Statistics in Data Analysis

All you ever wanted to know about statistics but never dared to ask

part 2

Pawel Brückman de Renstrom
(pawel.bruckman@ifj.edu.pl)

March 12, 2025

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Questions from the previous lecture

1. Using the Kolmogorov axioms, show that:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

.

2. What is the standard deviation of the sample mean $\bar{x}$, i.e. calculate $Var\langle \bar{x} \rangle \equiv \langle (\bar{x} - \mu)^2 \rangle$.
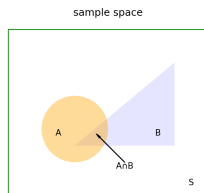   (Hint: On the way, you'll need to prove that $\langle x_i x_j \rangle_{i \neq j} = \mu^2$.)

Solutions to be sent to me before the next lecture

Solutions

Show that: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Using the Kolmogorov axioms, show that:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

It is enough to note that:

$$A \cup B = A \cup (\bar{A} \cap B) \quad \text{and} \quad B = (A \cap B) \cup (\bar{A} \cap B)$$

and use the 2nd Kolmogorov's axiom about probability of disjoint subsets twice:

$$P(A \cup B) = P(A \cup (\bar{A} \cap B) = P(A) + P(\bar{A} \cap B)$$

$$P(B) = P((A \cap B) \cup (\bar{A} \cap B)) = P(A \cap B) + P(\bar{A} \cap B)$$

From where we get:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \therefore$$

Solutions

What is the standard deviation of the sample mean $\bar{x}$, i.e. calculate $Var(\bar{x}) \equiv \langle (\bar{x} - \mu)^2 \rangle$.

$$
\begin{aligned}
Var(\bar{x}) \equiv \sigma_{\bar{x}}^2 &\equiv \langle (\bar{x} - \mu)^2 \rangle \\
&= \langle (\frac{1}{n} \sum_i x_i - \mu)^2 \rangle \\
&= \frac{1}{n^2} \sum_i \langle x_i^2 \rangle + \frac{1}{n^2} \sum_{i \neq j} \langle x_i x_j \rangle - 2\mu \langle \bar{x} \rangle + \mu^2 \\
&= \frac{1}{n^2} n \langle x^2 \rangle + \frac{n(n-1)}{n^2} \langle x_i x_j \rangle_{i \neq j} - 2\mu \langle \bar{x} \rangle + \mu^2 \\
&= \frac{\langle x^2 \rangle}{n} + \frac{n-1}{n} \mu^2 - \mu^2 = \frac{\langle x^2 \rangle - \mu^2}{n} = \frac{\sigma^2}{n} \qquad \therefore
\end{aligned}
$$

$$
\langle x_i x_j \rangle_{i \neq j} = \int \int x_i x_j g(x_i, x_j) \mathrm{d}x_i \mathrm{d}x_j = \int \int x_i x_j f(x_i) f(x_j) \mathrm{d}x_i \mathrm{d}x_j
$$

$$
= \left(\int x f(x) \mathrm{d}x \right)^2 = \mu^2 \qquad \therefore \qquad \text{also:} \ \ \langle x^2 \rangle = \sigma^2 + \mu^2
$$

Expectation value for the variance estimators s^2 and S^2

$$E[s^2] = \frac{1}{n-1} \sum_i E[(x_i - \bar{x})^2] = \frac{1}{n-1} \sum_i E[x_i^2 - 2x_i\bar{x} + \bar{x}^2] =$$

$$= \frac{1}{n-1} \sum_i \left(E[x_i^2] - \frac{2}{n} E\left[x_i \sum_j x_j\right] + \frac{1}{n^2} E\left[\sum_k x_k \sum_j x_j\right] \right) =$$

$$= \frac{1}{n-1} \sum_i \left(E[x_i^2] - \frac{2}{n} \sum_j E[x_i x_j] + \frac{1}{n^2} \sum_{k,j} E[x_k x_j] \right) =$$

$$=^* \frac{1}{n-1} \sum_i \left(\mu^2 + \sigma^2 - \frac{2}{n}(\mu^2 + \sigma^2 + (n-1)\mu^2) + \frac{1}{n^2} \left[(n^2-n)\mu^2 + n(\mu^2 + \sigma^2) \right] \right) =$$

