Hybrid high-energy factorization and evolution at NLO

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based on	`arxiv:2501.02619`
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Motivation

Many papers by my collaborator and me presenting calculations employing (hybrid) high-energy-(k_T)-factorization, **at LO QCD**.

[1] A. van Hameren, H. Kakkad, P. Kotko, K. Kutak, and S. Sapeta, *Searching for saturation in forward dijet production at the LHC*, *Eur. Phys. J. C* **83** (2023), no. 10 947, [2306.17513].

[2] I. Ganguli, A. van Hameren, P. Kotko, and K. Kutak, *Forward γ+jet production in proton-proton and proton-lead collisions at LHC within the FoCal calorimeter acceptance*, *Eur. Phys. J. C* **83** (2023), no. 9 868, [2306.04706].

[3] M. A. Al-Mashad, A. van Hameren, H. Kakkad, P. Kotko, K. Kutak, P. van Mechelen, and S. Sapeta, *Dijet azimuthal correlations in p-p and p-Pb collisions at forward LHC calorimeters*, *JHEP* **12** (2022) 131, [2210.06613].

[4] A. van Hameren, P. Kotko, K. Kutak, S. Sapeta, and E. Żarów, *Probing gluon number density with electron-dijet correlations at EIC*, *Eur. Phys. J. C* **81** (2021), no. 8 741, [2106.13964].

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[10] A. van Hameren, P. Kotko, K. Kutak, and S. Sapeta, *Broadening and saturation effects in dijet azimuthal correlations in p-p and p-Pb collisions at $\sqrt{s} = 5.02$ TeV*, *Phys. Lett. B* **795** (2019) 511–515, [1903.01361].

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Presenting results at leading order in perturbation theory implies the promise that the accuracy can be improved by calculating higher orders.

In collinear factorizaiton, calculations have been completely automized to NLO, with automation to NNLO in sight.

I consider it my obligation to achieve NLO precision in k_T-factorization.

Motivation (Why k_T-factorization?)

High-energy factorization (or k_T-factorization)

$$d\sigma^{LO} = \int_0^1 dx \int \frac{d^2k_\perp}{\pi} F(x,k_\perp) \int_0^1 d\bar{x} \int \frac{d^2\bar{k}_\perp}{\pi} F(\bar{x},\bar{k}_\perp)\, d\hat{\sigma}^{LO}_{gg}(x,k_\perp,\bar{x},\bar{k}_\perp)$$

- originally for heavy quark production (Collins, Ellis 1991, Catani Ciafaloni, Hautmann 1991 1994)
- unintegrated PDF (UPDF) $F(x,k_\perp)$ resums $\ln(1/x)$
- partonic cross section depends explicitly on $k_\perp$ (and $\bar{k}_\perp$) $\Rightarrow$ not trivial to define, requires initial-state "offshell" or "spacelike" gluons, or "reggeons"
- partonic cross section contains certain higher-twist corrections
- used for studying gluon saturation, particular the "hybrid" form for ITMD factorization (Kotko, Kutak, Marquet, Petreska, Sapeta, AvH 2015, Altinoluk, Boussarie, Kotko 2019)

$$d\sigma^{LO} = \sum_{\bar{\imath}} \sum_{j} \int_0^1 dx \int \frac{d^2k_\perp}{\pi} F^{(j)}(x,k_\perp) \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, d\hat{\sigma}^{LO,(j)}_{g\bar{\imath}}(x,k_\perp,\bar{x})$$

- having non-zero $k_\perp$ in the initial state has advantages when exact kinematics is required

Collinear factorization at NLO

general Sudakov decomposition: $K^\mu = x_K P^\mu + \bar{x}_K \bar{P}^\mu + K_\perp^\mu$

positive-rapidity initial state: $k_i^\mu = x P^\mu$

negative-rapidity initial-state: $k_{\bar{\imath}}^\mu = \bar{x}\bar{P}^\mu$

$$d\sigma^{\mathrm{LO}} = \sum_{i,\bar{\imath}} \int dx\, d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x})\, dB_{i\bar{\imath}}(x, \bar{x})$$

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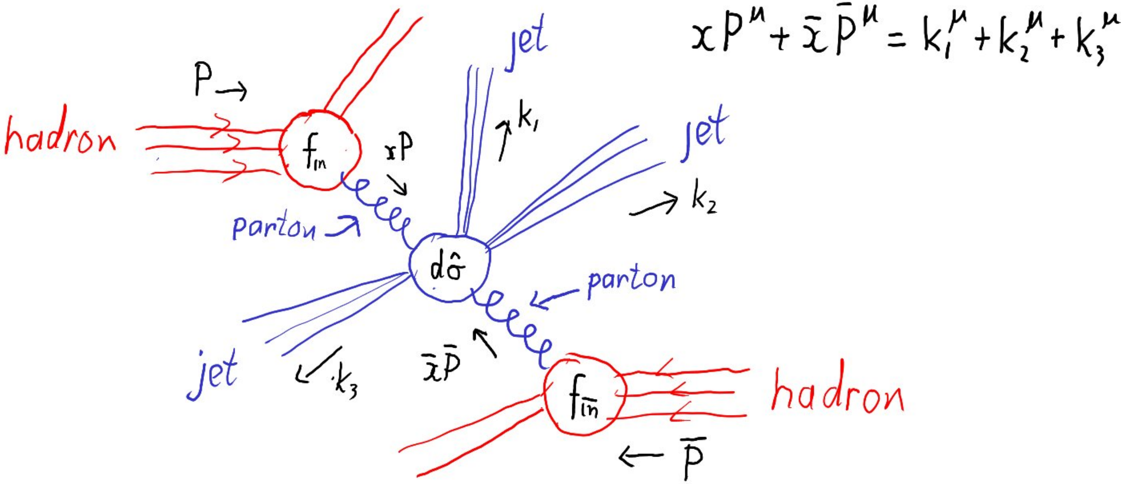
for example: 3 jets



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$$a_\epsilon = \frac{\alpha_s}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$$

$$d\sigma^{\mathrm{LO}} = \sum_{i,\bar{\imath}} \int dx\, d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x})\, dB_{i\bar{\imath}}(x,\bar{x})$$

$$d\sigma^{\mathrm{NLO}} = \sum_{i,\bar{\imath}} \int dx\, d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x}) \Big\{ \Big[a_\epsilon\, dV_{i\bar{\imath}}(x,\bar{x}) \; + \; a_\epsilon\, dR_{i\bar{\imath}}(x,\bar{x}) \Big]$$

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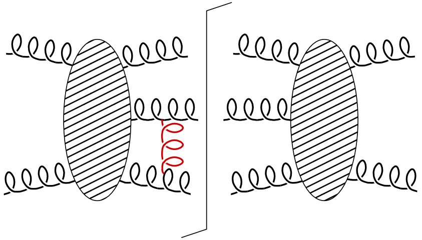
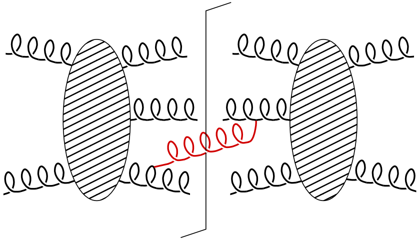
Virtual contribution,
containing the interference of
1-loop graphs with tree-level graphs



Real contribution,
containing the square of
real-radiation graphs,
integrated over the radiation



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$$- a_\epsilon \left[\frac{1}{\epsilon} \sum_{i'} \int_x^1 \frac{dz}{z} \mathcal{P}_{ii'}(z) \frac{f_{i'}(x/z)}{f_i(x)} + \frac{1}{\epsilon} \sum_{\bar{\imath}'} \int_{\bar{x}}^1 \frac{d\bar{z}}{\bar{z}} \mathcal{P}_{\bar{\imath}\bar{\imath}'}(\bar{z}) \frac{f_{\bar{\imath}'}(\bar{x}/\bar{z})}{f_{\bar{\imath}}(\bar{x})} \right] dB_{i\bar{\imath}}(x,\bar{x})$$

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$$d\sigma^{LO} = \sum_{i,\bar{\imath}} \int dx\,d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x})\, dB_{i\bar{\imath}}(x,\bar{x})$$

$$\begin{aligned} d\sigma^{NLO} = \sum_{i,\bar{\imath}} \int dx\,d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x}) \Big\{ \Big[a_\epsilon\, dV_{i\bar{\imath}}(x,\bar{x}) + a_\epsilon\, dR_{i\bar{\imath}}(x,\bar{x}) \Big]_{\text{finite}} \\ - a_\epsilon \Big[\frac{1}{\epsilon}\sum_{i'}\int_x^1 \frac{dz}{z}\mathcal{P}_{ii'}(z)\frac{f_{i'}(x/z)}{f_i(x)} + \frac{1}{\epsilon}\sum_{\bar{\imath}'}\int_{\bar{x}}^1 \frac{d\bar{z}}{\bar{z}}\mathcal{P}_{\bar{\imath}\bar{\imath}'}(\bar{z})\frac{f_{\bar{\imath}'}(\bar{x}/\bar{z})}{f_{\bar{\imath}}(\bar{x})} \Big] dB_{i\bar{\imath}}(x,\bar{x}) \\ + a_\epsilon \Big[\frac{\delta f_i\;(x,\mu_F)}{f_i(x)} + \frac{\delta f_{\bar{\imath}}\;(\bar{x},\mu_{\bar{F}})}{f_{\bar{\imath}}(\bar{x})} \Big] dB_{i\bar{\imath}}(x,\bar{x}) \Big\} \end{aligned}$$

$$f_i(x) \to f_i(x,\mu_F) = f_i(x) + a_\epsilon\, \delta f_i(x,\mu_F) + \mathcal{O}(\alpha_s^2)$$

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$$+ a_\epsilon \left[\ln\frac{\mu^2}{\mu_F^2} \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, \frac{f_{i'}(x/z)}{f_i(x)} + \ln\frac{\mu^2}{\mu_{\bar{F}}^2} \sum_{\bar{\imath}'} \int_{\bar{x}}^1 \frac{d\bar{z}}{\bar{z}}\, \mathcal{P}_{\bar{\imath}\bar{\imath}'}(\bar{z})\, \frac{f_{\bar{\imath}'}(\bar{x}/\bar{z})}{f_{\bar{\imath}}(\bar{x})} \right] dB_{i\bar{\imath}}(x,\bar{x})$$
$$+ a_\epsilon \left[\frac{\delta f_i^{\mathrm{fin}}(x,\mu_F)}{f_i(x)} + \frac{\delta f_{\bar{\imath}}^{\mathrm{fin}}(\bar{x},\mu_{\bar{F}})}{f_{\bar{\imath}}(\bar{x})} \right] dB_{i\bar{\imath}}(x,\bar{x}) \Bigg\}$$

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$$\delta f_i(x,\mu_F) = \delta f_i^{\mathrm{fin}}(x,\mu_F) + \frac{1}{\epsilon}\left(\frac{\mu^2}{\mu_F^2}\right)^\epsilon \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, f_{i'}(x/z)$$

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$$+ a_\epsilon \left[\frac{\delta f_i^{\text{fin}}(x,\mu_F)}{f_i(x)} + \frac{\delta f_{\bar{\imath}}^{\text{fin}}(\bar{x},\mu_{\bar{F}})}{f_{\bar{\imath}}(\bar{x})} \right] dB_{i\bar{\imath}}(x,\bar{x}) \Bigg\}$$

