Spin-entanglement in Hyperon Decays

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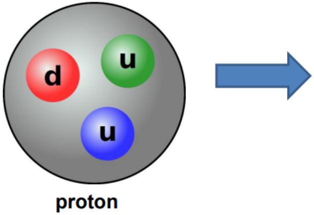
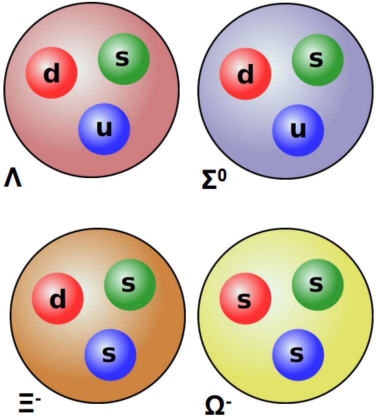
Hyperons

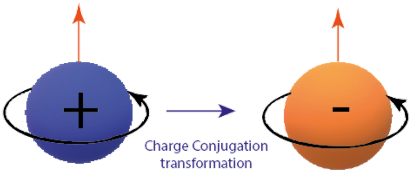
What happens if we replace one of the light quarks in the proton with one - or many - heavier quark(s)?

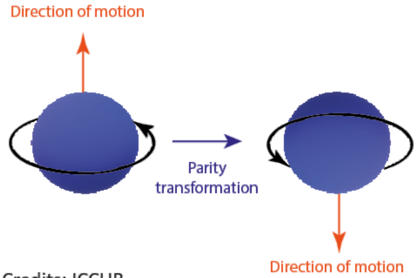




Credits: K. Schönning

CP violation

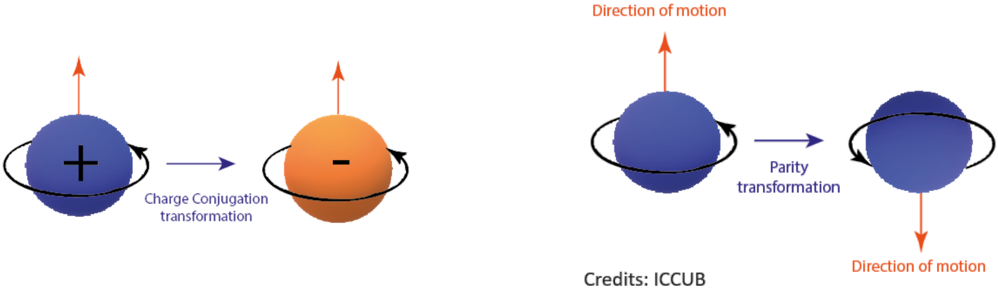




Credits: ICCUB

1. CP = combination of Charge-conjugation and Parity

CP violation



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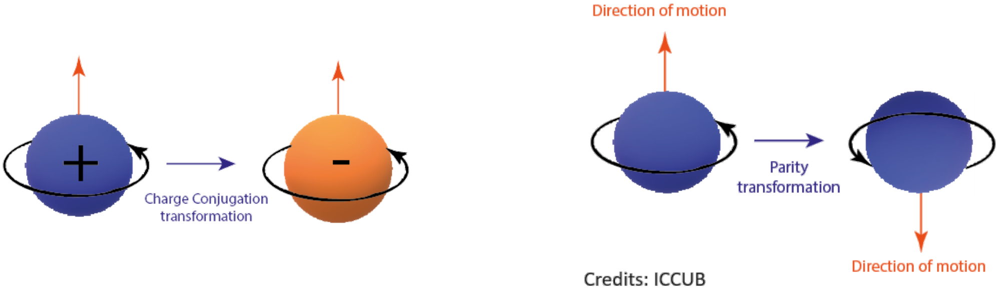

