the corresponding geometries appearing in any two-loop scattering amplitude. Our work will allow for a reduction in the computational complexity required for providing high-precision predictions for future high-multiplicity collider observables, both analytically and numerically.

The finite basis problem

1-loop n-point : linearly related to subsectors of 5-point



$$\mathcal{I}_{1,n} = \sum_{i_1=1}^{n} \frac{\mathcal{N}_{\text{tadpole},i_1}}{\mathcal{D}_{i_1}} + \sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n} \frac{\mathcal{N}_{\text{bubble},i_1,i_2}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}} + \sum_{\substack{i_1,i_2,i_3=1\\ i_j\neq i_m}}^{n} \frac{\mathcal{N}_{\text{triangle},i_1,i_2,i_3}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}} + \sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_j\neq i_m}}^{n} \frac{\mathcal{N}_{\text{box},i_1,i_2,i_3,i_4}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}\mathcal{D}_{i_4}} + \sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_j\neq i_m}}^{n} \frac{\mathcal{N}_{\text{pentagon},i_1,i_2,i_3,i_4,i_5}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}\mathcal{D}_{i_4}\mathcal{D}_{i_5}},$$

beyond 1-loop : new integrals with each new leg ?

- 1-loop : “no” i.e. up to 5-point [Passarino, Veltman, Ossola, Papadopoulos, Pittau]
- 2-loop massless planar in d=4-2ϵ : “also no” i.e. up to 11 denominators [Gluza, Kajda, Kosower 1009.0472]
- 2-loop in d=d_0-2ϵ : “also no” [Kleiss, Malamos, Papadopoulos, Verheyen 1206.4180]
- 2-loop in d=4 : “also no” i.e. up to 8 denominators [Feng, Huang 1209.3747]
- L-loop in d=4 : “also no" [Bourjaily, Herrmann, Langer, Trnka 2007.13905]
- **L-loop in d=d_0-2ϵ : “also no"** [**PB**, Tong-Zhi Yang 2408.06325]
  (for integer d_0 & propagator powers)

1-loop example

- consider an integral $\mathcal{I}_{1,6} = \dfrac{1}{k^2(k+p_1)^2(k+p_{12})^2(k+p_{123})^2(k+p_{1234})^2(k+p_{12345})^2}$

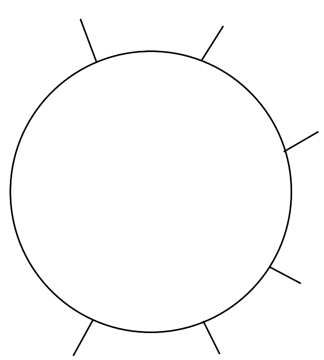
- only 4 momenta **span** the external space $p_5 = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + \beta_4 p_4$

- then, the integrand becomes $\mathcal{I}_{1,6} = \dfrac{1}{k^2(k+p_1)^2(k+p_{12})^2(k+p_{123})^2(k+p_{1234})^2(k+\sum_{i=1}^4 z_i p_i)^2}$

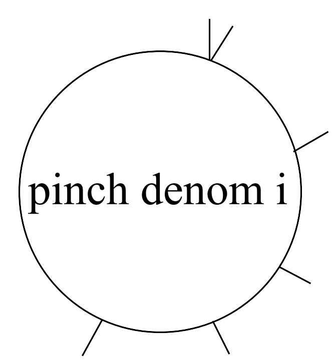
- thus, only 5 linearly independent propagators at 1 loop

$$(k+\sum_{i=1}^{4} z_i p_i)^2 = \alpha_0 + \alpha_1 k^2 + \alpha_2 (k+p_1)^2 + \alpha_3 (k+p_{12})^2 + \alpha_4 (k+p_{123})^2 + \alpha_5 (k+p_{1234})^2$$

- therefore, can partial fraction



$$= -\sum_{i=1}^{6} \frac{\alpha_i}{\alpha_0}$$



2-loop point-by-point

- independent external momenta $\{p_i\}$: (n-1) in Conventional Dimensional Regularization scheme (CDR)
  at most 4 in 't Hooft-Veltman scheme (tHV)

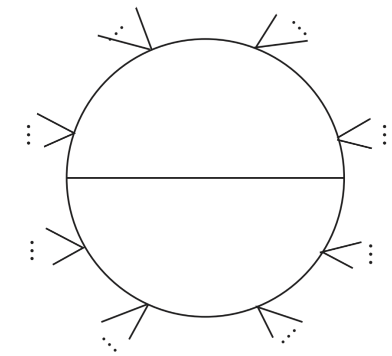
- scalar products $\{k_i \cdot k_j\,,\, k_i \cdot p_j\}$ involving loop momenta $\{k_1, k_2\}$: **at most 11** in tHV

- generalized propagators : denominators & Irreducible Scalar Products (ISPs)

kinematics	denominators	CDR ISPs	**tHV ISPs**	generalized CDR propagators	**generalized tHV propagators**
2-loop 4-point	7	2	2	9	9
2-loop 5-point	8	3	3	11	**11**
2-loop 6-point	9	4	2	13	**11**
2-loop 7-point	10	5	1	15	**11**
2-loop 8-point	**11**	6	**0**	17	**11**
2-loop 9-point	12	7	**0**	19	**11**

only 11 independent $\Rightarrow$ **partial fraction**

Improving the IBP efficiency beyond 5-point



- explicit example : consider the 2-loop 8-point topology
- it has 17 CDR generalized propagators

$\{k_1-k_2, k_1, k_2, k_1+p_1, k_1+p_{12}, k_1+p_{123}, k_1+p_{1234}, k_2+p_{1234}, k_2+p_{12345}, k_2+p_{123456}, k_2+p_{1234567}, k_1-p_5, k_1-p_6, k_1-p_7, k_2-p_1, k_2-p_2, k_2-p_3\}$

- apply tHV momentum decomposition $p_{j>4} = \sum_{i=1}^{4} p_i \frac{\det(\{p_1,\ldots,\hat{p}_i,p_j,\ldots,p_4\}\cdot\{p_1,\ldots,p_4\})}{\det(\{p_1,\ldots,p_4\}\cdot\{p_1,\ldots,p_4\})}$
- then, only **11** independent **tHV** generalized propagators

$\{k_1-k_2, k_1, k_2, k_1+p_1, k_1+p_{12}, k_1+p_{123}, k_1+p_{1234}, k_2+p_{1234}, k_2+\sum_{j=1}^{4} p_j\, z_{9,j}, k_2+\sum_{j=1}^{4} p_j\, z_{10,j}, k_2+\sum_{j=1}^{4} p_j\, z_{11,j}\}$

	CDR	tHV	R_t
D2	31.4m/3126MI/13.9G	6.8m/2368MI/1.88G	4.6
D3	51.5m/3302MI/18.19G	10.1m/2368MI/2.9G	5.1
N3	115.2m/4497MI/25.7G	5.7m/2358MI/2.59G	20.2
N4	321.3m/6742MI/56.4G	7.6m/2368MI/4.3G	42.3
N5	908.9m/9779MI/137G	12.5m/2368MI/7.35G	72.7
N6	-	20.1m/2368MI/10.34G	-

- 1-off-shell all-internally-massless
- no cut propagators
- one numerical probe, one finite field
- with FIRE6 [Smirnov, Chukharev 1901.07808]