$$= \frac{1}{n-1} \sum_i \left(0 \times \mu^2 + \frac{n-1}{n}\sigma^2 \right) = \frac{1}{n-1} n \frac{n-1}{n} \sigma^2 = \sigma^2, \qquad \therefore \tag{1}$$

$$E[S^2] = \frac{1}{n} \sum_i E[(x_i - \mu)^2] = \frac{1}{n} \sum_i E[x_i^2 - 2x_i\mu + \mu^2] =^* \frac{1}{n} \sum_i \left(\mu^2 + \sigma^2 - 2\mu^2 + \mu^2 \right) =$$

$$= \frac{1}{n} n\sigma^2 = \sigma^2, \qquad \therefore \qquad \text{* by virtue of identities used on the previous slide.}$$

covariance & correlation

Let $a(\mathbf{x})$ and $b(\mathbf{x})$ be two functions of random variables $\mathbf{x} = (x_1, x_2, ..., x_n)$.

covariance matrix

$$
\begin{aligned}
V_{ab} = \mathrm{cov}[a, b] &= E[(a - \mu_a)(b - \mu_b)] = \\
&= E[ab] - E[a\mu_b] - E[\mu_a b] + E[\mu_a \mu_b] = \\
&= E[ab] - \mu_a \mu_b - \mu_a \mu_b + \mu_a \mu_b = E[ab] - \mu_a \mu_b = \\
&= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} a(\mathbf{x}) b(\mathbf{x}) f(\mathbf{x}) dx_1 ... dx_n - \mu_a \mu_b
\end{aligned}
\tag{3}
$$

Note: $E\left[E[a(\mathbf{x})]\right] = E[a(\mathbf{x})]$ as $\int_S f(\mathbf{x}) d\mathbf{x} \equiv 1$.

variance & correlation coefficient

$$
V_{aa} = \mathrm{cov}[a, a] = \sigma_a^2 \qquad\qquad \rho_{a,b} = \frac{V_{ab}}{\sigma_a \sigma_b}.
\tag{4}
$$

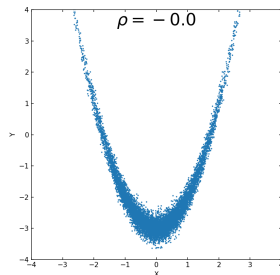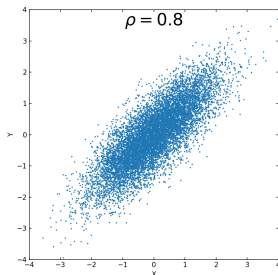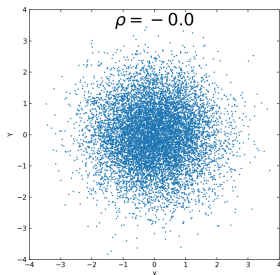Note that $-1 \leq \rho_{a,b} \leq 1$.

covariance & correlation

word of caution

For independent variables x and y the joint p.d.f. satisfies $f(x,y) = g(x)h(y)$ and hence:

$$E[xy] = E[x]E[y] = \mu_x \mu_y$$

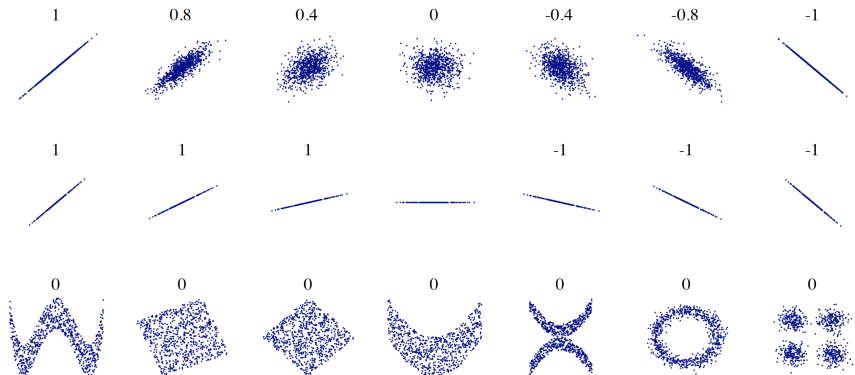From the definition of covariance we get $V_{x,y} \equiv 0$.