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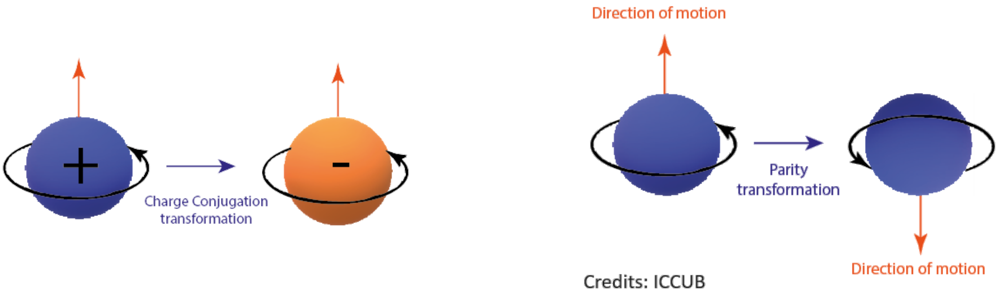

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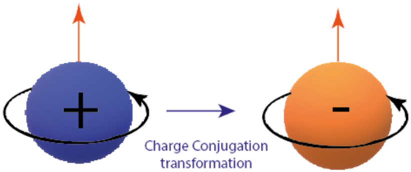

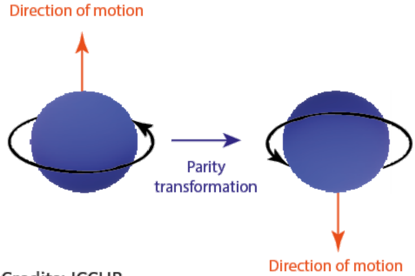

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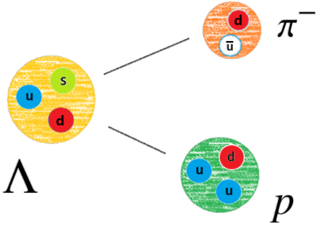
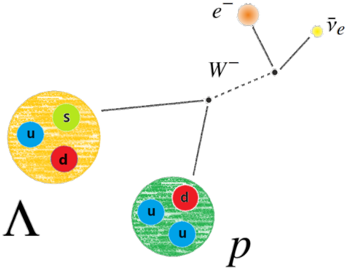
Hyperon Decays, compared





Nonleptonic decays $B \to b\pi$

Semileptonic decays $B \to bl\bar{\nu}_l$

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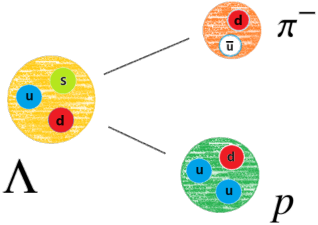

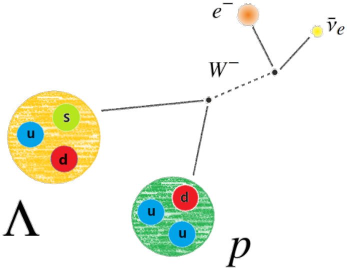

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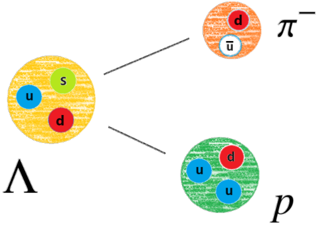

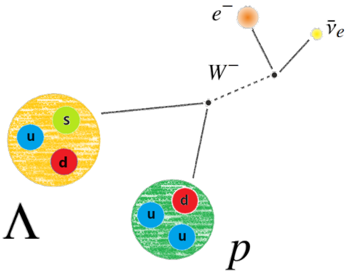

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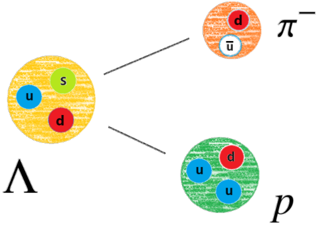