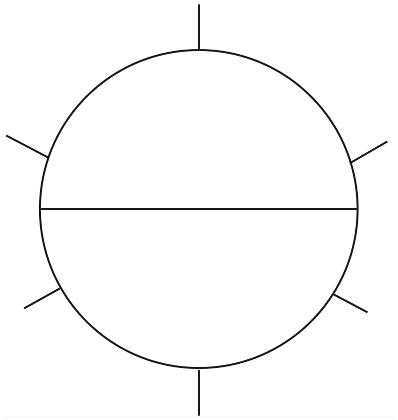
Improving the numerical evaluation efficiency

Auxiliary Mass Flow (AMFlow)

[Liu, Ma 2201.11669]

- numerical evaluator of Feynman integrals
- provides very high precision
- requires multiple internal IBP reductions

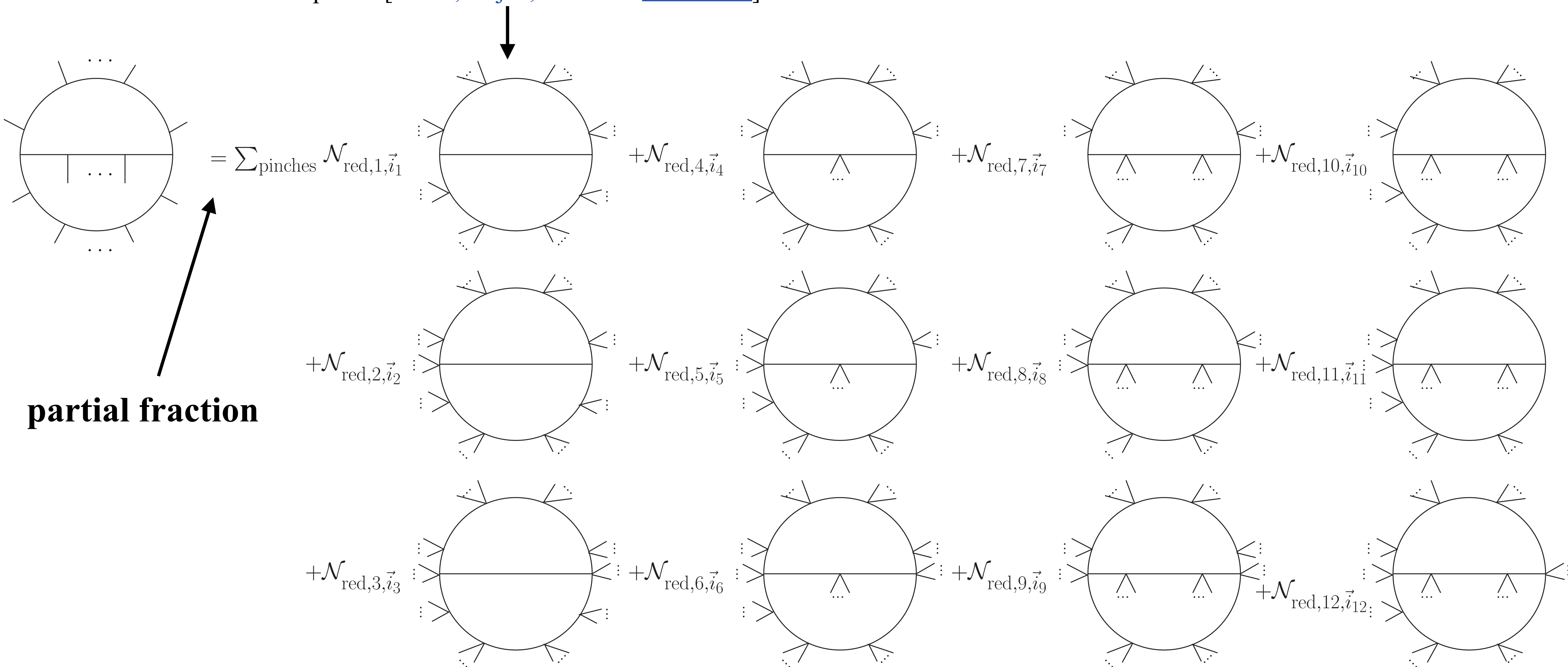
tHV for IBP



- decompose external legs in 4 dimensions
- beyond 5-point, IBP performance improved
- explicit example : 2-loop 6-point 2-off-shell all-internally-massless
- AMFlow setup : no cut, 1-digit numerical sample, 20-digit precision, 5 leading ϵ^n orders, 16 cores, Kira backend [Klappert, Lange, Maierhöfer, Usovitsch 2008.06494]
- tHV : **30h**
- CDR : **didn't finish** in 1 week

Finite basis topologies at 2 loops

planar [Gluza, Kajda, Kosower 1009.0472]