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$$\delta f_i(x,\mu_F) = \delta f_i^{\text{fin}}(x,\mu_F) + \frac{1}{\epsilon}\left(\frac{\mu^2}{\mu_F^2}\right)^\epsilon \sum_{i'} \int_x^1 \frac{dz}{z} \mathcal{P}_{ii'}(z)\, f_{i'}(x/z)$$

$$\frac{d\sigma}{d\ln\mu_F^2} = 0 \quad\Rightarrow\quad \frac{d\delta f_i^{\text{fin}}(x,\mu_F)}{d\ln\mu_F^2} = \sum_{i'} \int_x^1 \frac{dz}{z} \mathcal{P}_{ii'}(z)\, f_{i'}(x/z) \quad\Rightarrow\quad \frac{df_i(x,\mu_F)}{d\ln\mu_F^2} = a_\epsilon \sum_{i'} \int_x^1 \frac{dz}{z} \mathcal{P}_{ii'}(z)\, f_{i'}(x/z,\mu_F)$$

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$$a_\epsilon = \frac{\alpha_s}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$$

$$d\sigma^{LO} = \sum_{i,\bar{\imath}} \int dx\, d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x})\, dB_{i\bar{\imath}}(x,\bar{x})$$

$$d\sigma^{NLO} = \sum_{i,\bar{\imath}} \int dx\, d\bar{x}\, f_i \ \cdots \ (\bar{z})\, \frac{f_{\bar{\imath}'}(\bar{x}/\bar{z})}{f_{\bar{\imath}}(\bar{x})} \Big]\, dB_{i\bar{\imath}}(x,\bar{x})$$

We want to establish the same for hybrid k_T-factorization, which has leading-order formula

$$d\sigma^{LO} = \sum_{\bar{\imath}} \int dx \int \frac{d^2k_\perp}{\pi} \int d\bar{x}\, F(x,k_\perp)\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x,k_\perp,\bar{x})$$

and involves both a UPDF and matrix elements explicitly depending on $k_\perp$.

$$f_i(x) \to f_i(x,\mu_F) = f_i(x) + a_\epsilon\, \delta f_i(x,\mu_F) + \mathcal{O}(\alpha_s^2) \qquad \delta f_i(x,\mu_F) = \delta f_i^{fin}(x,\mu_F) + \frac{-}{\epsilon}\left(\frac{\cdots}{\mu_F^2}\right) \sum_{i'} \int_x^1 \frac{dz}{z} \mathcal{P}_{ii'}(z)\, f_{i'}(x/z)$$

$$\frac{d\sigma}{d\ln\mu_F^2} = 0 \quad \Rightarrow \quad \frac{d\delta f_i^{fin}(x,\mu_F)}{d\ln\mu_F^2} = \sum_{i'} \int_x^1 \frac{dz}{z} \mathcal{P}_{ii'}(z)\, f_{i'}(x/z) \quad \Rightarrow \quad \frac{df_i(x,\mu_F)}{d\ln\mu_F^2} = a_\epsilon \sum_{i'} \int_x^1 \frac{dz}{z} \mathcal{P}_{ii'}(z)\, f_{i'}(x/z,\mu_F)$$

Off-shell, or space-like, matrix elements with the auxiliary parton method

A space-like gluon is indicated with "$\star$" and has momentum

$$k_\star^\mu = xP^\mu + k_\perp^\mu \;.$$

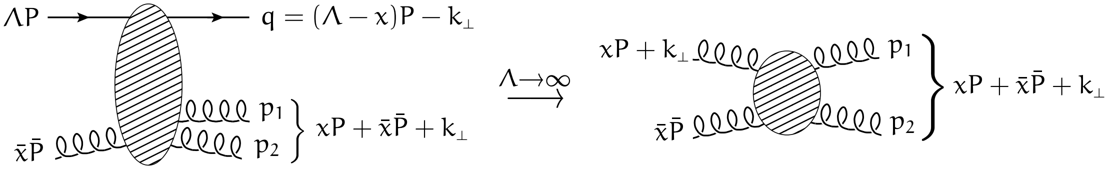
A tree-level matrix element with a space-like initial-state gluon is understood to be defined with the help of auxiliary partons as (AvH, Kotko, Kutak 2012)

$$\frac{x^2|k_\perp|^2}{g_s^2 C_i \Lambda^2}\,\big|\overline{M}_{i\bar{\imath}}\big|^2(k_\Lambda, k_{\bar{\imath}}; q_\Lambda, \{p\}_n) \;\overset{\Lambda\to\infty}{\longrightarrow}\; \big|\overline{M}_{\star\bar{\imath}}\big|^2(k_\star, k_{\bar{\imath}}; \{p\}_n)$$

with

$$k_\Lambda^2 = q_\Lambda^2 = 0 \quad , \quad k_\Lambda^\mu - q_\Lambda^\mu \overset{\Lambda\to\infty}{=} xP^\mu + k_\perp^\mu \;.$$

The example of auxiliary quarks, for the gluonic contribution to dijet production, and neglecting momentum components of $\mathcal{O}(\Lambda^{-1})$.



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with

$$k_\Lambda^2 = q_\Lambda^2 = 0 \quad , \quad k_\Lambda^\mu - q_\Lambda^\mu \overset{\Lambda\to\infty}{=} xP^\mu + k_\perp^\mu \ .$$

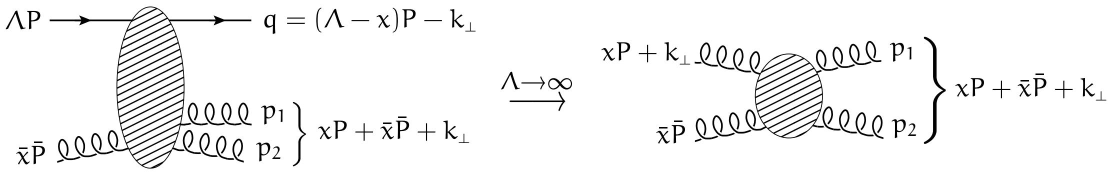
The matrix element $\left|\overline{M}_{\star\bar{\imath}}\right|^2$ is independent of the type of auxiliary parton i used, partly thanks to the color correction factor $C_i (= 2C_A, 2C_F)$.

The factor $1/g_s^2$ corrects the power of the coupling constant.

The factor $|k_\perp|^2$ assures a smooth on-shell limit $|k_\perp| \to 0$.

The factor x assures that $M_{\star\bar{\imath}}$ only depends on xP, not x separately.

The example of auxiliary quarks, for the gluonic contribution to dijet production, and neglecting momentum components of $\mathcal{O}(\Lambda^{-1})$.



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with

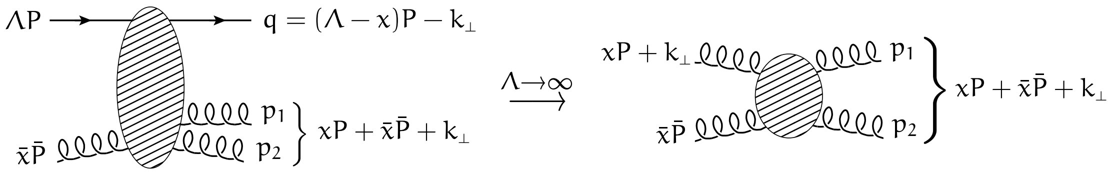
$$k_\Lambda^2 = q_\Lambda^2 = 0 \quad , \quad k_\Lambda^\mu - q_\Lambda^\mu \overset{\Lambda\to\infty}{=} xP^\mu + k_\perp^\mu \ .$$

The limit can be applied on expressions of matrix elements...

...but the result of the limit can also be obtained directly by using eikonal Feynman rules for the auxiliary parton.

And this is (at tree level) identical to using Lipatov's effective action.

The example of auxiliary quarks, for the gluonic contribution to dijet production, and neglecting momentum components of $\mathcal{O}(\Lambda^{-1})$.



Embedding in collinear factorization (CF)

Consider hadron collisions, with production of the final state of interest $\mathcal{H}$:

$$h(\lambda P) + h(\bar{P}) \to \mathcal{H} + \mathcal{X}$$

We assume that there is a natural rapidity Y_μ associated with $\mathcal{H}$, which separates the event into "target" and "projectile" parts. Then we can define

$$x = \sum_j \theta(y_j < Y_\mu)\,\frac{p_j\cdot\bar{P}}{P\cdot\bar{P}} \quad , \quad k_\perp = -\sum_j \theta(y_j < Y_\mu)\,p_{j\perp}$$

We can associate rapidity scale μ_Y (Collins-Soper scale) to the rapidity Y_μ as

$$\mu_Y = \nu x\,e^{-Y_\mu} \quad \Leftrightarrow \quad Y_\mu = \ln\frac{\nu x}{\mu_Y} \quad , \quad \nu^2 = (P+\bar{P})^2$$

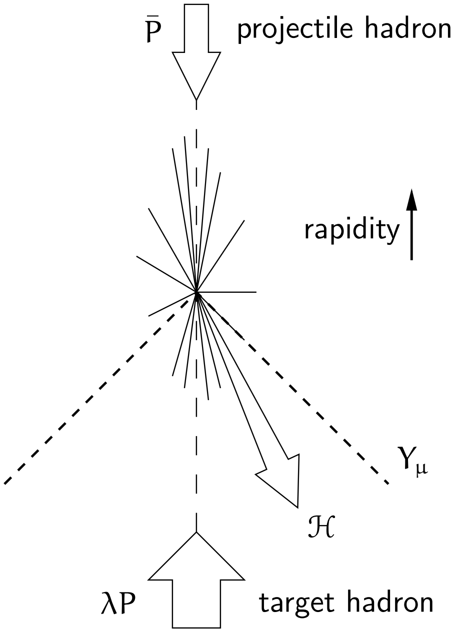
Due to IRC-safety of variables x and $k_\perp$, the hadronic differential cross section

$$\frac{d\sigma_\lambda^{\mathrm{CF}}}{dx\,d^2k_\perp}(x,k_\perp,\dots) = \sum_{i,\bar{\imath}}\int_0^1 dX\,f_i(X)\int_0^1 d\bar{x}\,f_{\bar{\imath}}(\bar{x})\,\frac{d\hat{\sigma}_{i\bar{\imath}}^{\mathrm{CF}}}{dx\,d^2k_\perp}(\lambda X,\bar{x};x,k_\perp,\dots)$$

should be computable in CF, at least up to NLO, and in the limit:

$$\lambda\to\infty \quad , \quad x,k_\perp\text{– fixed}\,.$$

$\lambda X = \Lambda$



kT-factorizable cross section

$\Lambda = \lambda X$ must go to ∞

absolute kinematic minimum: $X > x/\lambda$

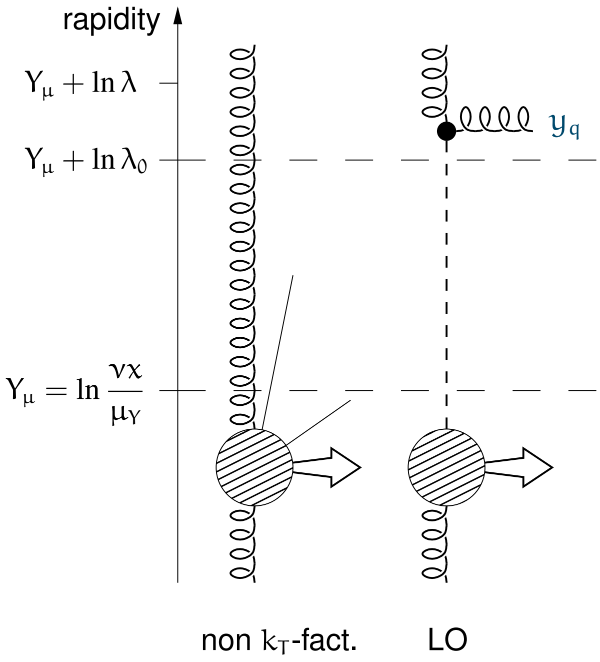
$$d\sigma^{\mathrm{LO}} = \int_{x/\lambda}^{\delta_0} dX\, f(X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X) + \int_{\delta_0}^{1} dX\, f(X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X)$$

non k_T-factorizable

$$\delta_0 = \frac{\lambda_0 x |k_\perp|}{\lambda \mu_Y} \quad , \quad \lambda \succ \lambda_0 \to \infty$$

kT-factorizable cross section

$\Lambda = \lambda X$ must go to ∞



absolute kinematic minimum: $X > x/\lambda$

$$y_q = \ln\frac{\nu(\lambda X - x)}{|k_\perp|} > Y_\mu + \ln\lambda_0 \quad \Rightarrow \quad X > \delta_0$$