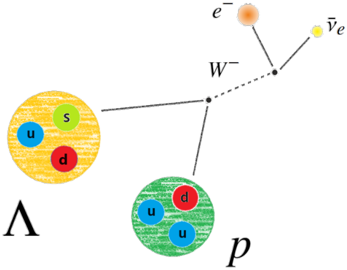

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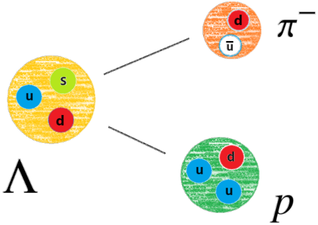

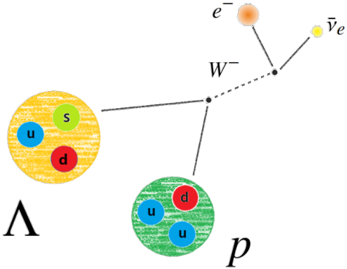

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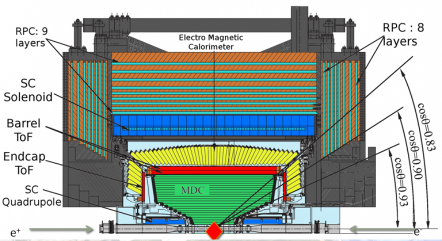
Production framework

Spin-entangled hyperon-antihyperon pairs produced at e^+e^- **colliders** (e.g. BESIII).

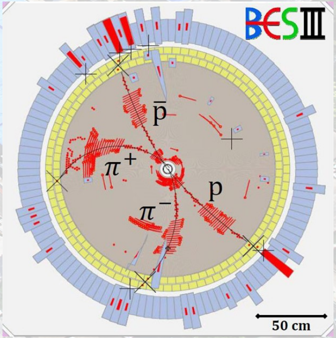
BESIII @ BEPCII

- Beijing **Electron-Positron** Collider (BEPCII)
  - e^+e^- collider: 1.85 GeV < E_{CMS} < 4.95 GeV
  - $L_{\mathrm{peak}} = 10^{33}$ cm^{-2} s^{-1}
  - Data taking since 2009

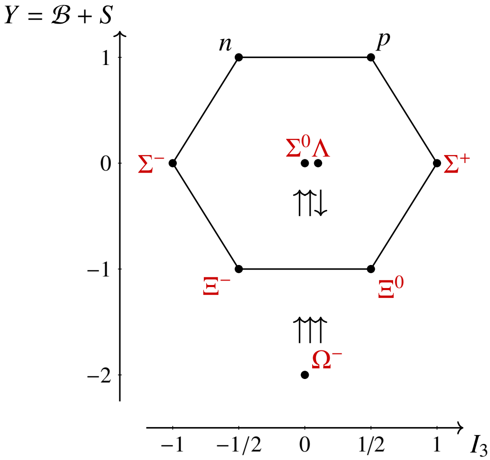
- Beijing **Spectrometer** (BESIII)
  - Optimized for flavor physics
  - Covering 93% of 4π solid angle
  - 1.0 T super-conducting solenoid
  - Momentum resolution: $\sigma(p)/p = 0.5\%$ at 1 GeV/c
  - Time resolution: 68(65) ps in the barrel (end cap)



[Nucl. Instrum. Meth. A598 (2009) 7]



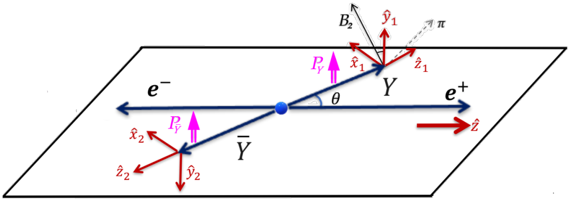
Lowest-lying hyperons



Y	Mass [GeV/c^2]	$\Delta S = 1$ **decays** (Br)
Λ (uds)	1.116	$p\pi^-$(64.1%) $n\pi^0$(35.9%)
Σ^+ (uus)	1.189	$p\pi^0$(51.5%) $n\pi^+$(48.4%)
Σ^- (dds)	1.197	$n\pi^-$(99.8%)
Ξ^0 (uss)	1.315	$\Lambda\pi^0$(99.5%)
Ξ^- (dss)	1.322	$\Lambda\pi^-$(99.9%)
Ω^- (sss)	1.672	ΛK^-(67.7%) $\Xi^0\pi^-$(24.3%) $\Xi^-\pi^0$(8.55%)