The inverse cannot be inferred, though! I.e. $V_{x,y} = 0$ does not imply independence of the variables!
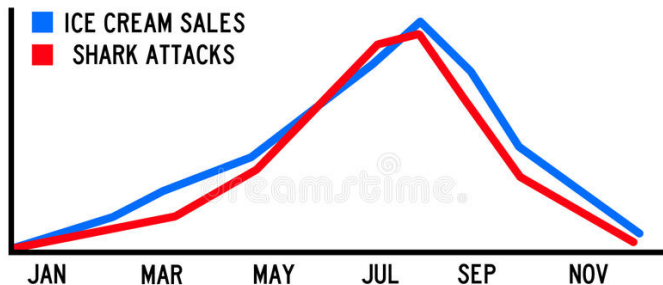
word of caution

correlated and uncorrelated variables (2D), examples

Correlation vs causation

The *hidden variable*

CORRELATION IS NOT CAUSATION!

ICE CREAM SALES
SHARK ATTACKS

JAN MAR MAY JUL SEP NOV

Both ice cream sales and shark attacks increase when the weather is hot and sunny, but they are not caused by each other (they are caused by good weather, with lots of people at the beach, both eating ice cream and having a swim in the sea)

Derived random variable
mean of derived random variable

Let $y(\mathbf{x})$ be a function of n random variables $\mathbf{x} = (x_1, x_2, ..., x_n)$.
We know how to rigorously determine the p.d.f. of y. However, if the exact form of $f(\mathbf{x})$ is unknown and we only know the means and variances, we can approximate these properties for y:

$$y(\mathbf{x}) \approx y(\mu) + \sum_{i=1}^{n} \left[\frac{\partial y}{\partial x_i}\right]_{\mathbf{x}=\mu} (x_i - \mu_i)$$

mean value

$$E[y(\mathbf{x})] \approx E[y(\mu)] = y(\mu), \tag{5}$$

as $E[x_i - \mu_i] \equiv 0$.

Error propagation
variance of derived random variable

$$E[y^2(\mathbf{x})] \approx y^2(\mu) + 2y(\mu)\sum_{i=1}^{n}\left[\frac{\partial y}{\partial x_i}\right]_{\mathbf{x}=\mu}\overbrace{E[x_i-\mu_i]}^{0} + E\left[\left(\sum_{i=1}^{n}\frac{\partial y}{\partial x_i}_{\mathbf{x}=\mu}(x_i-\mu_i)\right)^2\right] =$$

$$= y^2(\mu) + \sum_{i,j=1}^{n}\left[\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j}\right]_{\mathbf{x}=\mu}V_{ij}. \tag{6}$$

(co)variance

$$\sigma_y^2 = E[y^2] - (E[y])^2 \approx \sum_{i,j=1}^{n}\left[\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j}\right]_{\mathbf{x}=\mu}V_{ij}. \tag{7}$$
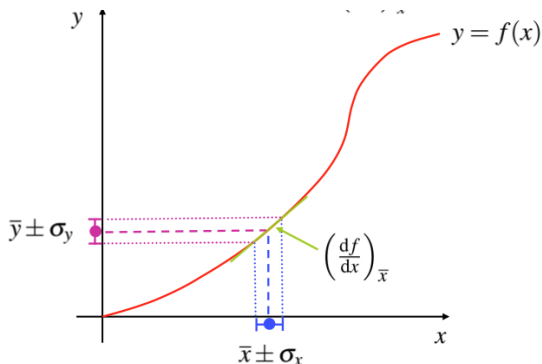
and analogously:

$$U_{kl} = \text{cov}[y_k, y_l] \approx \sum_{i,j=1}^{n}\left[\frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j}\right]_{\mathbf{x}=\mu}V_{ij}, \quad \text{in short} \quad U = AVA^T. \tag{8}$$