The rapidity y_q of the auxiliary parton must be large enough in order to guarantee that X is large enough.

$$d\sigma^{LO} = \int_{x/\lambda}^{\delta_0} dX\, f(X)\, d\hat{\sigma}^{LO}(\lambda X) + \int_{\delta_0}^{1} dX\, f(X)\, d\hat{\sigma}^{LO}(\lambda X)$$

non k_T-factorizable

$$\delta_0 = \frac{\lambda_0 x |k_\perp|}{\lambda \mu_Y} \quad , \quad \lambda \succ \lambda_0 \to \infty$$

kT-factorizable cross section

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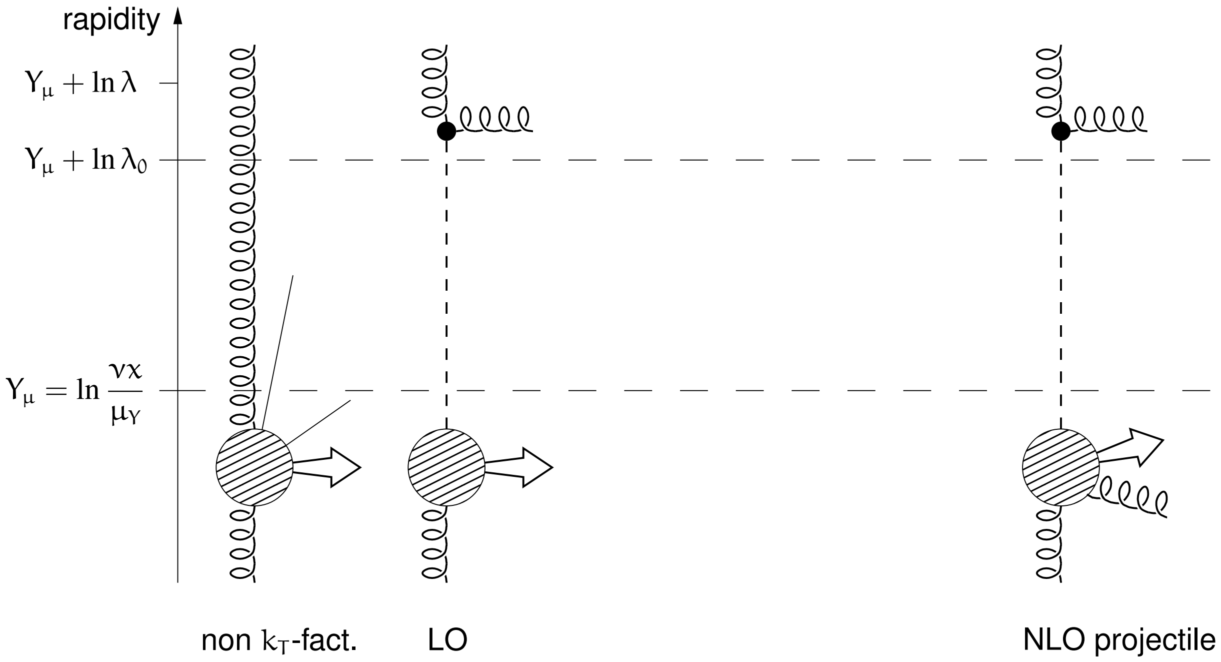

kT-factorizable cross section

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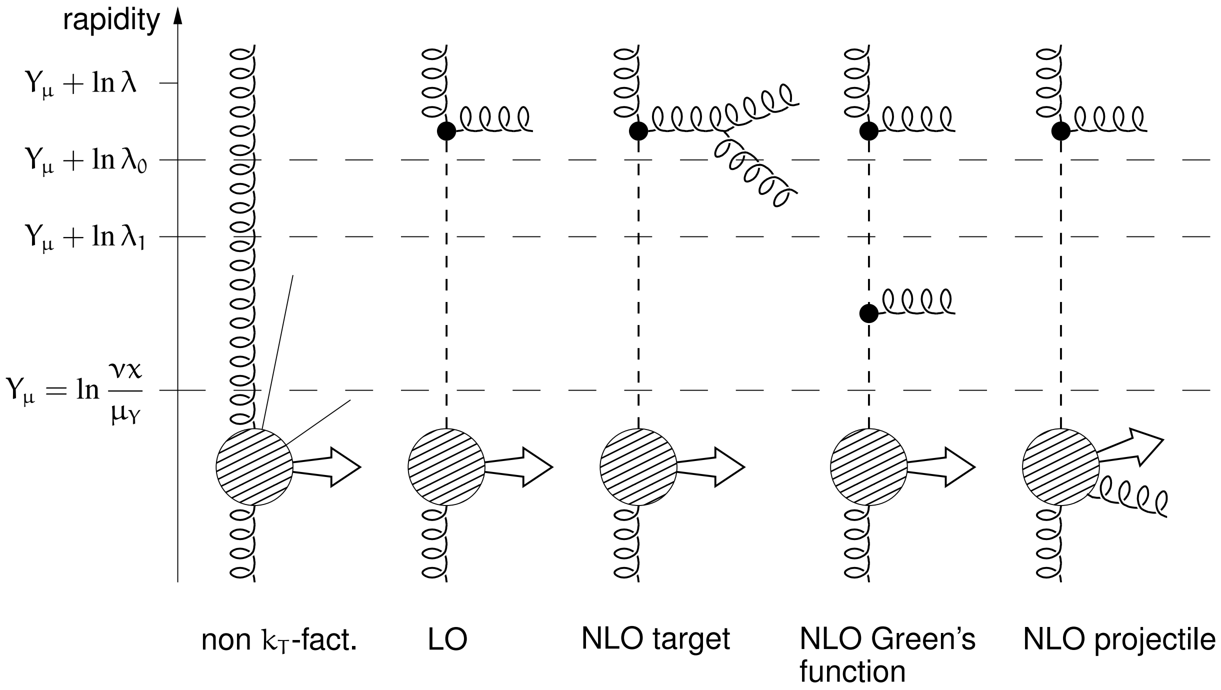


$\lambda \succ \lambda_0 \succ \lambda_1 \to \infty$

It turns out that another rapidity separator λ_1 is needed to consistently define the target and Green's function contributions.

Born contribution

Given the rapidity separator Y_μ, which separates the event into "target" and "projectile" parts, we define

$$x = \sum_j \theta(y_j < Y_\mu)\,\frac{p_j\cdot\bar{P}}{P\cdot\bar{P}} \quad , \quad k_\perp = -\sum_j \theta(y_j < Y_\mu)\,p_{j\perp}$$

The k_T-factorizable Born level differential cross section is defined as

$$\frac{d\sigma_\lambda^{CF,B}(\{p\}_n)}{dx\,d^2k_\perp} = \sum_{i,\bar{\imath}} \int_{\delta_0}^1 dX\, f_i(X) \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, \frac{d\hat{\sigma}^B_{i\bar{\imath}}(\lambda X, \bar{x}; \{p\}_n)}{dx\,d^2k_\perp}$$

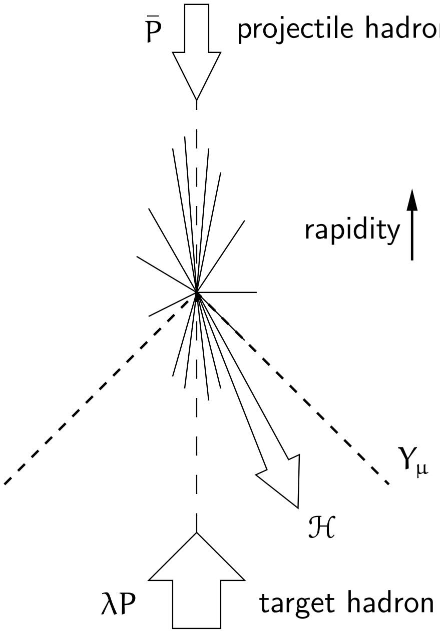
$\lambda \to \infty$ is guaranteed to be equivalent to $\Lambda = \lambda X \to \infty$, and we get

$$\frac{d\sigma_{\lambda\to\infty}^{CF,B}(\{p\}_n)}{dx\,d^2k_\perp} = F^{LO}(\delta_0, k_\perp) \sum_{\bar{\imath}} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x, k_\perp, \bar{x}; \{p\}_n) \ ,$$

where $dB_{\star\bar{\imath}}$ is the usual Born-level "partonic off-shell" cross section, and with

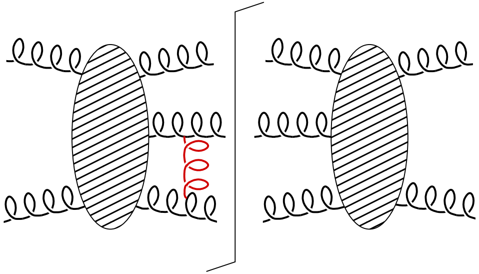
$$F^{LO}(\delta_0, k_\perp) = \sum_i \int_{\delta_0}^1 dX\, f_i(X)\, \frac{\alpha_s C_i}{2\pi^2 |k_\perp|^2} \ ,$$

a "proto UPDF". The Born contribution is independent of Y_μ.