Lowest-lying hyperons at BESIII

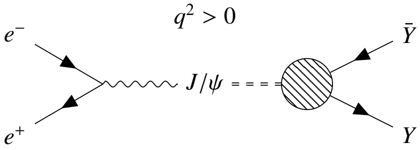
- World's largest charmonia sample – $10^{10} J/\psi,\ 3 \times 10^{9}\ \psi(2S)$
- Baryon-antibaryon pairs produced in **spin-entangled**, possibly **polarized** state.

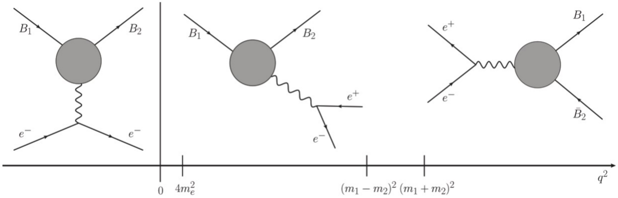


	Decay	$Br(\times 10^{-4})$	$\epsilon(\%)$	$N_{obs} \times 10^3$	Reference
	$\Lambda\bar{\Lambda}$	19.43(33)	42.37(14)	441	PRD 95 (2017) 5, 052003
	$\Sigma^0\bar{\Sigma}^0$	11.64(23)	17.83(06)	111	''
J/ψ	$\Sigma^+\bar{\Sigma}^-$	10.61(36)	24.1(7)	87	JHEP 11 (2021) 226
	$\Xi^0\bar{\Xi}^0$	11.65(43)	14.05(04)	135	PLB 770 (2017) 217-225
	$\Xi^-\bar{\Xi}^+$	10.40(74)	18.40(04)	43	PRD 93 (2016) 7, 072003
	$\Lambda\bar{\Lambda}$	3.97(12)	42.83(34)	31	PRD 95 (2017) 5, 052003
	$\Sigma^0\bar{\Sigma}^0$	2.44(11)	14.79(12)	6.6	''
	$\Sigma^+\bar{\Sigma}^-$	2.52(10)	18.6(5)	5.4	JHEP 11 (2021) 226
$\psi(2S)$	$\Sigma^-\bar{\Sigma}^+$	2.82(09)	5.26(5)	6.6	JHEP 12 (2022) 016
	$\Xi^0\bar{\Xi}^0$	2.73(13)	14.10(04)	11	PLB 770 (2017) 217-225
	$\Xi^-\bar{\Xi}^+$	2.78(15)	18.04(04)	5.3	PRD 95 (2017) 5, 052003
	$\Omega^-\bar{\Omega}^+$	0.585(28)	15.39(32)	4	PRL 126 (2021) 9, 092002

Baryon polarization

Produced $B\bar{B}$ in $e^+e^- \to \gamma^*$ reaction can be **polarized**.





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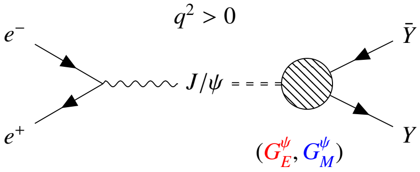

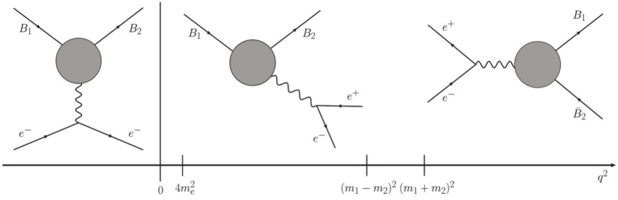

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- Sachs **form-factors** $G_{E,M}^\psi$ parametrize the $\psi \to Y\bar{Y}$ vertex.