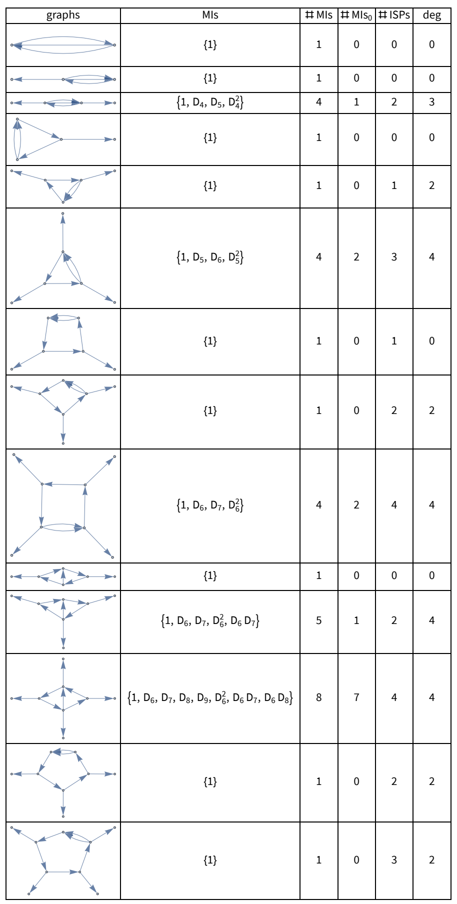
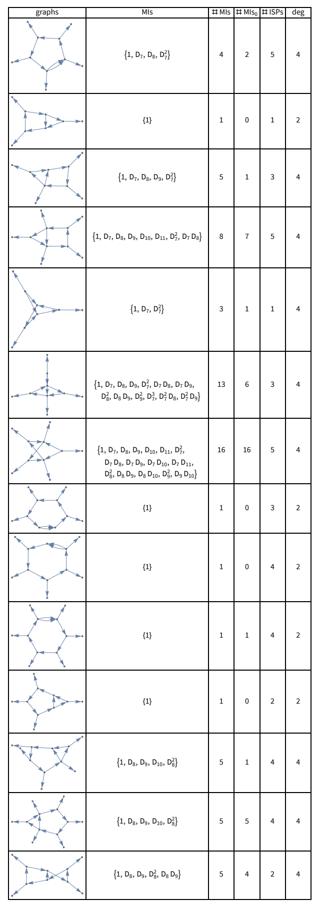
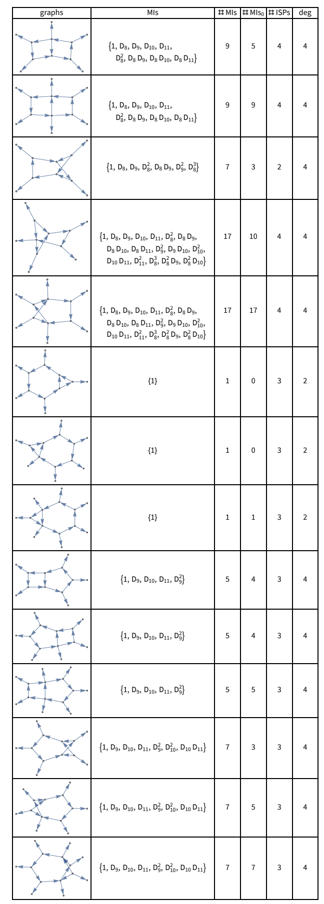
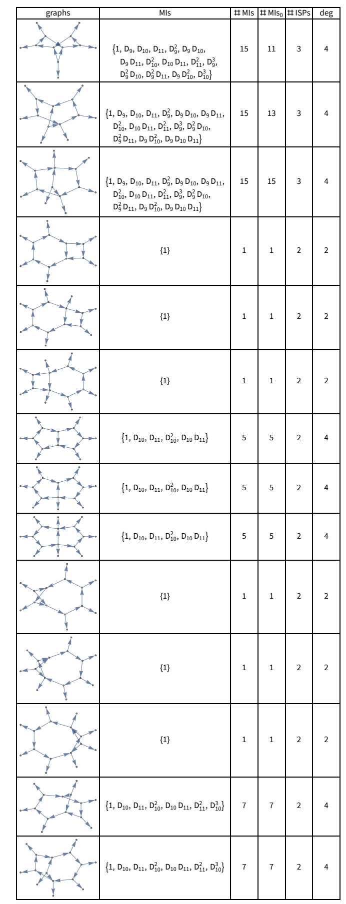
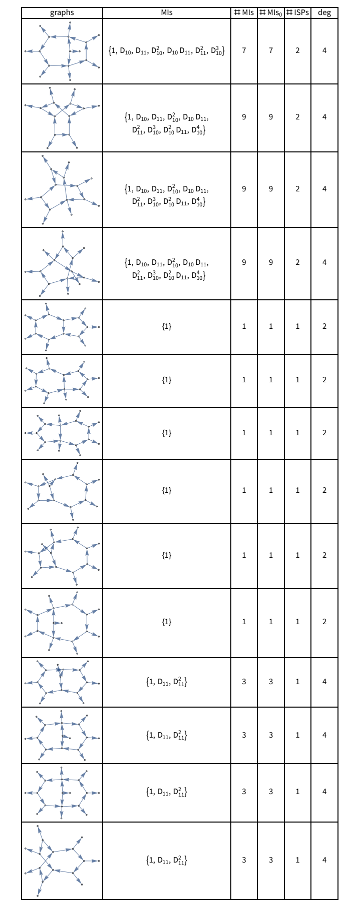
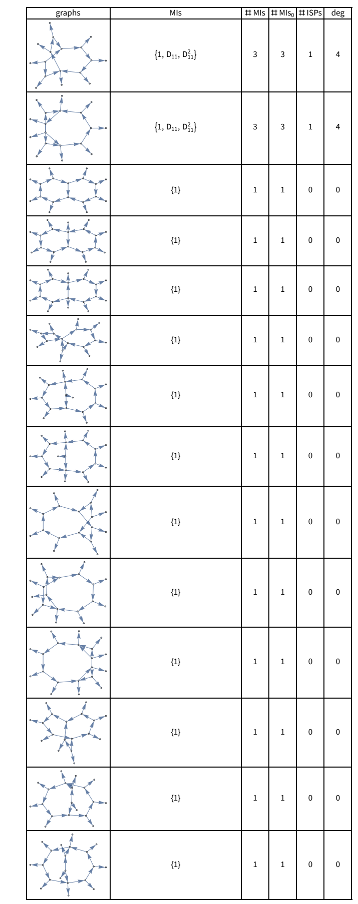


lower dimensions : less denominators e.g. 7 in d=2-2ϵ

higher loops : 12p@3L, 17p@4L, ... (loop-by-loop approach analogous but much more topologies)

Functionally distinct finite basis

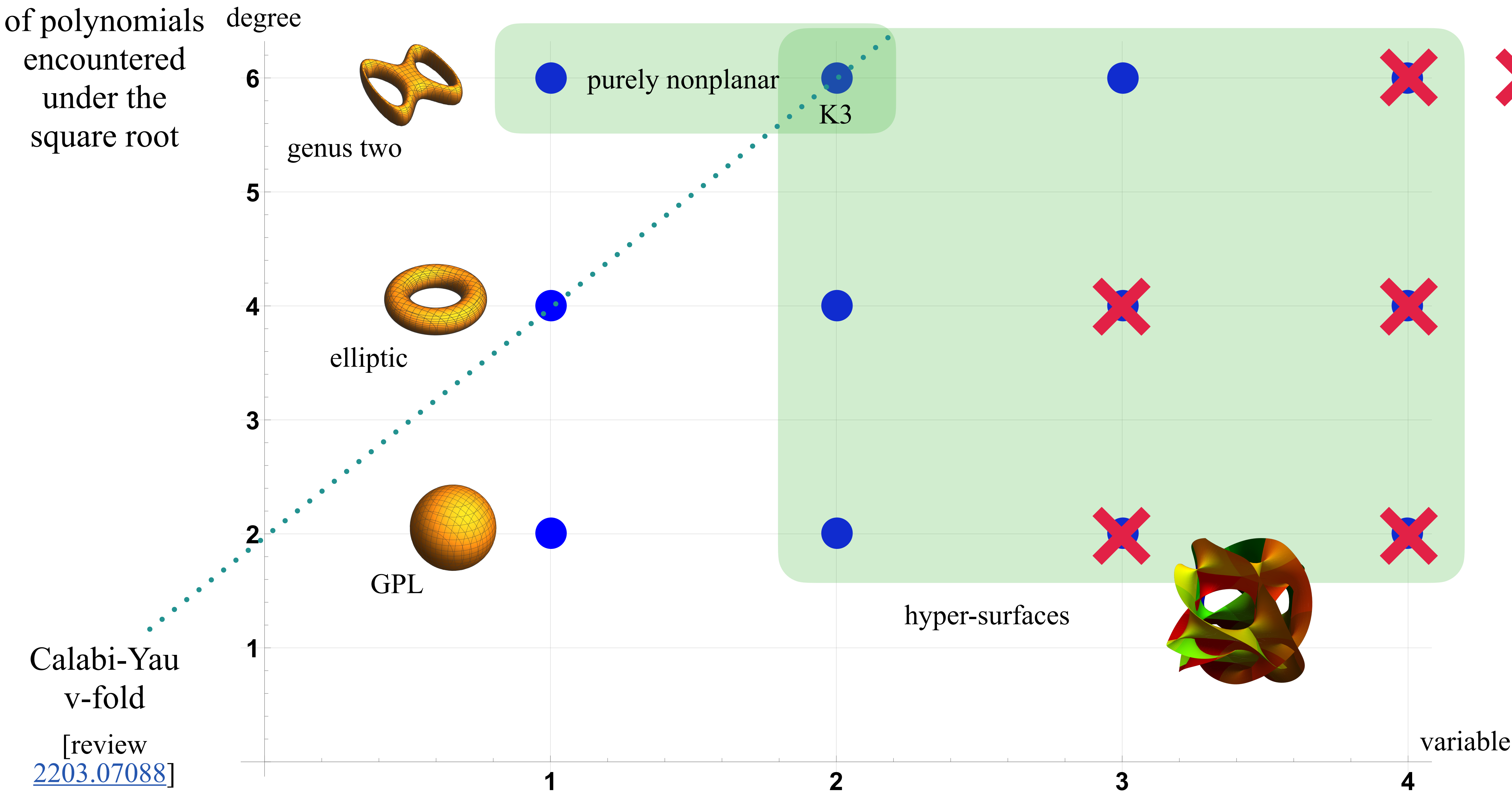
84 graphically distinct **subsectors**
of the 12 finite basis topologies