Virtual contribution

$$\frac{d\sigma_\lambda^{CF,V}(\{p\}_n)}{dx\,d^2k_\perp} \xrightarrow{\lambda\to\infty} \sum_{i,\bar{\imath}} \int_{\delta_0}^{1} dX\, f_i(X)\, \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x}) \Big[dV^{unf}_{i\bar{\imath}}(\epsilon, \lambda X; x, k_\perp, \bar{x}; \{p\}_n) + dV^{fam}_{\star\bar{\imath}}(\epsilon; x, k_\perp, \bar{x}; \{p\}_n) \Big]$$



Virtual contribution

$$\frac{d\sigma_{\lambda}^{CF,V}(\{p\}_n)}{dx\,d^2k_\perp} \xrightarrow{\lambda\to\infty} \sum_{i,\bar{\imath}} \int_{\delta_0}^{1} dX\, f_i(X)\, \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x}) \Big[dV_{i\bar{\imath}}^{unf}(\epsilon, \lambda X; x, k_\perp, \bar{x}; \{p\}_n) + dV_{\star\bar{\imath}}^{fam}(\epsilon; x, k_\perp, \bar{x}; \{p\}_n) \Big]$$

$$dV_{i\bar{\imath}}^{unf}(\epsilon, \Lambda) = dB_{\star\bar{\imath}} \times a_\epsilon N_c \left(\frac{\mu^2}{|k_\perp|^2}\right)^{\epsilon} \left[\frac{2}{\epsilon}\ln\frac{\Lambda}{x} + \bar{\mathcal{V}}_i\right]$$

$$\bar{\mathcal{V}}_{q/\bar{q}} = \frac{1}{\epsilon}\frac{13}{6} + \frac{\pi^2}{3} + \frac{80}{18} + \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2} + \frac{3}{2}\frac{1}{\epsilon} + 4\right] - \frac{n_f}{N_c}\left[\frac{2}{3}\frac{1}{\epsilon} + \frac{10}{9}\right]$$

$$\bar{\mathcal{V}}_g = -\frac{1}{\epsilon^2} + \frac{\pi^2}{3}$$

Virtual contribution

$$\frac{d\sigma_{\lambda}^{\mathrm{CF,V}}(\{p\}_n)}{dx\,d^2k_\perp} \xrightarrow{\lambda\to\infty} \sum_{i,\bar{\imath}} \int_{\delta_0}^{1} dX\, f_i(X)\, \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x}) \Big[dV_{i\bar{\imath}}^{\mathrm{unf}}(\epsilon, {\color{red}\lambda X}; x, k_\perp, \bar{x}; \{p\}_n) + dV_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon; x, k_\perp, \bar{x}; \{p\}_n) \Big]$$

$$dV_{i\bar{\imath}}^{\mathrm{unf}}(\epsilon,\Lambda) = dB_{\star\bar{\imath}} \times a_\epsilon N_c {\color{red}\left(\frac{\mu^2}{|k_\perp|^2}\right)^{\epsilon}} \left[\frac{2}{\epsilon}\ln\frac{{\color{red}\Lambda}}{x} + \bar{\mathcal{V}}_i\right]$$

$$\bar{\mathcal{V}}_{q/\bar{q}} = \frac{1}{\epsilon}\frac{13}{6} + \frac{\pi^2}{3} + \frac{80}{18} + \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2} + \frac{3}{2}\frac{1}{\epsilon} + 4\right] - \frac{n_f}{N_c}\left[\frac{2}{3}\frac{1}{\epsilon} + \frac{10}{9}\right]$$

$$\bar{\mathcal{V}}_g = -\frac{1}{\epsilon^2} + \frac{\pi^2}{3}$$

$dV_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon)$ is simply the rest of the virtual contribution.
It does exhibit auxiliary parton universality,
does not depend on Λ,
and has a smooth on-shell limit for $|k_\perp| \to 0$.
After UV subtraction in the $\overline{\mathrm{MS}}$ scheme on the familiar virtual contribution $dV_{\star\bar{\imath}}^{\mathrm{fam}}$,

$$\gamma_g = \frac{\beta_0}{2} = \frac{11N_c}{6} - \frac{2T_R n_f}{3}$$

$$dV_{\star\bar{\imath}}^{\mathrm{fam,UV\text{-}sub}}(\epsilon) = dV_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon) - dB_{\star\bar{\imath}} \times a_\epsilon \frac{\gamma_g}{\epsilon} \times [\text{Born-level-power-of-}\alpha_s]\ ,$$

the pole part follows the well-known universal formula for one-loop amplitudes (Kunszt *et al.* 1994 , Catani 1998bh)

Virtual contribution

$$\frac{d\sigma_{\lambda}^{\mathrm{CF,V}}(\{p\}_n)}{dx\,d^2k_\perp} \xrightarrow{\lambda\to\infty} \sum_{i,\bar{\imath}} \int_{\delta_0}^{1} dX\, f_i(X)\, \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x}) \Big[dV_{i\bar{\imath}}^{\mathrm{unf}}(\epsilon, \lambda X; x, k_\perp, \bar{x}; \{p\}_n) + dV_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon; x, k_\perp, \bar{x}; \{p\}_n) \Big]$$

$$dV_{i\bar{\imath}}^{\mathrm{unf}}(\epsilon, \Lambda)$$

$$dV_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon)$$

$$\gamma_g = \frac{\beta_0}{2} = \frac{11N_c}{6} - \frac{2T_R n_f}{3}$$

Virtual contribution

$$\frac{d\sigma_{\lambda}^{\mathrm{CF,V}}(\{p\}_n)}{dx\,d^2k_\perp} \xrightarrow{\lambda\to\infty} \sum_{i,\bar{\imath}} \int_{\delta_0}^{1} dX\, f_i(X)\, \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x}) \Big[dV_{i\bar{\imath}}^{\mathrm{unf}}(\epsilon, \lambda X; x, k_\perp, \bar{x}; \{p\}_n) + dV_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon; x, k_\perp, \bar{x}; \{p\}_n) \Big]$$

$$dV_{i\bar{\imath}}^{\mathrm{unf}}(\epsilon, \Lambda) - dB_{\star\bar{\imath}} \times a_\epsilon \left[\left(\frac{\mu^2}{\mu_Y^2}\right)^{\epsilon} \frac{N_c}{\epsilon^2} + \frac{\gamma_g}{\epsilon} \qquad \right]$$

$$dV_{\star\bar{\imath}}^{\mathrm{proj}}(\epsilon, \mu_Y) \qquad = dV_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon) \quad + dB_{\star\bar{\imath}} \times a_\epsilon \left[\left(\frac{\mu^2}{\mu_Y^2}\right)^{\epsilon} \frac{N_c}{\epsilon^2} + \frac{\gamma_g}{\epsilon} \right].$$

$$\gamma_g = \frac{\beta_0}{2} = \frac{11N_c}{6} - \frac{2T_R n_f}{3}$$

Move (soft-)collinear divergence, associated with the positive-rapidity initial state, from the familiary to the unfamiliar contribution. Amount of moved soft divergence set by μ_Y.

Virtual contribution

$$\frac{d\sigma_{\lambda}^{\mathrm{CF,V}}(\{p\}_n)}{dx\,d^2k_\perp} \xrightarrow{\lambda\to\infty} \sum_{i,\bar{\imath}}\int_{\delta_0}^{1} dX\, f_i(X)\,\frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\Big[dV_{i\bar{\imath}}^{\mathrm{unf}}(\epsilon,\lambda X;x,k_\perp,\bar{x};\{p\}_n) + dV_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon;x,k_\perp,\bar{x};\{p\}_n)\Big]$$

$$\begin{aligned}
dV_{i\bar{\imath}}^{\mathrm{targ}}(\epsilon,\Lambda,\lambda_1,\mu_Y) &= dV_{i\bar{\imath}}^{\mathrm{unf}}(\epsilon,\Lambda) - dB_{\star\bar{\imath}}\times a_\epsilon\left[\left(\frac{\mu^2}{\mu_Y^2}\right)^{\epsilon}\frac{N_c}{\epsilon^2}+\frac{\gamma_g}{\epsilon}+\left(\frac{\mu^2}{|k_\perp|^2}\right)^{\epsilon}\frac{2N_c\ln\lambda_1}{\epsilon}\right]\\
dV_{\star\bar{\imath}}^{\mathrm{Green}}(\epsilon,\lambda_1) &= dB_{\star\bar{\imath}}\times a_\epsilon\left[\left(\frac{\mu^2}{|k_\perp|^2}\right)^{\epsilon}\frac{2N_c\ln\lambda_1}{\epsilon}\right]\\
dV_{\star\bar{\imath}}^{\mathrm{proj}}(\epsilon,\mu_Y) &= dV_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon) + dB_{\star\bar{\imath}}\times a_\epsilon\left[\left(\frac{\mu^2}{\mu_Y^2}\right)^{\epsilon}\frac{N_c}{\epsilon^2}+\frac{\gamma_g}{\epsilon}\right].
\end{aligned}$$

$$\gamma_g = \frac{\beta_0}{2} = \frac{11N_c}{6} - \frac{2T_R n_f}{3}$$

Move (soft-)collinear divergence, associated with the positive-rapidity initial state, from the familiary to the unfamiliar contribution. Amount of moved soft divergence set by μ_Y.

Move an amount of soft divergence set by λ_1 from the unfamiliar contribution to the Green's function contribution.

Virtual contribution

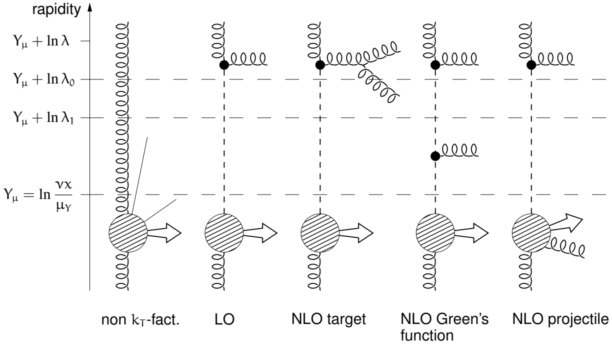
$$\frac{d\sigma_\lambda^{\mathrm{CF,V}}(\{p\}_n)}{dx\,d^2k_\perp} \xrightarrow{\lambda\to\infty} \sum_{i,\bar\imath}\int_{\delta_0}^{1} dX\, f_i(X)\,\frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_0^1 d\bar x\, f_{\bar\imath}(\bar x)\Big[dV_{i\bar\imath}^{\mathrm{unf}}(\epsilon,\lambda X;x,k_\perp,\bar x;\{p\}_n) + dV_{\star\bar\imath}^{\mathrm{fam}}(\epsilon;x,k_\perp,\bar x;\{p\}_n)\Big]$$

$$dV_{i\bar\imath}^{\mathrm{targ}}(\epsilon,\Lambda,\lambda_1,\mu_Y) = dV_{i\bar\imath}^{\mathrm{unf}}(\epsilon,\Lambda) - dB_{\star\bar\imath}\times a_\epsilon\left[\left(\frac{\mu^2}{\mu_Y^2}\right)^\epsilon\frac{N_c}{\epsilon^2}+\frac{\gamma_g}{\epsilon}+\left(\frac{\mu^2}{|k_\perp|^2}\right)^\epsilon\frac{2N_c\ln\lambda_1}{\epsilon}\right]$$