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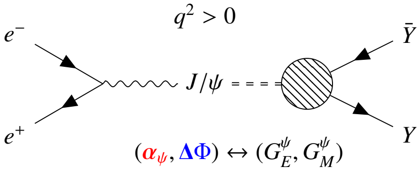

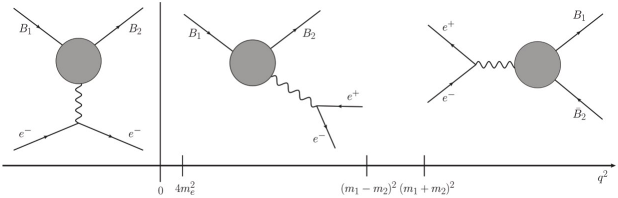

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- Sachs **form-factors** $G_{E,M}^\psi$ parametrize the $\psi \to Y\bar{Y}$ vertex.
- Annihilation process: time-like $q^2 > M_Y^2$, i.e. **complex form-factors**.

$$\alpha_\psi = \frac{q^2|G_M^\psi|^2 - 4M_Y^2|G_E^\psi|^2}{q^2|G_M^\psi|^2 + 4M_Y^2|G_E^\psi|^2}, \qquad \frac{G_E^\psi}{G_M^\psi} = e^{i\Delta\Phi}\left|\frac{G_E^\psi}{G_M^\psi}\right|$$

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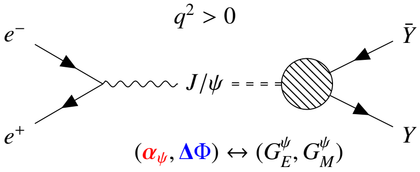
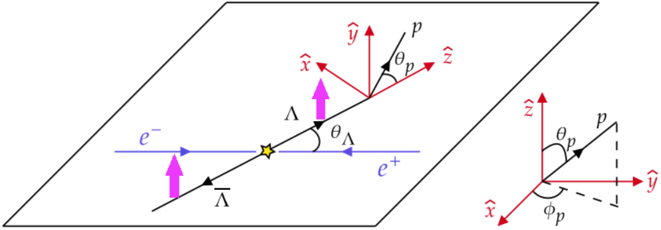




$e^+e^- \to J/\psi \to \Lambda\bar{\Lambda}$ process [BESIII, PRL 129 (2022) 131801]

$$\mathbf{P}_\Lambda = \sqrt{1-\alpha_\psi{}^2}\,\frac{\sin(\Delta\Phi)\cos\theta_\Lambda\sin\theta_\Lambda}{1+\alpha_\psi\cos^2\theta_\Lambda}\hat{\mathbf{y}}$$

Starting point

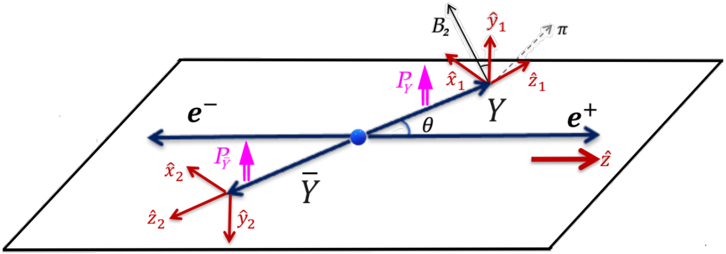
Goal

Spin properties of $e^+e^- \to B\bar{B}$

$\Longleftarrow$

Tool

All info is encoded in the **spin density matrix**.



- Production defines CM frame;
- for subsequent decays, we need to go to B, $\bar{B}$ helicity frames $\Longrightarrow$ **helicity formalism**

Helicity formalism

[M. Jacob and G.C. Wick, Ann. Phys. 7, 404 (1959)]

- Spin quantization axis = flight direction $\Longrightarrow$ helicity is boost-invariant.