347 functionally distinct **Master** Integrals
(274 for massless internal lines)

Spanning the space of special functions at 2 loops

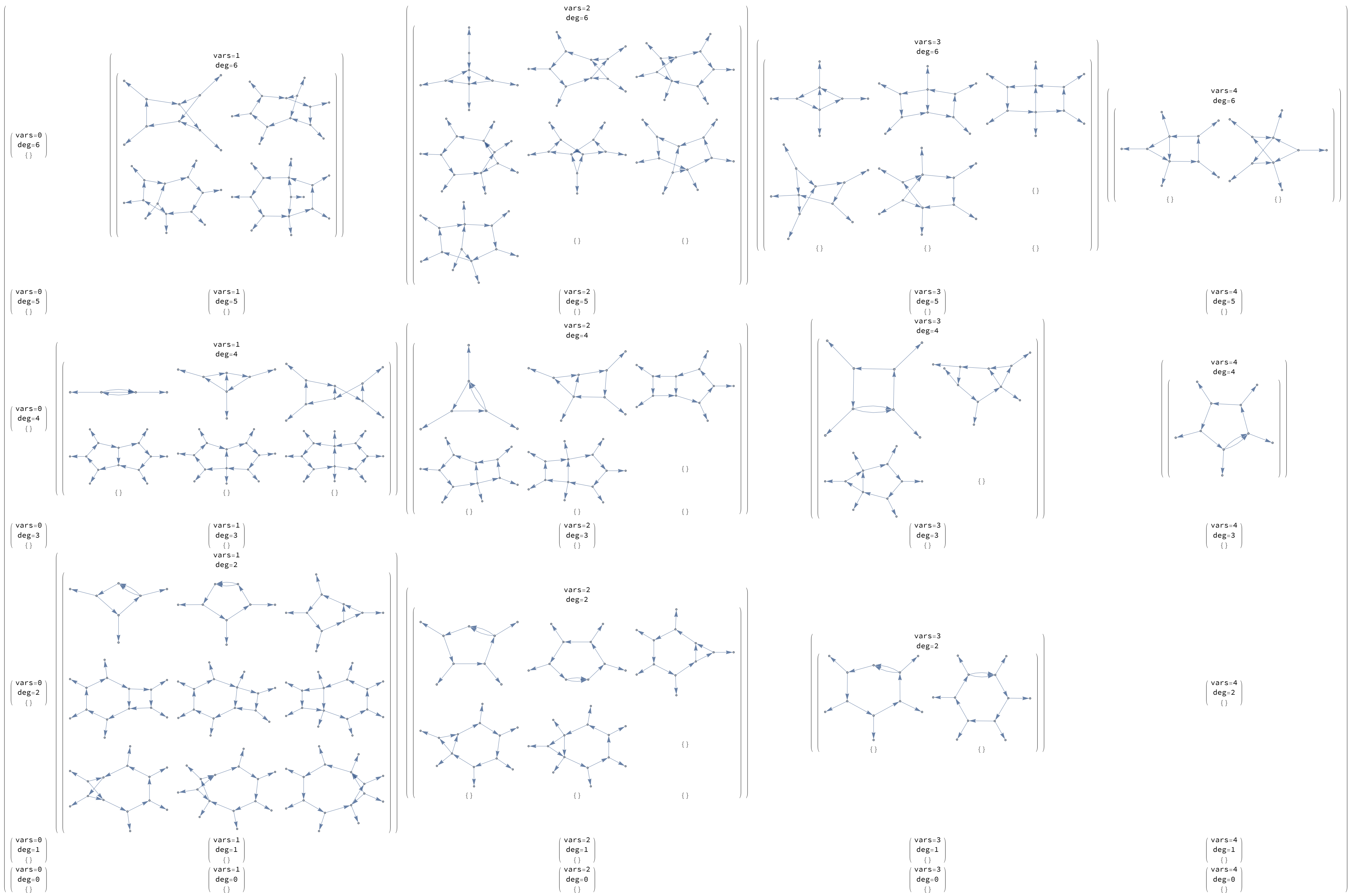
integrate out 1 Baikov variable on the maximal cut of each sector



ongoing :
✕ = vanish in Loop-By-Loop Baikov representation
[Frellesvig, Papadopoulos 1701.07356]

there are also 3 sectors with iterated roots

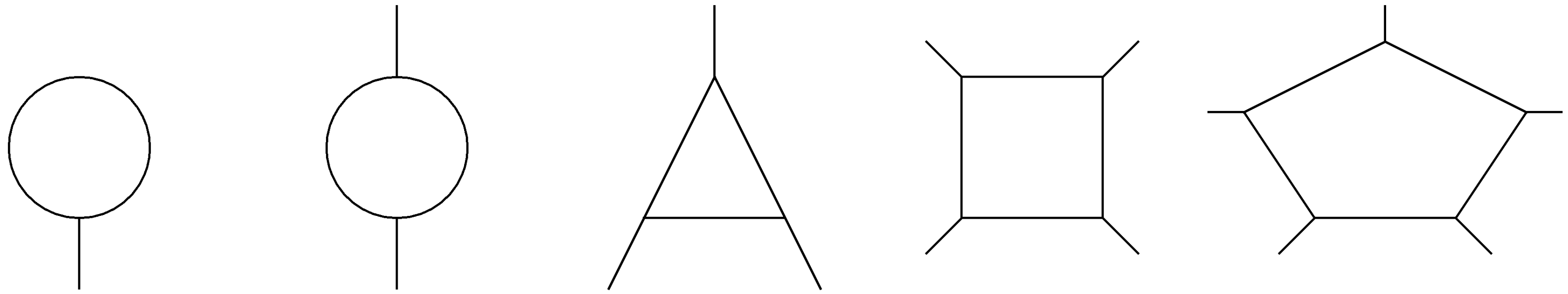
Spanning the space of special functions at 2 loops



which of them contribute to the amplitude **finite part** ?

1-loop evanescent sector

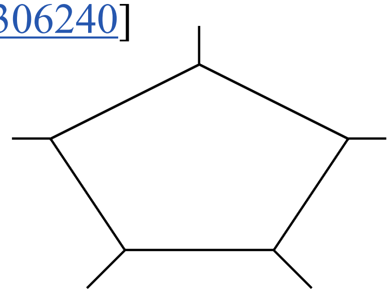
- functionally distinct basis : tadpole, bubble, triangle, box, pentagon



- note : only 4 independent propagators in d=4
- **pentagon** : the one Master Integral in its top sector can be chosen to be **evanescent** $\sim \mathcal{O}(\epsilon)$

[Bern, Dixon, Kosower 9306240]

- indeed : dimension-shift relation [Baikov, Tarasov 1996]

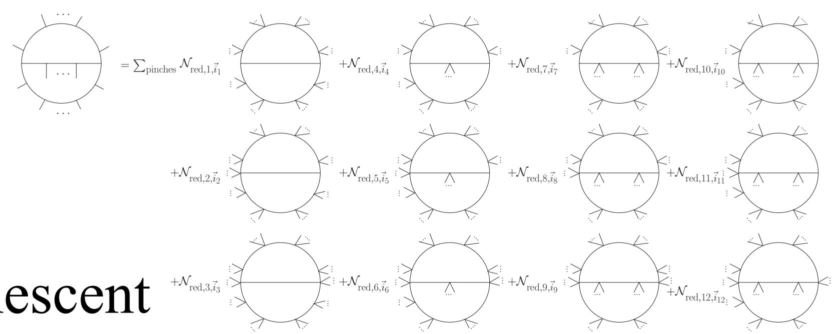


$$I_5^{(4D,\text{sing})} = \epsilon\, a^{(\text{fin})}\, I_5^{(6D,\text{fin})} + \text{subsectors}^{(\text{sing})}$$