Error propagation

A simple 1D illustration

In the simplest case $y = f(x)$, it is easy to see the origin of $\sigma_y = \left(\dfrac{\mathrm{d}f}{\mathrm{d}x}\right)_{\bar{x}} \sigma_x$:
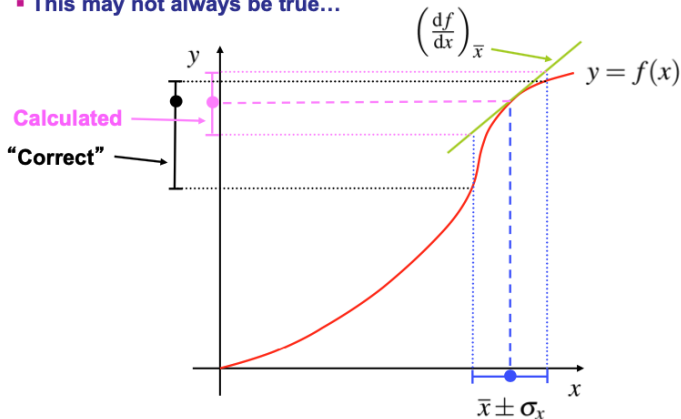
Error propagation

Watch out for traps...

In the previous example validity of the *linear expansion* was assumed, i.e. we considered higher order terms in the Taylor expansion to be negligible.

- **This may not always be true...**

Error propagation

Watch out for traps...

- We measure the transverse momentum of a track (p_T) from the fitted track curvature which is inversely proportional to the radius of curvature (R) of the track in the solenoidal magnetic field:

$$R = 0.3B(\text{T})p_T(\text{GeV})$$

- We obtain a symmetric (Gaussian) uncertainty on $1/R$. Now we calculate the error on p_T. For simplicity, let us take $p_T = 1/x$, and we know σ_x:

$$\frac{\mathrm{d}p_T}{\mathrm{d}x} = -\frac{1}{x^2} = -p_T^2 \quad \text{hence} \quad \sigma_{p_T} = p_T^2 \sigma_x$$

- Take the measured x to be 0.01 ± 0.005 GeV^{-1}.
- We obtain: $p_T = 100 \pm 50$ GeV.
- The real variation corresponding to the uncertainty on x is: $p_T = 100 + 100 - 33$ GeV.
- The two results are very different (the latter being correct).

De-correlation

unitary rotation in the $\mathbf{x}_n$ space

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $V_{ij} = \text{cov}[x_i, x_j]$ their (symmetric) covariance matrix. One can always find a linear transformation of $\mathbf{x}$ that diagonalizes the covariance:

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \qquad \text{cov}[y_i, y_j] = \text{cov}\left[\sum_{k=1}^{n} A_{ik} x_k \sum_{l=1}^{n} A_{jl} x_l\right] = AVA^T = U, \quad (9)$$

which is a special case of error propagation (exact, thanks to linear nature of the transformation!). The problem boils down to diagonalising the the matrix V, i.e. finding eigenvectors $\mathbf{r^i}$ and their corresponding eigenvalues λ_i satysfying the eigenequation:

$$V\mathbf{r^i} = \lambda_i \mathbf{r^i} \tag{10}$$

(note that: $\quad \lambda_i \mathbf{r^i}^T \mathbf{r^j} = \mathbf{r^i}^T V \mathbf{r^j} = \lambda_j \mathbf{r^i}^T \mathbf{r^j} \stackrel{\lambda_i \neq \lambda_j}{\Longrightarrow} \mathbf{r^i}^T \mathbf{r^j} = \delta_{ij} \quad$ if $\mathbf{r^i}$ normalised).

$$A \equiv \begin{pmatrix} \boxed{\quad \mathbf{r^1} \quad} \\ \boxed{\quad \mathbf{r^2} \quad} \\ ..... \\ ..... \\ \boxed{\quad \mathbf{r^n} \quad} \end{pmatrix} \qquad U = AVA^T = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & ... & \\ 0 & & ... & \\ & & & \lambda_n \end{pmatrix} \tag{11}$$

De-correlation

example: rotation in the 2D space

$$V = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{12}$$

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{13}$$

$$\theta = \frac{1}{2}\arctan\left(\frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}\right) \tag{14}$$

Verify this result!
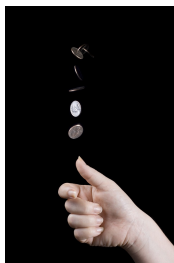
NOTE: Decorrelation will not
necessarily make the variables
independent!

Tossing a coin
Binomial distribution

- Tossing a coin can yield two distinct results (usually with equal probability).