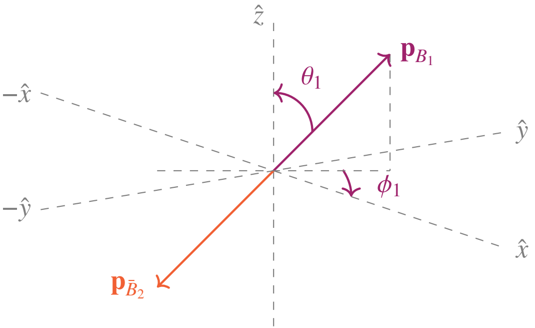
Before

Boost (L_z) to rest frame of mother (B, $\bar{B}$):

- Perform 3 rotations around the coordinate axes:
  1. $-\phi_1$ around $\hat{z}$;
  2. $-\theta_1$ around $\hat{y}$;
  3. $+\phi_1$ around $\hat{z}$ (extra);

After

$\mathbf{p}_{B(\bar{B})}$ is aligned with $\hat{z}$.



A bit of theory

Helicity states:

$$|p, \theta_1, \phi_1, \lambda_1, \lambda_2\rangle := R(\phi_1, \theta_1, 0)\,|p, \lambda_1, \lambda_2\rangle$$

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$$|p, \theta_1, \phi_1, \lambda_1, \lambda_2\rangle := R(\phi_1, \theta_1, 0)\,|p, \lambda_1, \lambda_2\rangle$$

Production spin-density matrix:

$$\rho_{B_1,\bar{B}_2}^{\lambda_1,\lambda_2;\lambda_1',\lambda_2'} \propto \sum_{k=\pm 1} \langle \theta_1, 0, \lambda_1, \lambda_2 | S | 0, 0, \lambda, -\lambda\rangle \langle 0, 0, \lambda, -\lambda | S^\dagger | \theta_1, 0, \lambda_1', \lambda_2'\rangle$$
$$\propto A_{\lambda_1,\lambda_2} A^*_{\lambda_1',\lambda_2'}\, \rho_1^{\lambda_1-\lambda_2,\lambda_1'-\lambda_2'}(\theta_1)$$

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Transition amplitude matrix

$$J = 1/2: \qquad A_{\lambda_1,\lambda_2} = \begin{pmatrix} h_1 & h_2 \\ h_2 & h_1 \end{pmatrix}$$

$$h_1 = \sqrt{\frac{1-\alpha_\psi}{2}}; \quad h_2 = \sqrt{1+\alpha_\psi}\,e^{-i\Delta\Phi}$$

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e^+e^- **spin-density** matrix

$$\rho_1^{i,j}(\theta_1) = \sum_{k=\pm1} [\mathcal{D}^1_{k,i}(0,\theta_1,0)]^* \mathcal{D}^1_{k,j}(0,\theta_1,0)$$

$$\mathcal{D}^j_{m'm}(\alpha,\beta,\gamma) = \langle jm'|R(\alpha,\beta,\gamma)|jm\rangle$$

This is the result!

[F. Tabakin and R. A. Eisenstein, Phys. Rev. C 31, 1857 (1985)], [E. Perotti et al., Phys. Rev. D 99, 056008 (2019)]

Spin-density matrix for $1/2 + \overline{1/2}$:

$$\rho_{B,\bar{B}} = \frac{1}{4} \sum_{\mu,\bar{\nu}=0}^{3} C_{\mu\bar{\nu}}(\theta)\, \sigma_{\mu}^{B} \otimes \sigma_{\bar{\nu}}^{\bar{B}}$$

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$$\boxed{C_{\mu\nu}(\theta)} \propto \begin{pmatrix} 1+\alpha_\psi \cos^2\theta & 0 & \beta_\psi \sin\theta\cos\theta & 0 \\ 0 & \sin^2\theta & 0 & \gamma_\psi \sin\theta\cos\theta \\ -\beta_\psi \sin\theta\cos\theta & 0 & \alpha_\psi \sin^2\theta & 0 \\ 0 & -\gamma_\psi \sin\theta\cos\theta & 0 & -\alpha_\psi - \cos^2\theta \end{pmatrix}$$

$$\beta_\psi = \sqrt{1-{\alpha_\psi}^2}\,\sin(\Delta\Phi), \quad \gamma_\psi = \sqrt{1-{\alpha_\psi}^2}\,\cos(\Delta\Phi)$$