- amplitude finite part : no contribution from the Master Integral in the pentagon top sector

$$\mathcal{A} = a_i\, M_{\leq 4\text{den},i} + \mathcal{O}(\epsilon)$$

2-loop evanescent sectors



- note : only 8 independent propagators in d=4
- consider : the 12 finite basis topologies
- will show : all of their subsectors with more than 8 denominators are evanescent
- IBP setup : drop all subsectors with 8 or less denominators, fix a generic example numerical probe for a mixed on/off-shell all-internally-massless kinematics

recently proven for 2-loop 6-point planar massless [Abreu, Monni, Page, Usovitsch 2412.19884]

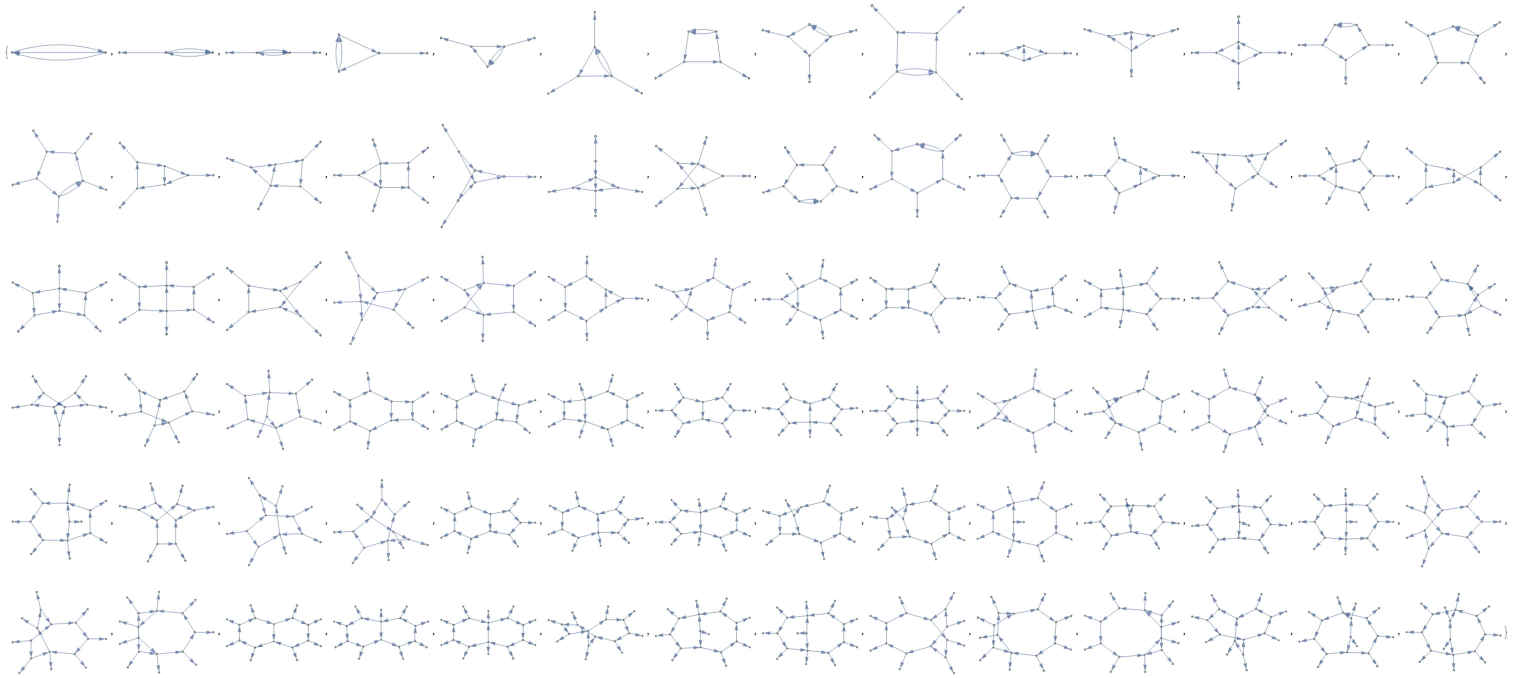
- Master Integrals : choose them to be finite
  IR : dimensionally shift Laporta Masters into 6 dimensions
  UV : linearize all the numerators in Laporta Masters
- IBP reduction : evanescence shown for all integrals allowed in a renormalizable theory

$$I^{(4D,\text{sing})} = \epsilon\, a_i^{(\text{fin})} M_i^{(6D,\text{fin})} + \text{subsectors}^{(\text{sing})}$$

- **amplitude** finite part : no contribution from Master Integrals with more than **8 denominators**

$$\mathcal{A} = a_i\, M_{\leq 8\text{den},i} + \mathcal{O}(\epsilon)$$

Towards computing all 2-loop basis integrals



to sum up

- 374 functionally distinct Master Integrals
- 242 contributing to the finite amplitude part
- 169 if only massless internal lines

however

- for practical ϵ-expansion, combinatorial growth for all degenerate values of kinematics
- do we **really** need to compute them in dimensional regularization ?
- a.k.a. the locally finite basis problem

Exploiting the singular structure of amplitudes

[**PB** 2504.xxxxx]

Making amplitudes locally finite

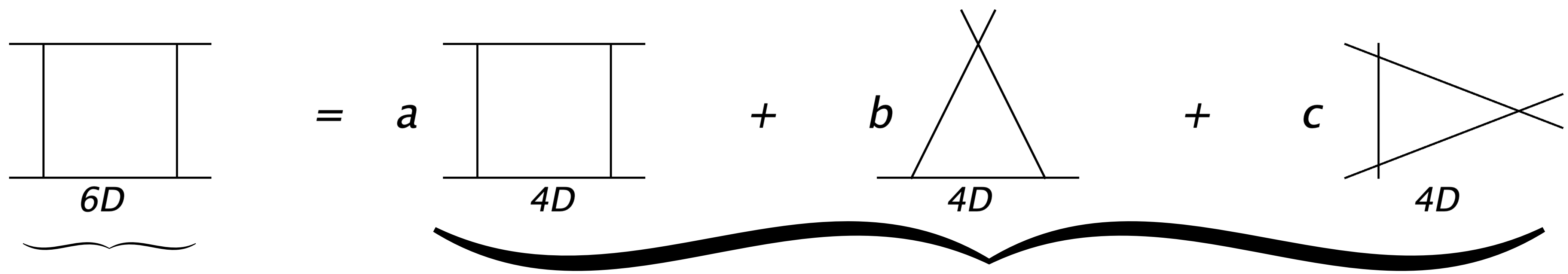
numerical approaches

- real-virtual matching for loop-induced amplitudes [Anastasiou, Karlen, Sterman, Venkata 2403.13712]
- Loop Tree Duality for local numerical evaluation [Kermanschah, Vicini 2407.18051]

analytic approaches

- locally finite Masters for bare amplitudes [Manteuffel, Panzer, Schabinger 1510.06758]
- locally finite integrals for bare amplitudes [Gambuti, Kosower, Novichkov, Tancredi 2311.16907]
- locally finite Masters for **amplitude finite part** [**PB** 2504.xxxxx]

1-loop example



locally finite

globally finite

$\mathcal{O}(\epsilon^0)$ $\mathcal{O}(\epsilon^{-2})$ $\mathcal{O}(\epsilon^{-2})$ $\mathcal{O}(\epsilon^{-2})$

can put d=6 at the integrand level
without any regulators
and the numerical evaluation will converge

cannot put d=4 at the integrand level
but need an ϵ regulator
because the singularities
only happen to cancel when combined together