- What is the probability of scoring n heads in N trials?

$$P(n(N,p)) = \overbrace{\frac{N!}{n!(N-n)!}}^{\text{number of sequences}} \underbrace{p^n(1-p)^{N-n}}_{P \text{ of a particular sequence}} \, .$$

(15)

- The expectation value:

$$E[n(N,p)] = \sum_{n=0}^{N} n\frac{N!}{n!(N-n)!}p^n(1-p)^{N-n} = Np,$$

(16)

which agrees with our intuition, e.g. for a fair coin ($p = 0.5$) we expect heads and tail in 50/50 proportion.

Think of N independent trials, each with expectation value $E[1(1,p)] = p$.

Binomial distribution
Is it a proper p.d.f., i.e. normalised?

1. To start with, recall the binomial theorem:

$$(a + b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)^* a^k b^{n-k} \tag{17}$$

2. Now we can use the above to show that the binomial distribution is normalised:

$$\sum_{n=0}^{N} P(n(N,p)) = \sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) p^n (1-p)^{N-n} = (p+q)^N = 1^N = 1 \qquad \therefore \tag{18}$$

$$* \qquad \frac{n!}{k!(n-k)!} \equiv \left(\begin{array}{c} n \\ k \end{array} \right)$$

Binomial distribution

$\langle n \rangle$ - rigorous calculation

$$\langle n \rangle = \sum_{n=0}^{N} n P(n(N,p)) = \sum_{n=0}^{N} n \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$$= Np \sum_{n=1}^{N} \frac{(N-1)!}{(n-1)!(N-n)!} p^{n-1} (1-p)^{N-n}$$

$$= Np \sum_{n=0}^{N-1} \frac{(N-1)!}{n!(N-1-n)!} p^n (1-p)^{N-1-n} \qquad (19)$$

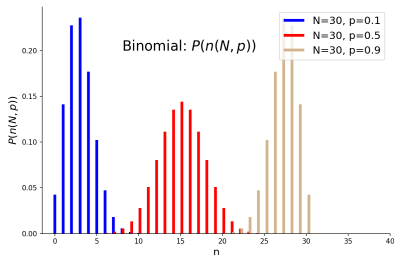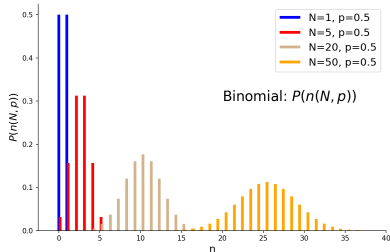$$= Np \underbrace{\sum_{n=0}^{N-1} P(n(N-1,p))}_{\text{normalised}} = Np \qquad \therefore$$

Binomial distribution

- The variance is:

$$Var[n(N,p)] = E[n^2(N,p)] - (E[n(N,p)])^2 = Np(1-p), \qquad (20)$$

which for a fair coin yields $1/4$ of the number of trials N.
This can be rigorously calculated, but can be thought of in terms of error propagation: $Var[1(1,p)] = p(1-p)$.

Binomial distribution

$Var[n(N, p)]$ - rigorous calculation

$$Var[n(N, p)] = \langle (n - \langle n \rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2 \tag{21}$$

$$\langle n^2 \rangle = \sum_{n=0}^{N} n^2 P(n(N, p)) = \sum_{n=0}^{N} n^2 \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$$= Np \sum_{n=1}^{N} n \frac{(N-1)!}{(n-1)!(N-n)!} p^{n-1} (1-p)^{N-n}$$

$$= Np \sum_{n=0}^{N-1} (n+1) \frac{(N-1)!}{n!(N-1-n)!} p^n (1-p)^{N-1-n}$$

$$= Np \left[\sum_{n=0}^{N-1} n \frac{(N-1)!}{n!(N-1-n)!} p^n (1-p)^{N-1-n} + \sum_{n=0}^{N-1} \frac{(N-1)!}{n!(N-1-n)!} p^n (1-p)^{N-1-n} \right]$$

$$= Np \left[(N-1)p + 1 \right] = Np(Np - p + 1) \tag{22}$$

$$Var[n(N, p)] = \langle n^2 \rangle - \langle n \rangle^2 = Np(Np - p + 1) - (Np)^2 = Np(1-p) \qquad \therefore \tag{23}$$