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$\frac{d\sigma}{d\Omega}$ $\bar{B}$ polarization

$$C_{\mu\nu}(\theta) \propto \begin{pmatrix} 1+\alpha_\psi \cos^2\theta & 0 & \beta_\psi \sin\theta\cos\theta & 0 \\ 0 & \sin^2\theta & 0 & \gamma_\psi \sin\theta\cos\theta \\ -\beta_\psi \sin\theta\cos\theta & 0 & \alpha_\psi \sin^2\theta & 0 \\ 0 & -\gamma_\psi \sin\theta\cos\theta & 0 & -\alpha_\psi - \cos^2\theta \end{pmatrix}$$

B polarization spin-correlation terms

$$\beta_\psi = \sqrt{1-\alpha_\psi{}^2}\,\sin(\boldsymbol{\Delta\Phi}), \quad \gamma_\psi = \sqrt{1-\alpha_\psi{}^2}\,\cos(\boldsymbol{\Delta\Phi})$$

Let's go back to something simpler

Spin-density matrix for $e^+e^- \to \tau^+\tau^-$ (also $1/2 + \overline{1/2}$!):

$$\rho_{B,\bar{B}} = \frac{1}{4}\sum_{\mu,\bar{\nu}=0}^{3} \boxed{C_{\mu\bar{\nu}}(\theta)}\, \sigma_\mu^B \otimes \sigma_{\bar{\nu}}^{\bar{B}}, \qquad \text{pointlike } \tau : \Delta\Phi = 0,\ G_{E,M}^{\psi} \to 1$$

Let's go back to something simpler

Spin-density matrix for $e^+e^- \to \tau^+\tau^-$ (also $1/2 + \overline{1/2}$!):

$$\rho_{B,\bar{B}} = \frac{1}{4}\sum_{\mu,\bar{\nu}=0}^{3} \boxed{C_{\mu\bar{\nu}}(\theta)}\,\sigma_\mu^B \otimes \sigma_{\bar{\nu}}^{\bar{B}}, \qquad \text{pointlike } \tau : \Delta\Phi = 0,\ G_{E,M}^{\psi} \to 1$$

$$\boxed{C_{\mu\nu}(\theta)} \propto \begin{pmatrix} 1+\alpha_\psi \cos^2\theta & 0 & \beta_\psi \sin\theta\cos\theta & 0 \\ 0 & \sin^2\theta & 0 & \gamma_\psi \sin\theta\cos\theta \\ -\beta_\psi \sin\theta\cos\theta & 0 & \alpha_\psi \sin^2\theta & 0 \\ 0 & -\gamma_\psi \sin\theta\cos\theta & 0 & -\alpha_\psi - \cos^2\theta \end{pmatrix}$$

$$\beta_\psi = \sqrt{1-\alpha_\psi{}^2}\,\sin(\Delta\Phi), \quad \gamma_\psi = \sqrt{1-\alpha_\psi{}^2}\,\cos(\Delta\Phi)$$

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$$C_{\mu\nu}(\theta) \propto \begin{pmatrix} 1+\eta\cos^2\theta & 0 & 0 & 0 \\ 0 & \sin^2\theta & 0 & \sqrt{1-\eta^2}\sin\theta\cos\theta \\ 0 & 0 & \eta\sin^2\theta & 0 \\ 0 & -\sqrt{1-\eta^2}\sin\theta\cos\theta & 0 & -\eta-\cos^2\theta \end{pmatrix}$$

$$\alpha_\psi \to \eta = \frac{q^2-4m_\tau^2}{q^2+4m_\tau^2}, \quad \beta_\psi \to 0, \quad \gamma_\psi \to \sqrt{1-\eta^2}$$