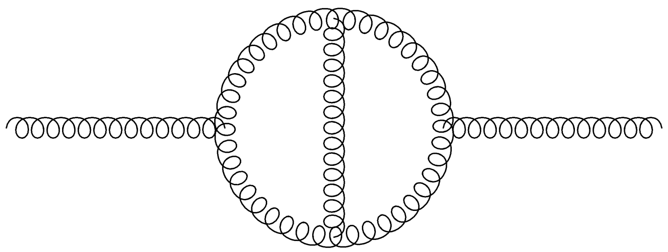
Amplitude singularities are universal

$$\mathscr{A}_{\text{bare}} = \sum \text{FeynDiags} = \mathscr{A}^{(\text{sing})} + \mathscr{A}^{(\text{fin})} + \mathcal{O}(\epsilon)$$

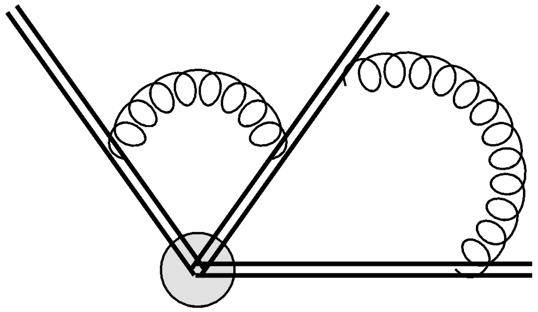
[Catani, Becher, Neubert]

predicted by

- UV : high-energy



- IR : low-energy soft and collinear



singularities **cancel** against real radiation
=> only care about the **finite** part of the amplitude

Amplitude's globally finite part

$$\mathscr{A}^{(\text{fin})} = \mathscr{A}_{\text{bare}} - \mathscr{A}^{(\text{sing})} = \mathcal{O}(\epsilon^0)$$

- bare amplitude arises from Feynman diagrams, so can IBP reduce it to Masters M_i

$$\mathscr{A}_{\text{bare}} = b_i M_i$$

- now, want to do the same for the finite part
- to this end, find a set of integrals I_i with ϵ-independent coefficients a_i such that they match all of the predicted amplitude poles in ϵ

$$\mathscr{A}^{(\text{sing})} = a_i I_i$$

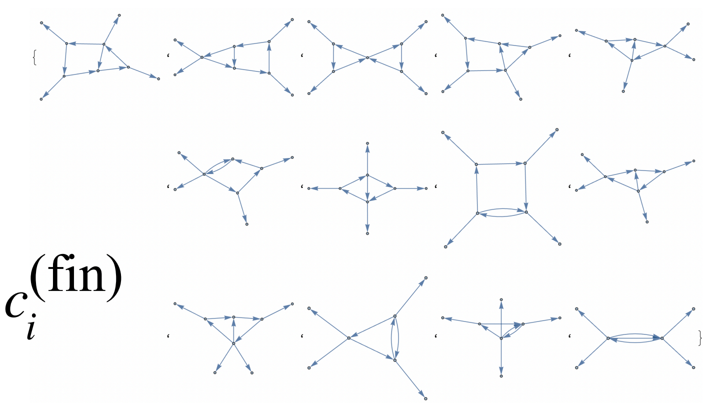
- then, the whole finite part can also be IBP reduced to a Master basis

$$\mathscr{A}^{(\text{fin})} = c_i M_i$$

- since the coefficients c_i and Masters M_i can be singular term by term, the whole expression is only **globally** finite (note that the obtained finite part depends on the renormalization and regularization scheme choice)

Amplitude's locally finite part

$$\mathscr{A}^{(\text{fin})} = c_i M_i$$



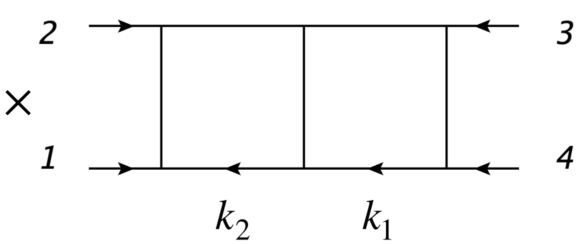
- now, want the globally finite part of the amplitude to become also locally finite
- to this end, find a Master basis of locally finite integrals $M_i^{(\text{fin})}$ with finite coefficients $c_i^{(\text{fin})}$
- for example, focus on 2-loop QCD leading color contribution to the color factor $(T^{a_3}T^{a_4})_{i_1\bar{i}_2}$ of

$$q_1\bar{q}_2 \to g_3 g_4$$

- consider the following Master basis (and the corresponding subtraction terms)

[Gambuti, Kosower, Novichkov, Tancredi 2311.16907]

$$\vec{F} = \left\{ G\begin{pmatrix} k_2\, p_1\, p_2 \\ k_1\, p_3\, p_4 \end{pmatrix}, G\begin{pmatrix} k_1\, p_1\, p_2\, p_3 \\ k_2\, p_1\, p_2\, p_3 \end{pmatrix}, G(k_1\, p_3\, p_4)\, G\begin{pmatrix} k_2\, p_1\, p_2 \\ p_1\, p_2\, p_4 \end{pmatrix}, G(k_1\, p_3\, p_4)\,(k_2-p_1)^2, (k_2-p_1)^2\,(k_1+p_4)^2 \right\} \times$$



$$\vec{M}^{(\text{fin})} = \{F_1, F_2, k_2\cdot p_1\, F_2, F_3, k_1\cdot p_1\, F_3, k_2\cdot p_1\, F_3, F_4, F_5\} \cup \{F_1, F_2, k_2\cdot p_1\, F_2, F_3, k_1\cdot p_1\, F_3\}\,|_{x1234}$$

- in this basis, for gluon helicity states :

  $(-,-)$ and $(+,+)$: $c_i \sim \mathcal{O}(\epsilon^0)$ => $M_i^{(\text{fin})}$ form a **finite** basis => can numerically put d=4

  $(-,+)$ and $(+,-)$: $c_i \sim \mathcal{O}(\epsilon^{-1})$ => $M_i^{(\text{fin})}$ form a **quasi-finite** basis => must expand $M_i^{(\text{fin})}$ to $\mathcal{O}(\epsilon^1)$

Futher questions

$$\mathscr{A}^{(\text{fin})} = c_i^{(\text{fin})} M_i^{(\text{fin})}$$

- does locally finite Master basis
  always **exists** ?   YES   [Chetyrkin, Faisst, Sturm, Tentyukov 0601165]

- how to **choose** locally finite Master basis and subtraction integrals
  s.t. coefficients finite ?

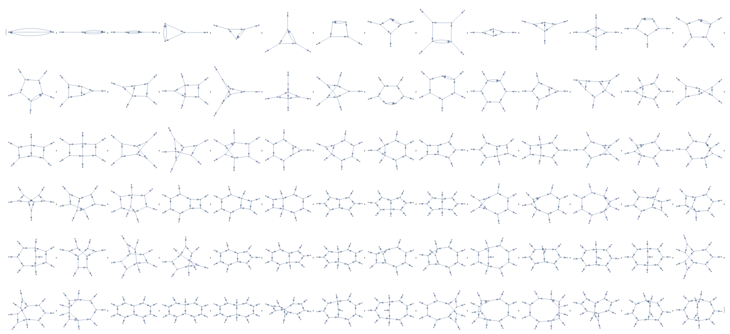
- how to choose locally finite Master basis and subtraction integrals
  s.t. virtual correction **minimized** so real corrections dominate ?