Binomial distribution
Simple example

Suppose you are assessing efficiency of a certain process (vaxine effectiveness, event selection, what have you...) and you observe n out of N passing the test. What is the efficiency and its uncertainty?
This is a binomial process (fixed number of trials).
The best estimate of the efficiency is:

$$\varepsilon = \frac{n}{N} \qquad (\langle\varepsilon\rangle = \frac{\langle n\rangle}{N} = \frac{Np}{N} = p)$$

How about the straightforward estimation of the variance:

$$\sigma^2 = \frac{\varepsilon(1-\varepsilon)}{N}$$

$$\langle\sigma^2\rangle = \frac{\langle n\rangle}{N^2} - \frac{\langle n^2\rangle}{N^3} = \frac{Np - p(Np - p + 1)}{N^2} = \frac{N+1}{N^2}p(1-p) = \frac{N+1}{N}Var(\varepsilon)$$

Multinomial distribution

generalization of binomial

- Let us extend the process to $m > 2$ outcomes, e.g. rolling a dice.
- The only requirement is to have $\sum_{i=1}^{m} p_i = 1$.
- Probability distribution of a given sequence is given by:

$$P(n_1...n_m(N, p_1...p_m)) = \frac{N!}{n_1!...n_m!} p_1^{n_1}...p_m^{n_m}. \tag{24}$$

Can you derive the above?

- One can calculate covariance from the joint probability distribution to get:

$$V_{ij} = E[(n_i - E[n_i])(n_j - E[n_j])] = -Np_i p_j \tag{25}$$

Note that for binomial $\rho_{1,2} = \frac{-Np(1-p)}{\sqrt{Np(1-p)}\sqrt{N(1-p)p}} = -1$ (dice: $\rho_{k,l} = ?$)

Multinomial distribution
generalization of binomial

- Let us extend the process to $m > 2$ outcomes, e.g. rolling a dice.
- The only requirement is to have $\sum_{i=1}^{m} p_i = 1$.
- Probability distribution of a given sequence is given by:

$$P(n_1...n_m(N, p_1...p_m)) = \frac{N!}{n_1!...n_m!} p_1^{n_1}...p_m^{n_m}. \tag{26}$$

Can you derive the above?
- One can calculate covariance from the joint probability distribution to get:

$$V_{ij} = E[(n_i - E[n_i])(n_j - E[n_j])] = -Np_i p_j \tag{27}$$

Note that for binomial $\rho_{1,2} = \frac{-Np(1-p)}{\sqrt{Np(1-p)}\sqrt{N(1-p)p}} = -1$ (dice: $\rho_{k,l} = -0.2$)

Multinomial distribution
generalization of binomial

- Let us extend the process to $m > 2$ outcomes, e.g. rolling a dice.
- The only requirement is to have $\sum_{i=1}^{m} p_i = 1$.

- Probability distribution of a single outcome is simply:

$$P(n_i(N, p_i)) = \frac{N!}{n_i!(N - n_i)!} p_i^{n_i}(1 - p_i)^{N - n_i}, \tag{28}$$

yielding $E[n_i] = Np_i$ and $V[n_i] = Np_i(1 - p_i)$.

Counting experiment

Do counting of a random process (e.g. number of cars passing by
the IFJ main entrance in 10'). We want to know the probability
distribution to find a certain number of occurences.

- A binomial limit when
  $$N \rightarrow \infty, \quad p = \varepsilon \rightarrow 0, \quad N\varepsilon = \mu = \text{const.}$$

-
$$P(n) = \left(\begin{array}{c} N \\ n \end{array} \right) \varepsilon^n (1-\varepsilon)^{N-n}$$
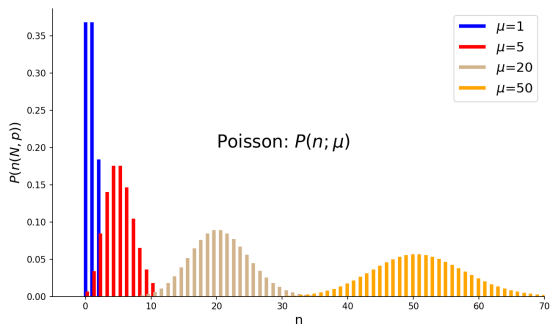
$$P(n; \mu) = \frac{N!}{(N-n)!n!} \left(\frac{\mu}{N} \right)^n \left(1 - \frac{\mu}{N} \right)^{N-n} = \frac{\mu^n}{n!} \frac{N!}{(N-n)!} \frac{(N-\mu)^{N-n}}{N^N}$$