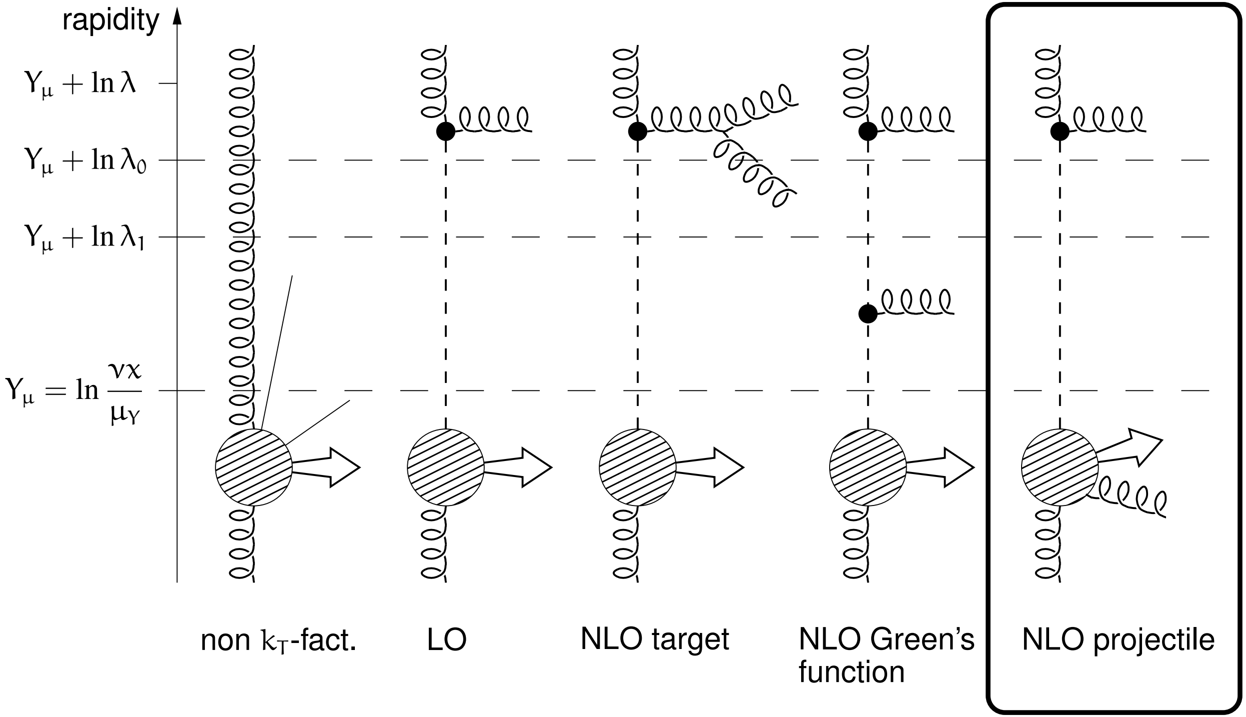
$$dV_{\star\bar\imath}^{\mathrm{Green}}(\epsilon,\lambda_1) = dB_{\star\bar\imath}\times a_\epsilon\left[\left(\frac{\mu^2}{|k_\perp|^2}\right)^\epsilon\frac{2N_c\ln\lambda_1}{\epsilon}\right]$$

$$dV_{\star\bar\imath}^{\mathrm{proj}}(\epsilon,\mu_Y) = dV_{\star\bar\imath}^{\mathrm{fam}}(\epsilon) + dB_{\star\bar\imath}\times a_\epsilon\left[\left(\frac{\mu^2}{\mu_Y^2}\right)^\epsilon\frac{N_c}{\epsilon^2}+\frac{\gamma_g}{\epsilon}\right].$$

$$\gamma_g = \frac{\beta_0}{2} = \frac{11N_c}{6} - \frac{2T_R n_f}{3}$$



Real projectile contribution

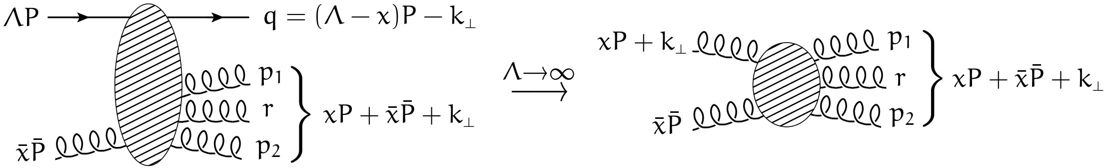


Real projectile contribution

Familiar real contribution

Includes collinear divergence associated with the positive-rapidity initial state.

No restriction in rapidity on the radiation.



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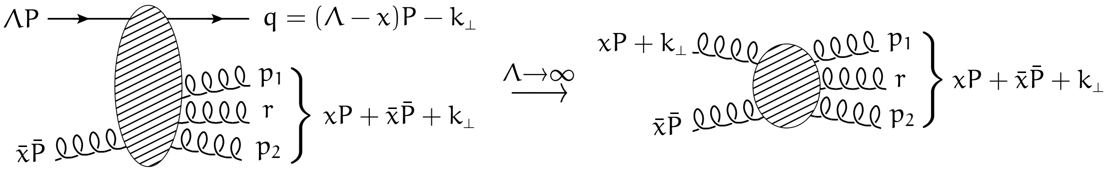
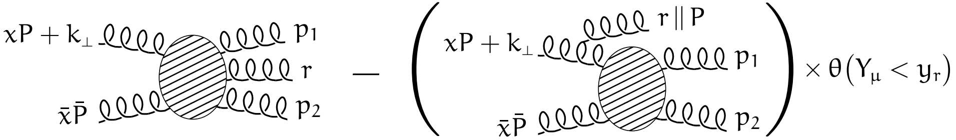


Real projectile contribution

Remove this collinear region, but only for the radiation rapidity above Y_μ.

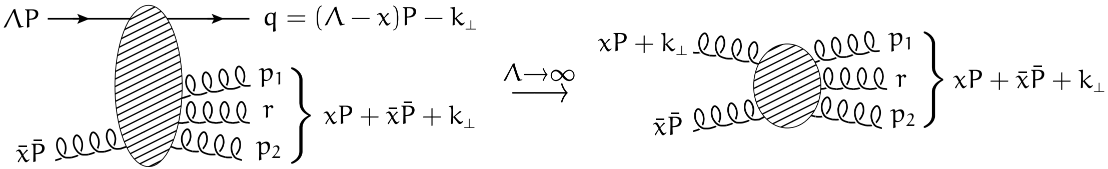


Real projectile contribution

<u>Familiar real contribution</u>

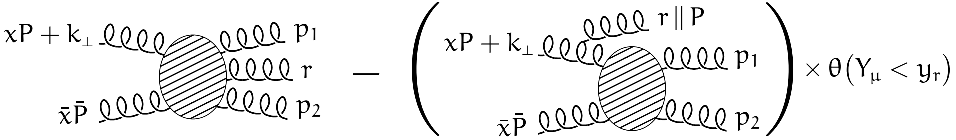
Includes collinear divergence associated with the positive-rapidity initial state.

No restriction in rapidity on the radiation.



<u>Real projectile contribution</u>

Remove this collinear region, but only for the radiation rapidity above Y_μ.



The radiative matrix element in this collinear region is given by

$$\left|\overline{M}_{\star\bar{\imath}}\right|^2(k_\star, k_{\bar{\imath}}; r, \{p\}_n) \xrightarrow{\bar{x}_r \to 0} g_s^2 \frac{4N_c}{|r_\perp|^2(1 - x_r/x)^2} \left|\overline{M}_{\star\bar{\imath}}\right|^2(k_\star - x_r P - r_\perp, k_{\bar{\imath}}; \{p\}_n)$$

$$\frac{2N_c}{z(1-z)}$$

Complete finite projectile contribution

The negative-rapidity-side PDF needs to be renormalized like usual to arrive at a finite result

$$f_{\bar{\imath}}(\bar{x}) \longrightarrow f_{\bar{\imath}}(\bar{x}, \mu_{\bar{F}}) + \frac{a_\epsilon}{\epsilon}\left(\frac{\mu^2}{\mu_{\bar{F}}^2}\right)^{\epsilon} \sum_{\bar{\imath}'} \left[\mathcal{P}_{\bar{\imath}\bar{\imath}'} \otimes f_{\bar{\imath}'}\right](\bar{x})$$

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$$\begin{aligned}\text{complete finite projectile contribution} =& \Big[\text{resolved contribution}\Big] \\ &+ \Big[\text{unresolved independent of } \mu_{\bar{F}}, \mu_Y\Big] \\ &+ \Big[\text{remnants of collinear cancellations/subtractions}\Big](\mu_{\bar{F}}, \mu_Y)\end{aligned}$$

Complete finite projectile contribution

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$$\text{complete finite projectile contribution} = \Big[\text{resolved contribution}\Big]$$
$$+\Big[\text{unresolved independent of } \mu_{\bar F}, \mu_Y\Big]$$
$$+\Big[\text{remnants of collinear cancellations/subtractions}\Big](\mu_{\bar F}, \mu_Y)$$

$$\mathcal{P}_\star(z) = \frac{2N_c}{z(1-z)_+}$$

contract $(x, k_\perp)$-dependent differential cross section with "test UPDF" $F(x, k_\perp)$

$$dB_{\star\bar\imath} \times a_\epsilon \left\{ \ln\frac{\mu^2}{\mu_{\bar F}^2} \sum_{\bar\imath'} \left[\mathcal{P}_{\bar\imath\bar\imath'} \otimes f_{\bar\imath'}\right](\bar x) \right.$$
$$\left. + \ln\frac{\mu^2}{\mu_Y^2}\left[\mathcal{P}_\star \otimes F\right](x, k_\perp) - \frac{2N_c}{\pi}\int_x^1 \frac{dz}{z^2(1-z)} \int \frac{d^2 r_\perp}{|r_\perp|^2}\left[F\left(\frac{x}{z}, k_\perp + r_\perp\right) - F\left(\frac{x}{z}, k_\perp\right)\right]\theta\left(|r_\perp| < \mu_Y \frac{1-z}{z}\right)\right\}$$

UPDF evolution from the projectile point of view

The cross section should be independent of μ_Y order by order in α_s.

Expand the UPDF as $\hat{F}(x,k_\perp;\mu_Y) = \hat{F}^{(0)}(x,k_\perp) + \frac{\alpha_s}{2\pi}\hat{F}^{(1)}(x,k_\perp;\mu_Y) + \mathcal{O}(\alpha_s)$ $\hat{F}(x,k_\perp;\mu_Y) = x\,F(x,k_\perp;\mu_Y)$

$$\frac{d\hat{F}^{(1)}(x,k_\perp;\mu_Y)}{d\ln\mu_Y^2} = \frac{N_c}{\pi}\int\frac{d^2r_\perp}{|r_\perp|^2}\left\{\hat{F}^{(0)}\left(x\left[1+\frac{|r_\perp|}{\mu_Y}\right],k_\perp + r_\perp\right)\theta\left(|r_\perp| < \mu_Y\frac{1-x}{x}\right) - \theta\big(\mu_Y - |r_\perp|\big)\,\hat{F}^{(0)}(x,k_\perp)\right\}$$

UPDF evolution from the projectile point of view

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Expand the UPDF as $\hat{F}(x,k_\perp;\mu_Y) = \hat{F}^{(0)}(x,k_\perp) + \frac{\alpha_s}{2\pi}\hat{F}^{(1)}(x,k_\perp;\mu_Y) + \mathcal{O}(\alpha_s)$ $\quad \hat{F}(x,k_\perp;\mu_Y) = x\,F(x,k_\perp;\mu_Y)$

$$\frac{d\hat{F}(x,k_\perp;\mu_Y)}{d\ln\mu_Y^2} = \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2r_\perp}{|r_\perp|^2}\left\{\hat{F}\left(x\left[1+\frac{|r_\perp|}{\mu_Y}\right],k_\perp+r_\perp;\mu_Y\right)\theta\left(|r_\perp|<\mu_Y\frac{1-x}{x}\right) - \theta\big(\mu_Y-|r_\perp|\big)\,\hat{F}(x,k_\perp;\mu_Y)\right\}$$

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$$\hat{F}(x,k_\perp;\mu_Y) = \int d^{2-2\epsilon}x_\perp\, e^{ik_\perp x_\perp}\tilde{F}(x,x_\perp;\mu_Y) \quad \xrightarrow{|x_\perp|\to 0} \quad \frac{d\tilde{F}(x,0;\mu_Y)}{d\ln\mu_Y^2} = \frac{\alpha_s N_c}{\pi}\int_x^1\frac{dz}{z(1-z)_+}\tilde{F}\left(\frac{x}{z},0;\mu_Y\right)$$

UPDF evolution from the projectile point of view

The cross section should be independent of μ_Y order by order in α_s.