- is it possible to only use **integrand**-level identities
  s.t. subtraction term matching avoided ?

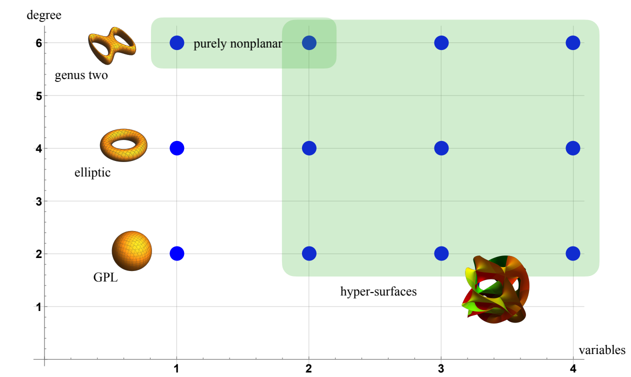
Conclusions

Conclusions

can **constrain** the analytic structure of amplitudes

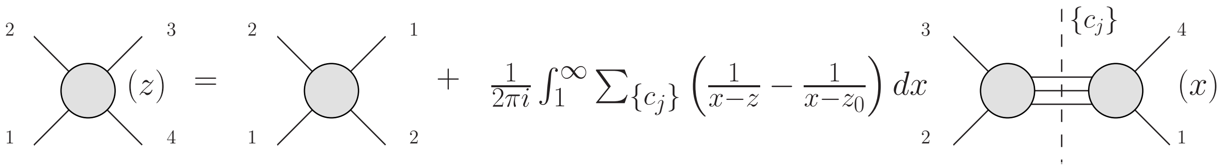


- tHV decomposition induces integrand reduction
- special **functions** for all 2-loop Masters analyzed
- IBP **efficiency** and numerical evaluation improved

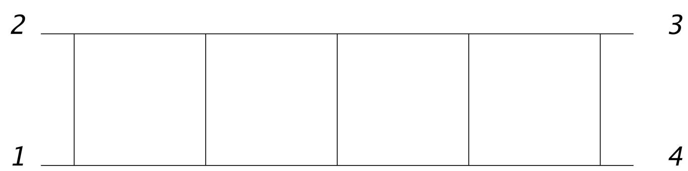


can **exploit** the analytic and singular structure of amplitudes

- dispersion relations in **Integrated Unitarity**



- can make an example 2-loop amplitude analytically **locally-finite**



$$\mathscr{A}^{(2,\mathrm{fin})}(q_1^- \bar{q}_2^+ g_3^- g_4^-) = c_i^{(\mathrm{fin})}\, M_i^{(\mathrm{fin})}$$

THANK YOU

Appendix

Cuts

- integral definition :

$$I_{\{n_i\}} = \int \left(\prod_{l=1}^{L} D^d k_l \right) \prod_{i=1}^{N} \mathscr{D}_i^{-n_i} \qquad D^d k_l = e^{\epsilon \gamma_E} \frac{d^d k_l}{i \pi^{d/2}}$$

- cut propagator :

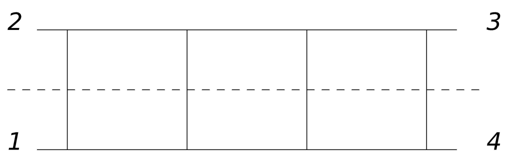
$$\frac{1}{\mathscr{D} + i\varepsilon} \to 2\pi i\, \delta^+(\mathscr{D}) \qquad \delta^+(q^2) = \delta(q^2)\, \theta(q_0)$$

- cut integral :

$$\text{Cut}_u I_{\{n_i\}} = \sum_{\{c_j\} \in \mathscr{C}_u} I_{\{n_i\};\{c_j\}} = \sum_{\{c_j\} \in \mathscr{C}_u} \int \left(\prod_{l=1}^{L} D^d k_l \right) \left(\prod_{i \notin \{c_j\}} \mathscr{D}_i^{-n_i} \right) \prod_{m \in \{c_j\}} \delta_{1,n_m} 2\pi i\, \delta^+(\mathscr{D}_m)$$

- less subsectors :

$$/2^C$$



- Integration-By-Parts identities (IBPs) :
  [Laporta 0102033]

$$\int \left(\prod_{l=1}^{L} D^d k_l \right) \frac{\partial}{\partial k_l^\mu} \left(q^\mu \prod_{i=1}^{N} \mathscr{D}_i^{-n_i} \right) = 0$$

[Chetyrkin, Tkachov 1981]

$$p_{j\,\mu}\, p_{l\,\nu} \left(p_n^\nu \frac{\partial}{\partial p_{n,\mu}} - p_n^\mu \frac{\partial}{\partial p_{n,\nu}} \right) I_{\{n_i\}} = 0$$

[Gehrmann, Remiddi 9912329]

- Differential Equations (DEQ) : $\partial_{x_n} M_i(\vec{x}, \epsilon) = A_{ij}(\vec{x}, \epsilon)\, M_j(\vec{x}, \epsilon)$ $\xrightarrow{\text{canonical}}$ $\partial_{x_n} M_i^c(\vec{x}, \epsilon) = \epsilon\, A_{ij}^c(\vec{x})\, M_j^c(\vec{x}, \epsilon)$

  [Kotikov 1991] [Henn 1304.1806]

  Master Integrals (MIs)

$\to$ can solve cut integrals with DEQ

Discontinuities

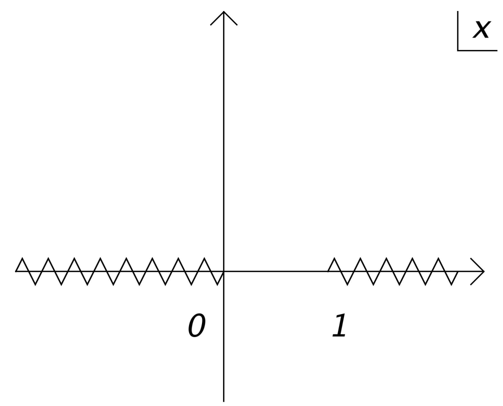
- discontinuity :

$$\mathrm{Disc}_0 f(x) = f(x+i\epsilon) - f(x-i\epsilon)$$

- from now on, focus on specific kinematics :

$$x = -\frac{t}{s}$$

4-point massless



- branch cuts :

$t>0$	$u>0$	$s>0$
$x<0$	$x>1$	pushed to ∞

- Harmonic Polylogarithms (HPLs) :
  [Remiddi, Vermaseren 9905237]