$$\stackrel{N \rightarrow \infty}{=} \frac{\mu^n}{n!} \left(\frac{N-\mu}{N} \right)^N = \frac{\mu^n}{n!} \left(1 - \frac{\mu}{N} \right)^N \stackrel{*}{=} \frac{\mu^n}{n!} e^{-\mu} \qquad \therefore \tag{29}$$

- Counting random process is described by the **Poisson** distribution:

$$\mathbf{P(n; \mu)} = \frac{\mu^{\mathbf{n}}}{\mathbf{n!}} \mathbf{e}^{-\mu} \tag{30}$$

$*$ $\left| \ln \left[(1 - \frac{\lambda}{x})^x = e^{-\lambda} \right] = -x \ln \left(1 - \frac{\lambda}{x} \right) \simeq \frac{\lambda}{1 - \frac{\lambda}{x}} \stackrel{x \rightarrow \infty}{\rightarrow} \lambda \Rightarrow \lim_{x \rightarrow \infty} \left(1 - \frac{\lambda}{x} \right)^x = e^{-\lambda} \right|$

Poisson distribution

Poisson: $P(n; \mu)$

For large values of μ Poisson distribution asymptotically tends to a Gaussian*
$$G(\mu, \sigma^2 = \mu)$$
* See the Central Limit Theorem later in this lecture.

$$E[n] = \sum_{n=0}^{\infty} n \frac{\mu^n}{n!} e^{-\mu} = \sum_{n=1}^{\infty} n \frac{\mu^n}{n!} e^{-\mu} = \mu \sum_{n=1}^{\infty} \frac{\mu^{n-1}}{(n-1)!} e^{-\mu} = \mu \sum_{k=0}^{\infty} \frac{\mu^k}{(k)!} e^{-\mu} = \mu \quad (31)$$

$$V[n] = E[n^2] - (E[n])^2 = E[n(n-1) + n] - (E[n])^2 = \mu^2 + \mu - \mu^2 = \mu \quad (32)$$

Hence, the well known $\sigma(N) = \sqrt{N}$ for event counting.

Raindrops
Uniform distribution

- Some processes have uniform probability over a limited range of parameter (raindrops on a window sill). Usually these are selected fiducial region of a wider distributed random process.

- Characterised by a continuous uniform p.d.f.

  - Must have finite range in order to allow normalisation.

$$f(x; \alpha, \beta) = \begin{cases} \dfrac{1}{\beta - \alpha} \text{ for } \alpha < x < \beta \\ 0 \text{ otherwise} \end{cases} \tag{33}$$

  - Mean and the variance are easily obtained:

$$E[x] = \int_\alpha^\beta \frac{x}{\beta - \alpha} \, dx = \tfrac{1}{2}(\alpha + \beta), \qquad V[x] = \int_\alpha^\beta [x - \tfrac{1}{2}(\alpha + \beta)]^2 \frac{1}{\beta - \alpha} \, dx = \tfrac{1}{12}(\beta - \alpha)^2. \tag{34}$$

- $f(x; 0, 1)$ (or simply $[0, 1]$) is commonly used in statistics, notably for base random number generators.

- For any continuous p.d.f. $f(x)$, $y = F(x)$ is distributed according to $[0, 1]$. (?) Hence, p.d.f. of $x = F^{-1}(y)$ will be $f(x)$ if y has a uniform distribution $[0, 1]$.

Questions

1. Suppose two independent measurements of the same quantity gave the following results:

$$x_1 \pm \sigma_1 \quad \text{and} \quad x_2 \pm \sigma_2$$

Take the weighted mean to be $\bar{x} = wx_1 + (1-w)x_2$. Find the w which minimizes the error on the mean, hence provide expressions for the weighted mean $\bar{x}$ and its variance $\sigma_{\bar{x}}^2$.

Solutions to be sent to me before the next lecture

Thank you

Back-up