Expand the UPDF as $$\hat{F}(x,k_\perp;\mu_Y) = \hat{F}^{(0)}(x,k_\perp) + \frac{\alpha_s}{2\pi}\hat{F}^{(1)}(x,k_\perp;\mu_Y) + \mathcal{O}(\alpha_s)$$ $\hat{F}(x,k_\perp;\mu_Y) = x\,F(x,k_\perp;\mu_Y)$

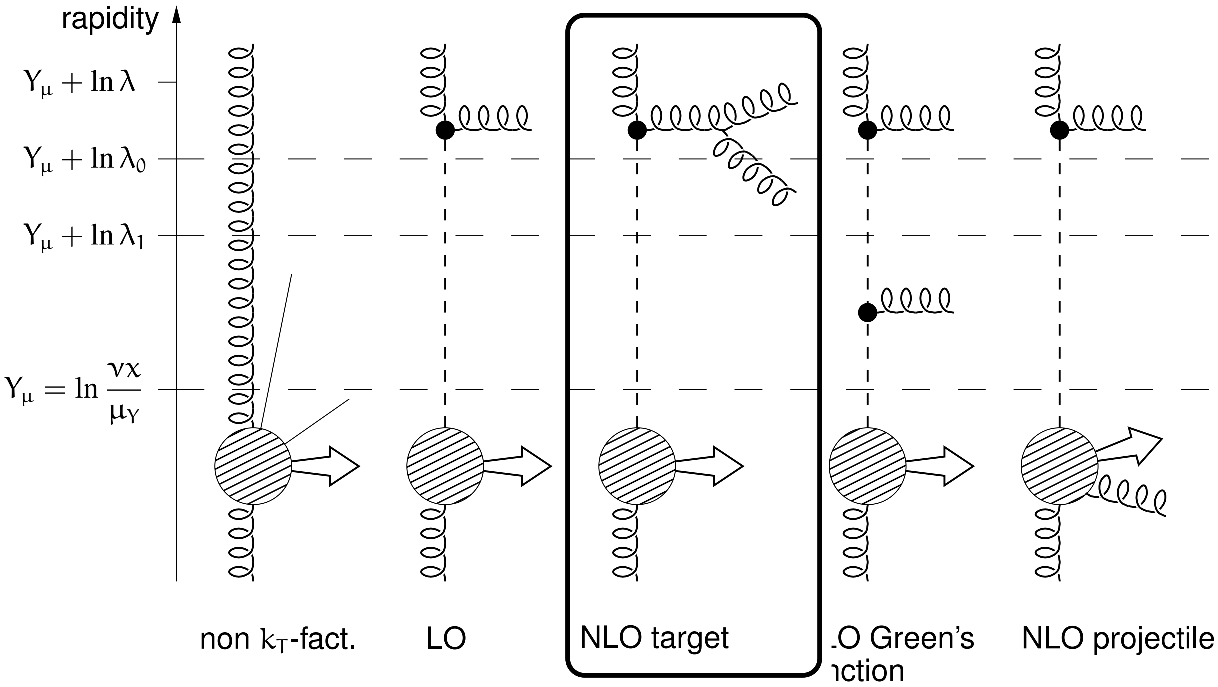
$$\frac{d\hat{F}(x,k_\perp;\mu_Y)}{d\ln\mu_Y^2} = \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2r_\perp}{|r_\perp|^2}\left\{\hat{F}\left(x\left[1+\frac{|r_\perp|}{\mu_Y}\right],k_\perp+r_\perp;\mu_Y\right)\theta\left(|r_\perp|<\mu_Y\frac{1-x}{x}\right) - \theta\big(\mu_Y-|r_\perp|\big)\,\hat{F}(x,k_\perp;\mu_Y)\right\}$$

$$\hat{F}(x,k_\perp;\mu_Y) = \int d^{2-2\epsilon}x_\perp\, e^{ik_\perp x_\perp}\tilde{F}(x,x_\perp;\mu_Y) \quad \overset{|x_\perp|\to 0}{\longrightarrow} \quad \frac{d\tilde{F}(x,0;\mu_Y)}{d\ln\mu_Y^2} = \frac{\alpha_s N_c}{\pi}\int_x^1\frac{dz}{z(1-z)_+}\tilde{F}\left(\frac{x}{z},0;\mu_Y\right)$$

$$|k_\perp| \ll \mu_Y \quad\Rightarrow\quad \frac{d}{d\ln\mu_Y^2}\tilde{F}(x,x_\perp;\mu_Y) = \frac{\alpha_s}{2\pi}\Big[-N_c\ln\big(\mu_Y^2\bar{x}_\perp^2\big)\Big]\tilde{F}(x,x_\perp;\mu_Y)$$ $\bar{x}_\perp = x_\perp/(2e^{-\gamma_E})$

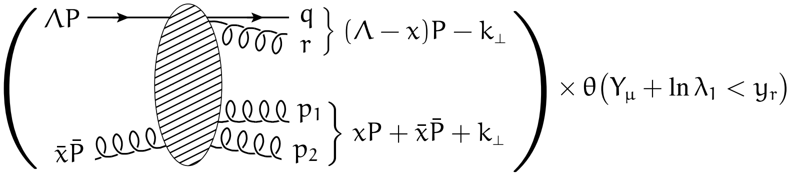
LO Collins-Soper-Sterman equation

Real target impact factor contribution



Real target impact factor contribution

Almost identical to the *unfamiliar* real contribution of AvH, Motyka, Ziarko 2022, but with a different phase space restriction: $y_r > Y_\mu + \ln\lambda_1$



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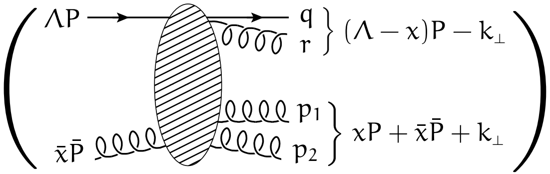


$\times\, \theta(Y_\mu + \ln\lambda_1 < y_r)$

For the matrix element in the "triple-Λ" limit we have:

$$\frac{x^2|k_\perp|^2}{g_s^4 C_i \Lambda^2}\,\big|\overline{M}_{i\bar{\imath}}\big|^2(k_\Lambda, k_{\bar{\imath}}; r_\Lambda, q_\Lambda, \{p\}_n) \overset{\Lambda\to\infty}{\longrightarrow} 2z(1-z)\,\mathcal{Q}_i(z, r_\perp)\,\big|\overline{M}_{\star\bar{\imath}}\big|^2(k_\star, k_{\bar{\imath}}; \{p\}_n)$$

where

$$\mathcal{Q}_i(z, r_\perp) = \mathcal{P}_i(z)\left(\frac{c_q\,|k_\perp|^2}{|r_\perp|^2|r_\perp + k_\perp|^2} + \frac{c_q(1-z)^2\,|k_\perp|^2}{|r_\perp + k_\perp|^2|r_\perp + zk_\perp|^2} + \frac{c_r z^2\,|k_\perp|^2}{|r_\perp|^2|r_\perp + zk_\perp|^2}\right),$$

with

$$k_\Lambda, q_\Lambda \text{ quarks, } r_\Lambda \text{ gluon:} \quad \mathcal{P}_i(z) = \frac{1}{z} + \frac{(1-\epsilon)z - 2}{2}\,, \quad c_q = N_c\,,\ c_r = \frac{-1}{N_c}$$

$$k_\Lambda, q_\Lambda, r_\Lambda \text{ gluons:} \quad \mathcal{P}_i(z) = \frac{1}{z} + \frac{1}{1-z} - 2 + z(1-z)\,, \quad c_q = c_r = N_c$$

$$k_\Lambda \text{ gluon, } q_\Lambda, r_\Lambda \text{ q-qbar pair:} \quad \mathcal{P}_i(z) = \frac{1}{2} - \frac{z(1-z)}{1-\epsilon}\,, \quad c_q = \frac{1}{2}\,,\ c_r = \frac{-1}{2N_c^2}$$

Auxiliary and radiative momenta

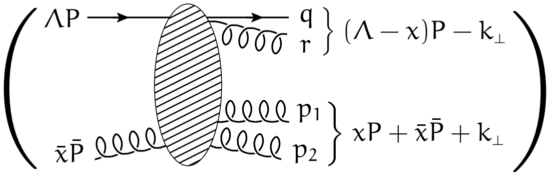
$$\begin{aligned} k_\Lambda^\mu &= \Lambda P^\mu \\ r_\Lambda^\mu &= z(\Lambda - x)P^\mu + r_\perp^\mu + \bar{x}_r \bar{P}^\mu \\ q_\Lambda^\mu &= (1-z)(\Lambda - x)P^\mu - k_\perp^\mu - r_\perp^\mu + \bar{x}_q \bar{P}^\mu \end{aligned}$$

where $\bar{x}_q, \bar{x}_r$ are such that $q^2 = r^2 = 0$, and vanish as $1/\Lambda$. These momenta satisfy.

$$k_\Lambda^\mu - r_\Lambda^\mu - q_\Lambda^\mu \overset{\Lambda\to\infty}{\longrightarrow} xP^\mu + k_\perp^\mu$$

Real target impact factor contribution

Almost identical to the *unfamiliar* real contribution of AvH, Motyka, Ziarko 2022, but with a different phase space restriction: $y_r > Y_\mu + \ln\lambda_1$



$\times\, \theta(Y_\mu + \ln\lambda_1 < y_r)$

$$d\mathcal{R}^{\text{targ}}_{i\bar{\imath}}(\epsilon,\Lambda,\lambda_1,\mu_Y) = dB_{\star\bar{\imath}} \times a_\epsilon N_c \left(\frac{\mu^2}{|k_\perp|^2}\right)^{\epsilon} \left[\frac{1}{\epsilon^2} - \frac{4}{\epsilon}\ln\frac{\Lambda\mu_Y}{\lambda_1 x|k_\perp|} + \bar{\mathcal{R}}_i\right]$$

$$\bar{\mathcal{R}}_{q/\bar{q}} = \frac{3}{\epsilon} - \frac{2\pi^2}{3} + \frac{7}{2} - \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 4\right]$$

$$\bar{\mathcal{R}}_g = \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\frac{11}{2} - \frac{2\pi^2}{3} + \frac{67}{9} - \frac{n_f}{N_c}\left[\frac{2}{3\epsilon} + \frac{10}{9} - \frac{1}{N_c^2}\left(\frac{1}{3\epsilon} - \frac{1}{6}\right)\right]$$