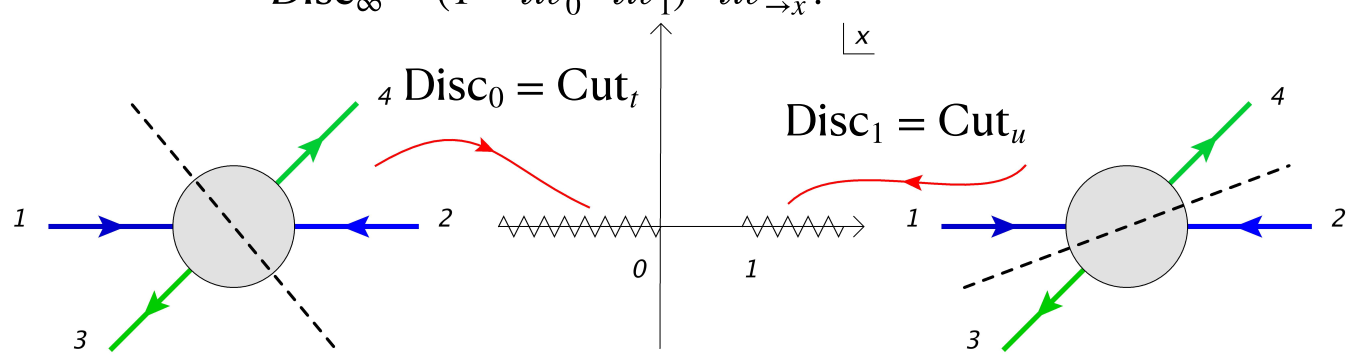
$$G(\alpha_n,\ldots,\alpha_1;x) = \int_0^x \frac{dz}{z-\alpha_n} G(\alpha_{n-1},\ldots,\alpha_1;z)$$

$$\alpha_k \in \{0,1\}$$

- discontinuities of HPLs algorithmic from monodromy matrices :
  [Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu 2007.13747]

$$\mathrm{Disc}_0 = (1-\mathcal{M}_0)\cdot\mathcal{M}_{\to x},$$
$$\mathrm{Disc}_1 = -(1-\mathcal{M}_1)\cdot\mathcal{M}_{\to x},$$
$$\mathrm{Disc}_\infty = (1-\mathcal{M}_0\cdot\mathcal{M}_1)\cdot\mathcal{M}_{\to x}.$$

- unitarity :



$(\mathrm{Disc}_\infty = \mathrm{Cut}_s)$

Monodromies

following :

[Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu 2007.13747]

- Harmonic Polylog : $G(\alpha_n, \ldots, \alpha_1; x)$ $\alpha_k \in \{0,1\}$
- vector of derivatives : $$\mathscr{V}_i = \begin{cases} 1 & \text{if } i = 0\,, \\ (-1)^{\#\text{ of }1}\, G(\alpha_{n+1-i}, \ldots, \alpha_n, x) & \text{if } n \geq i > 0 \end{cases}$$
- connection matrix : $$\omega_{ij} = \frac{dx}{x - \alpha_{n-i}} \delta_{i+1,j} \qquad \text{s.t.} \qquad d\mathscr{V} = \mathscr{V} \cdot \omega$$
- variation matrix : $$\mathscr{M}_\gamma = \mathscr{P}\, e^{\int_\gamma \omega}$$ collects all $n+1$ solutions for $\mathscr{V}$
- general solution : $$\left(\mathscr{M}_{\to x}\right)_{ij} = \sum_{k=0}^{n} (-1)^{\#\text{ of }1}\, G(\alpha_{n-i}, \ldots, \alpha_{n-i-k+1}, x)\, \delta_{i+k,j}$$ $$G(x) = 1$$
- monodromy matrices : $$\mathscr{M}_0 = \mathscr{M}_{\circlearrowleft_0}\,,$$ $$\mathscr{M}_1 = \mathscr{M}_{\to 1} \mathscr{M}_{\circlearrowleft_1} \mathscr{M}_{\to 1}^{-1}\,.$$
- discontinuities : $$\text{Disc}_0 = (1 - \mathscr{M}_0) \cdot \mathscr{M}_{\to x}\,,$$ $$\text{Disc}_1 = -\,(1 - \mathscr{M}_1) \cdot \mathscr{M}_{\to x}\,,$$ $$\text{Disc}_\infty = (1 - \mathscr{M}_0 \cdot \mathscr{M}_1) \cdot \mathscr{M}_{\to x}\,.$$

Ansatz matching

- example ansatz : $c_{1,1}\,G(1,1;x) + c_{1,0}\,G(1,0;x) + c_{0,1}\,G(0,1;x) + c_{0,0}\,G(0,0;x) + c_1\,G(1;x) + c_0\,G(0;x) + c$
- impose e.g. $\mathrm{Disc}_0 = 0$: $c_{1,1}\,G(1,1;x) + 0 + c_{0,1}\,G(0,1;x) + 0 + c_1\,G(1;x) + 0 + c$
- $\mathrm{Disc}_1 = 2\pi i\,(2\,G(1;x) + 3\,G(0;x) + 5\,\pi i)$ $= c_{1,1}\,(2\pi i\,(-G(1;x) + i\pi)) + c_{0,1}\,(-2\pi i\,G(0,x)) + c_1\,(-2\pi i) + 0$
- now only constant unconstrained : $-2\,G(1,1;x) - 3\,G(0,1;x) - 7\,i\pi\,G(1;x) + c$
- impose fixed value e.g. ζ_2 at x=0 : $-2\,G(1,1;x) - 3\,G(0,1;x) - 7\,i\pi\,G(1;x) + \zeta_2$

Explicit reduction coefficients

$$\mathcal{I}_{L,n} = \prod_{i=1}^{D(n,L)} \frac{1}{\mathcal{D}_i} = \sum_{\substack{i_1,\ldots,i_A=1 \\ i_j \neq i_m}}^{D(n,L)} \frac{c_{i_1,\cdots,i_A}}{\mathcal{D}_1 \cdots \hat{\mathcal{D}}_{i_1} \hat{\mathcal{D}}_{i_2} \cdots \hat{\mathcal{D}}_{iA} \cdots \mathcal{D}_{D(n,L)}}$$

with

$$c_{i_1,\cdots,i_A} = \frac{(-B_{i_1,\cdots,i_A})^A}{B_{0,\cdots,i_A} B_{i_1,0,\cdots,i_A} \cdots B_{i_1,\cdots,0}}$$

$$A = D(n,L) - D(N(L,d_0),L)$$

$$B_{i_1,\cdots,i_A} = \begin{vmatrix} \alpha_{D(N(L,d_0),L)+1,i_1} & \cdots & \alpha_{D(N(L,d_0),L)+1,i_A} \\ \alpha_{D(N(L,d_0),L)+2,i_1} & \cdots & \alpha_{D(N(L,d_0),L)+2,i_A} \\ \vdots & \vdots & \vdots \\ \alpha_{D(n,L),i_1} & \cdots & \alpha_{D(n,L),i_A} \end{vmatrix}$$

$$\mathcal{D}_{i>D(N(L,d_0),L)} = \alpha_{i,0} + \sum_{j=1}^{D(N(L,d_0),L)} \alpha_{i,j} \mathcal{D}_j$$

$$\alpha_{i,i} = -1$$

$$\alpha_{i,j>D(N(L,d_0),L)} = 0$$

partial fractions implemented in Mathematica package **Apart** [Feng 1204.2314]

Leading singularity

no ISPs ⇒ **maximal cut** localizes [Bosma, Sogaard, Zhang 1704.04255]

true for finite basis topologies t

e.g. at 2 loops

$$I^{(t)}_{\text{max}} = \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \prod_{i=1}^{11} 2\pi i\, \delta(\mathcal{D}_i^{(t)}) = c(d)\, \frac{G^{(5-d)/2}}{B_t^{(7-d)/2}}$$

with

$$c(d) = \frac{(2\pi i)^{11}}{4^d \pi^{9/2} \Gamma((d-4)/2)\Gamma((d-5)/2)}$$

$$G = \det(\{p_1, \ldots, p_4\} \cdot \{p_1, \ldots, p_4\})$$

$$B_t = \det(\{k_1, k_2, p_1, \ldots, p_4\} \cdot \{k_1, k_2, p_1, \ldots, p_4\})\,|_{\mathcal{D}_i^{(t)}=0,\ i=1,\ldots,11}$$

leading singularity for higher-loop finite basis topologies

$$\mathcal{S}_{t,L} = \frac{\sqrt{G}}{\left(\sqrt{B_{t,L}}\right)^{L+1}}$$