Real plus virtual target impact factor contribution

Combining the virtual target IF contribution and the real target IF contribution, we get

$$dV^{\mathrm{targ}}_{i\bar{\imath}}(\epsilon,\Lambda,\lambda_1,\mu_Y)+dR^{\mathrm{targ}}_{i\bar{\imath}}(\epsilon,\Lambda,\lambda_1,\mu_Y)=dB_{\star\bar{\imath}}\times a_\epsilon\,[\mathcal{V}+\mathcal{R}]^{\mathrm{targ}}_i(\epsilon,\Lambda,\lambda_1,\mu_Y)$$

with

$$[\mathcal{V}+\mathcal{R}]^{\mathrm{targ}}_i(\epsilon,\Lambda,\lambda_1,\mu_Y)=\frac{1}{\epsilon}\left(\frac{\mu^2}{|k_\perp|^2}\right)^{\epsilon}\left[\mathcal{J}_i-2N_c\ln\frac{\Lambda\mu_Y}{\lambda_1 x|k_\perp|}\right]+\frac{\gamma_g}{\epsilon}+2\gamma_g\ln\frac{\mu^2}{|k_\perp|^2}+2\mathcal{K}-\frac{N_c}{2}\ln^2\frac{\mu_Y^2}{|k_\perp|^2}$$

and

$$\mathcal{J}_{q/\bar{q}}=\frac{3N_c}{2}+\frac{N_c}{2}\epsilon\quad,\quad\mathcal{J}_g=\frac{11N_c}{6}+\frac{n_f}{3N_c^2}-\frac{n_f}{6N_c^2}\epsilon$$

$$\gamma_g=\frac{11N_c}{6}-\frac{2T_Rn_f}{3}$$
$$\mathcal{K}=N_c\left(\frac{67}{18}-\frac{\pi^2}{6}\right)-\frac{5n_f}{9}$$

The impact factor corrections from [Ciafaloni, Colferai 1998] are still there, like in the unfamiliar contribution from [AvH, Motyka, Ziarko 2022], but the logarithm is different.

Target collinear counter term

We need to include the collinear counter term for the target-side collinear PDF

$$\frac{d\sigma_{\lambda}^{\mathrm{CF,B}}(\{p\}_n)}{dx\,d^2k_\perp} = \sum_{i,\bar{\imath}} \boxed{\int_0^1 dX\, f_i(X)\,\theta(X > \delta_0)} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, \frac{d\hat{\sigma}_{i\bar{\imath}}^{\mathrm{B}}(\lambda X, \bar{x}; \{p\}_n)}{dx\,d^2k_\perp}$$

Target collinear counter term

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$$\boxed{\int_0^1 dX\, f_i(X)\,\theta(X > \delta_0) + \int_0^1 dX\, \frac{a_\epsilon}{\epsilon}\left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon} \sum_{i'} \left[\mathcal{P}_{ii'} \otimes f_{i'}\theta_{>\delta_0}\right](X)\,\theta(X > \delta_1)}$$

$$\frac{\lambda_0 x|k_\perp|}{\lambda\mu_Y} = \delta_0 \succ \delta_1 = \frac{\lambda_1 x|k_\perp|}{\lambda\mu_Y}$$

Target collinear counter term

We need to include the collinear counter term for the target-side collinear PDF

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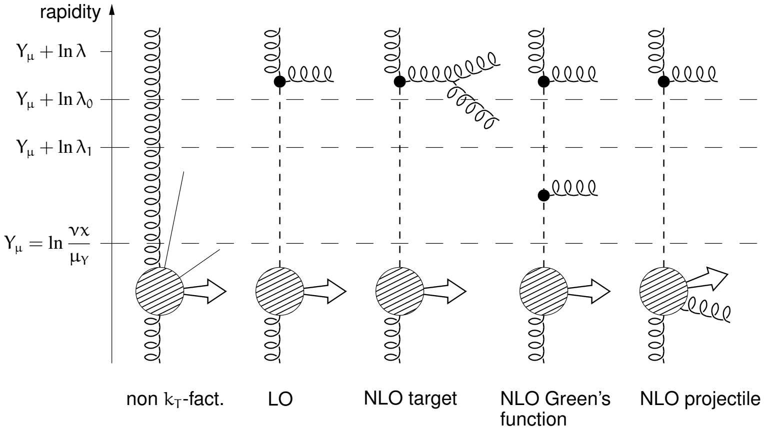
$$\boxed{\int_0^1 dX\, f_i(X)\,\theta(X>\delta_0) + \int_0^1 dX\, \frac{a_\epsilon}{\epsilon}\left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon} \sum_{i'} \big[\mathcal{P}_{ii'} \otimes f_{i'}\theta_{>\delta_0}\big](X)\,\theta(X>\delta_1)}$$

$$\frac{\lambda_0 x|k_\perp|}{\lambda\mu_Y} = \delta_0 \succ \delta_1 = \frac{\lambda_1 x|k_\perp|}{\lambda\mu_Y}$$



Target collinear counter term

We need to include the collinear counter term for the target-side collinear PDF

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$$\boxed{\int_0^1 dX\, f_i(X)\,\theta(X > \delta_0) + \int_0^1 dX\, \frac{a_\epsilon}{\epsilon}\left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon} \sum_{i'} \left[\mathcal{P}_{ii'} \otimes f_{i'}\theta_{>\delta_0}\right](X)\,\theta(X > \delta_1)}$$

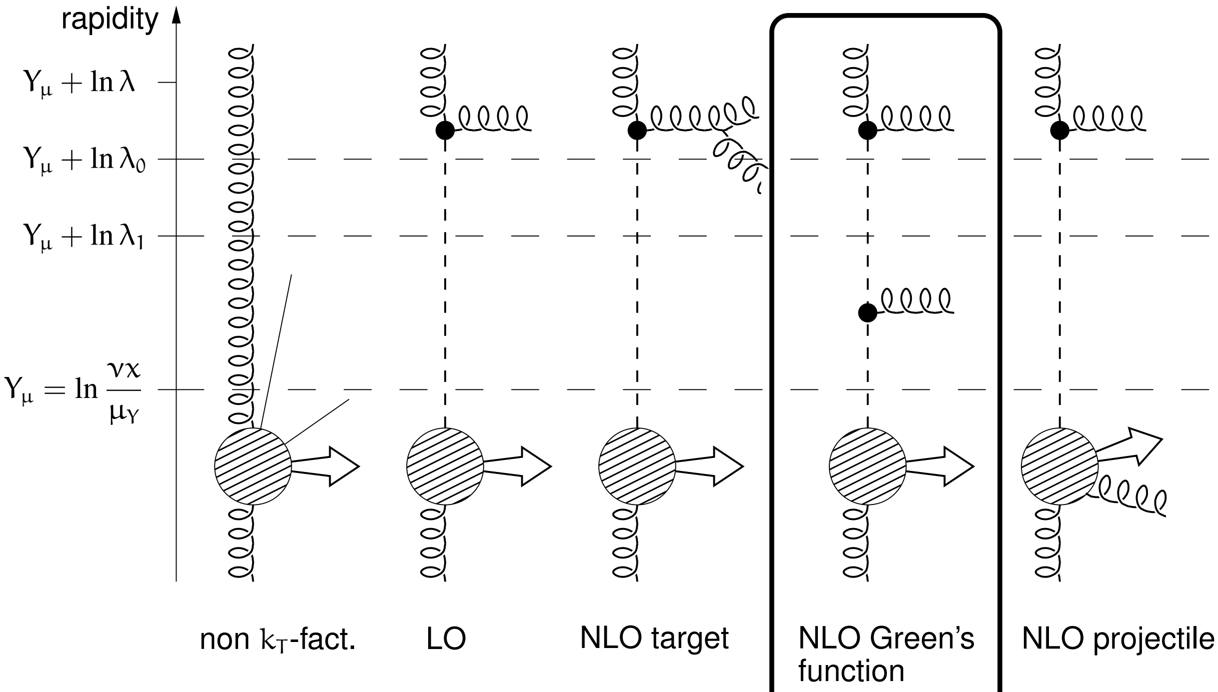
$$\frac{\lambda_0 x|k_\perp|}{\lambda \mu_Y} = \delta_0 \succ \delta_1 = \frac{\lambda_1 x|k_\perp|}{\lambda \mu_Y}$$

Complete finite target impact factor contribution

after proper coupling constant renormalization

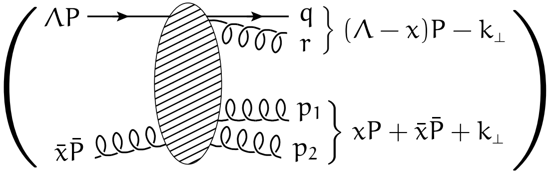
$$[\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\mathrm{targ}}(\Lambda, \lambda_1, \mu_Y, \mu_F; x, k_\perp) = \left[\mathcal{J}_i^{(0)} - 2N_c \ln\frac{\Lambda \mu_Y}{\lambda_1 x|k_\perp|}\right] \ln\frac{\mu_F^2}{|k_\perp|^2} + 2\gamma_g \ln\frac{\mu^2}{|k_\perp|^2} + 2\mathcal{K} + \mathcal{J}_i^{(1)} - \frac{N_c}{2}\ln^2\frac{\mu_Y^2}{|k_\perp|^2}$$

Real Green's function contribution



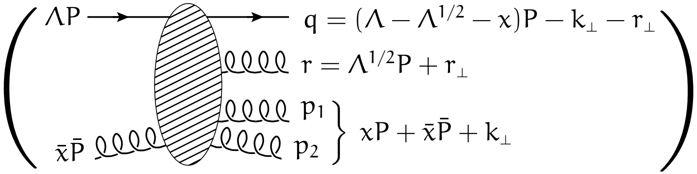
Real Green's function contribution

target



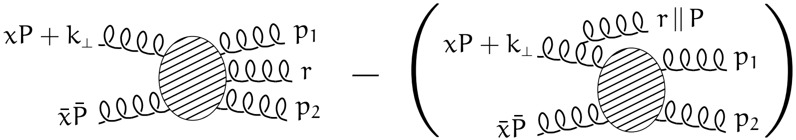
$\times\,\theta\big(Y_\mu + \ln\lambda_1 < y_r\big)$

Green's function



$\times\,\theta\big(Y_\mu < y_r < Y_\mu + \ln\lambda_1\big)$

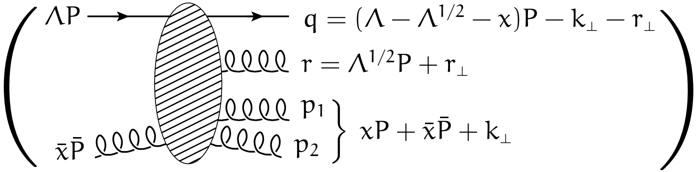
projectile



$\times\,\theta\big(Y_\mu < y_r\big)$

Real Green's function contribution

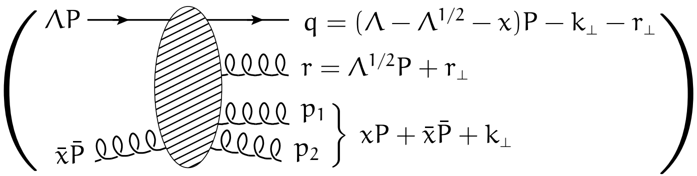
The radiation has rapidity between Y_μ and $Y_\mu + \ln\lambda_1$, with longitudinal momentum becoming infinite, but slower than the auxiliary partons.



$$\times\, \theta\big(Y_\mu < y_r < Y_\mu + \ln\lambda_1\big)$$

Real Green's function contribution

The radiation has rapidity between Y_μ and $Y_\mu + \ln\lambda_1$, with longitudinal momentum becoming infinite, but slower than the auxiliary partons.



$\times\, \theta(Y_\mu < y_r < Y_\mu + \ln\lambda_1)$

Multi-Regge kinematics (MRK):

$$
\begin{aligned}
k_\Lambda^\mu &= \Lambda P^\mu \,, \\
r_\Lambda^\mu &= \sqrt{\Lambda}\, P^\mu + r_\perp^\mu + \bar{x}_r \bar{P}^\mu \,, \quad \bar{x}_r \text{ such that } r^2 = 0 \\
q_\Lambda^\mu &= (\Lambda - \sqrt{\Lambda} - x) P^\mu - k_\perp^\mu - r_\perp^\mu + \bar{x}_q \bar{P}^\mu \,, \quad \bar{x}_q \text{ such that } q^2 = 0
\end{aligned}
$$

For the matrix element in this limit we have:

$$
\frac{x^2|k_\perp|^2}{g_s^4 C_i \Lambda^2} \left|\overline{M}_{i\bar{\imath}}\right|^2 (k_\Lambda, k_{\bar{\imath}}; r_\Lambda, q_\Lambda, \{p\}_n) \stackrel{\Lambda\to\infty}{\longrightarrow} 4N_c \frac{|k_\perp|^2}{|r_\perp|^2 |r_\perp + k_\perp|^2} \left|\overline{M}_{\star\bar{\imath}}\right|^2 (k_\star, k_{\bar{\imath}}; \{p\}_n) \,,
$$

for either case k_Λ, q_Λ quarks, r_Λ gluon, and $k_\Lambda, q_\Lambda, r_\Lambda$ gluons. The universal factor appearing is the square of Lipatov's vertex for the gluon emission in MRK. The limit $z \to 0$ of the target expressions gives the same result, since that limit overlaps with MRK if *e.g.* $z \sim 1/\sqrt{\Lambda}$.

Unintegrated PDF

$$\frac{d\sigma^{\mathrm{CF,B+NLO}}_{\lambda\to\infty,\mathrm{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\mathrm{LO+NLO}}(\cdot)}\sum_{\bar{\imath}}\int_0^1 d\bar{x}\,f_{\bar{\imath}}(\bar{x})\,dB_{\star\bar{\imath}}\big(x,k_\perp,\bar{x};\{p\}_n\big)$$

$$\sum_i \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_{\delta_0}^1 dX\,f_i(X,\mu_F)\Big[1 + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\mathrm{targ}}\big(\lambda X,\lambda_1,\mu_Y,\mu_F\,;x,k_\perp\big) + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]^{\mathrm{Green}}\big(\lambda_1,\mu_F\,;k_\perp\big)\Big]$$

Unintegrated PDF

$$\frac{d\sigma^{\mathrm{CF,B+NLO}}_{\lambda\to\infty,\mathrm{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\mathrm{LO+NLO}}(\cdot)}\sum_{\bar{\imath}}\int_0^1 d\bar{x}\,f_{\bar{\imath}}(\bar{x})\,dB_{\star\bar{\imath}}(x,k_\perp,\bar{x};\{p\}_n)$$

$$\sum_i \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_{\delta_0}^1 dX\,f_i(X,\mu_F)\Big[1 + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\mathrm{targ}}(\lambda X,\lambda_1,\mu_Y,\mu_F\,;x,k_\perp) + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]^{\mathrm{Green}}(\lambda_1,\mu_F\,;k_\perp)\Big]$$

$\lambda\to\infty$ for x-fixed, is equivalent to $x\to 0$ for $\lambda = 1$

set $\lambda_1 = (X\mu_Y)/(x|k_\perp|)$

set $\mu_Y = |k_\perp|$

$$F(x,k_\perp,\mu_Y=|k_\perp|) = \sum_i\int_x^1 dX\,f_i(X,\mu_F)\int d^{2-2\epsilon}k'_\perp\,I_i(k'_\perp,\mu_F)\,G\left(k'_\perp,k_\perp,\frac{X}{x},\mu_F\right)$$

$$\frac{d\sigma^{\text{CF,B+NLO}}_{\lambda\to\infty,\text{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\text{LO+NLO}}(\cdot)} \sum_{\bar{\imath}} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x, k_\perp, \bar{x}\,; \{p\}_n)$$

$$\sum_i \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2} \int_{\delta_0}^1 dX\, f_i(X,\mu_F) \Big[1 + a_\epsilon [\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\text{targ}}(\lambda X, \lambda_1, \mu_Y, \mu_F\,; x, k_\perp) + a_\epsilon [\mathcal{V}+\mathcal{R}-\mathcal{C}]^{\text{Green}}(\lambda_1, \mu_F\,; k_\perp) \Big]$$

$\lambda \to \infty$ for x-fixed, is equivalent to $x \to 0$ for $\lambda = 1$

set $\lambda_1 = (X\mu_Y)/(x|k_\perp|)$

set $\mu_Y = |k_\perp|$

$$F(x, k_\perp, \mu_Y = |k_\perp|) = \sum_i \int_x^1 dX\, f_i(X,\mu_F) \int d^{2-2\epsilon}k'_\perp\, I_i(k'_\perp, \mu_F)\, G\left(k'_\perp, k_\perp, \frac{X}{x}, \mu_F\right)$$

$$I_i(k_\perp, \mu_F) = \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2} \left\{ 1 + \frac{\alpha_s}{2\pi} \left[\mathcal{J}_i^{(0)} \ln\frac{\mu_F^2}{|k_\perp|^2} + 2\gamma_g \ln\frac{\mu^2}{|k_\perp|^2} + 2\mathcal{K} + \mathcal{J}_i^{(1)} \right] + \mathcal{O}(\alpha_s^2) \right\}$$

$$\mathcal{J}_{q/\bar{q}} = \frac{3N_c}{2} + \frac{N_c}{2}\epsilon$$

$$\mathcal{J}_g = \frac{11N_c}{6} + \frac{n_f}{3N_c^2} - \frac{n_f}{6N_c^2}\epsilon$$

Unintegrated PDF

$$\frac{d\sigma^{\mathrm{CF,B+NLO}}_{\lambda\to\infty,\mathrm{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\mathrm{LO+NLO}}(\cdot)} \sum_{\bar{\imath}} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x, k_\perp, \bar{x}; \{p\}_n)$$

$$\sum_i \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2} \int_{\delta_0}^1 dX\, f_i(X, \mu_F) \Big[1 + a_\epsilon [\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\mathrm{targ}}(\lambda X, \lambda_1, \mu_Y, \mu_F; x, k_\perp) + a_\epsilon [\mathcal{V}+\mathcal{R}-\mathcal{C}]^{\mathrm{Green}}(\lambda_1, \mu_F; k_\perp) \Big]$$

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$$F(x, k_\perp, \mu_Y = |k_\perp|) = \sum_i \int_x^1 dX\, f_i(X, \mu_F) \int d^{2-2\epsilon}k'_\perp\, I_i(k'_\perp, \mu_F)\, G\Big(k'_\perp, k_\perp, \frac{X}{x}, \mu_F\Big)$$

$$G(k'_\perp, k_\perp, y, \mu_F) = \delta^{(2-2\epsilon)}(k'_\perp - k_\perp) + a_\epsilon \int_y^1 \frac{dz}{z} \int d^{2-2\epsilon}q_\perp \Big[K_{\mathrm{BFKL}}(k'_\perp, q_\perp) - \theta(\mu_F^2 - |k'_\perp|^2)\, K_{\mathrm{BFKL}}(0, q_\perp) \Big] G\Big(q_\perp, k_\perp, \frac{y}{z}, \mu_F\Big)$$

$$K_{\mathrm{BFKL}}(k'_\perp, q_\perp) = 4N_c \frac{\mu^{2\epsilon}}{2\pi_\epsilon} \Big[\frac{1}{|k'_\perp - q_\perp|^2} + \delta^{(2-2\epsilon)}(k'_\perp - q_\perp) \frac{\pi_\epsilon}{\epsilon} |q_\perp^2|^{-\epsilon} \Big]$$

Unintegrated PDF

$$\frac{d\sigma^{\mathrm{CF,B+NLO}}_{\lambda\to\infty,\mathrm{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\mathrm{LO+NLO}}(\cdot)}\sum_{\bar{\imath}}\int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x,k_\perp,\bar{x};\{p\}_n)$$

$$\boxed{\sum_i \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_{\delta_0}^1 dX\, f_i(X,\mu_F)\Big[1 + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\mathrm{targ}}(\lambda X,\lambda_1,\mu_Y,\mu_F;x,k_\perp) + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]^{\mathrm{Green}}(\lambda_1,\mu_F;k_\perp)\Big]}$$

$\lambda\to\infty$ for x-fixed, is equivalent to $x\to 0$ for $\lambda=1$

set $\lambda_1 = (X\mu_Y)/(x|k_\perp|)$

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$$G(k'_\perp,k_\perp,y,\mu_F) = \delta^{(2-2\epsilon)}(k'_\perp-k_\perp)+a_\epsilon\int_y^1\frac{dz}{z}\int d^{2-2\epsilon}q_\perp\Big[K_{\mathrm{BFKL}}(k'_\perp,q_\perp)-\theta(\mu_F^2-|k'_\perp|^2)\,K_{\mathrm{BFKL}}(0,q_\perp)\Big]G\Big(q_\perp,k_\perp,\frac{y}{z},\mu_F\Big)$$

The initial condition $F(x,x_\perp,\mu_Y=|k_\perp|)$ resums $\ln(X/x)$,
while the μ_Y-evolution resums $\ln(\mu_Y/|k_\perp|)$.

Conclusions

- We have derived a scheme for NLO computations in HEF for arbitrary processes.
- The ambiguity of the separation between projectile and target contributions is resolved with a rapidity scale μ_Y, the evolution of UPDF with respect to which is similar to the CSS evolution.
- This result brings the notion of UPDF of HEF formalism closer to the notion of the TMD PDF in the standard TMD formalism.
- We have derived the matching formula between UPDF and collinear PDF at the NLO in α_s and generalized it to all orders at the scale $\mu_Y = |k_\perp|$, thus providing a first-principle initial condition for the UPDF evolution.
- The BFKL-Collins-Ellis evolution of the Green's function in this initial condition is resumming the logarithms of partonic center of mass energy $\ln(1/x) \sim \ln(\hat{s}/\mu^2)$, which corresponds to the original formulation of HEF.