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τ^+ polarization

$$C_{\mu\nu}(\theta) \propto \begin{pmatrix} 1+\eta\cos^2\theta & 0 & 0 & 0 \\ 0 & \sin^2\theta & 0 & \sqrt{1-\eta^2}\sin\theta\cos\theta \\ 0 & 0 & \eta\sin^2\theta & 0 \\ 0 & -\sqrt{1-\eta^2}\sin\theta\cos\theta & 0 & -\eta-\cos^2\theta \end{pmatrix}$$

τ^- polarization — spin-correlation terms

$$\alpha_\psi \to \eta = \frac{q^2 - 4m_\tau^2}{q^2 + 4m_\tau^2}, \quad \beta_\psi \to 0, \quad \gamma_\psi \to \sqrt{1-\eta^2}$$

Production spin-density matrix

[F. Tabakin and R. A. Eisenstein, Phys. Rev. C 31, 1857 (1985)], [E. Perotti et al., Phys. Rev. D 99, 056008 (2019)]

$B\bar{B}$ spin-density matrix for $1/2 + \overline{1/2}$

$$\rho_{B,\bar{B}} = \frac{1}{4} \sum_{\mu,\bar{\nu}=0}^{3} C_{\mu\bar{\nu}}(\theta)\, \sigma_{\mu}^{B} \otimes \sigma_{\bar{\nu}}^{\bar{B}}$$

Goal

Derive the **production** spin-density matrix $\rho_{B,\bar{B}}$ with e^- **beam polarization**.

Production spin-density matrix

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$B\bar{B}$ spin-density matrix for $1/2 + \overline{1/2}$

$$\rho_{B,\bar{B}} = \frac{1}{4} \sum_{\mu,\bar{\nu}=0}^{3} C_{\mu\bar{\nu}}(\theta)\, \sigma_{\mu}^{B} \otimes \sigma_{\bar{\nu}}^{\bar{B}}$$

Goal

Derive the **production** spin-density matrix $\rho_{B,\bar{B}}$ with e^- **beam polarization**.

Longitudinally polarized electron beam:

- Envisioned improvement @ next-generation e^+e^- colliders (e.g. Super Tau-Charm Factory in China).
- At J/ψ energies, $P_e = 0.8 - 0.9$ is achievable.

Production spin-density matrix

At current e^+e^- colliders, e^- beam is **not polarized**:

$$C_{\mu\nu}(\theta) \propto \begin{pmatrix} 1+\alpha_\psi \cos^2\theta & 0 & \beta_\psi \sin\theta\cos\theta & 0 \\ 0 & \sin^2\theta & 0 & \gamma_\psi \sin\theta\cos\theta \\ -\beta_\psi \sin\theta\cos\theta & 0 & \alpha_\psi \sin^2\theta & 0 \\ 0 & -\gamma_\psi \sin\theta\cos\theta & 0 & -\alpha_\psi - \cos^2\theta \end{pmatrix}$$

$$\beta_\psi = \sqrt{1-\alpha_\psi{}^2}\,\sin(\Delta\Phi), \quad \gamma_\psi = \sqrt{1-\alpha_\psi{}^2}\,\cos(\Delta\Phi)$$

Production spin-density matrix

If e^- beam is **polarized** [NS, A. Kupść, V. Batozskaya et al., PRD 105, 116022 (2022)]:

$$C_{\mu\nu}(\theta) \propto \begin{pmatrix} 1+\alpha_\psi \cos^2\theta & \gamma_\psi \boldsymbol{P_e} \sin\theta & \beta_\psi \sin\theta\cos\theta & (1+\alpha_\psi)\boldsymbol{P_e}\cos\theta \\ \gamma_\psi \boldsymbol{P_e}\sin\theta & \sin^2\theta & 0 & \gamma_\psi \sin\theta\cos\theta \\ -\beta_\psi \sin\theta\cos\theta & 0 & \alpha_\psi \sin^2\theta & -\beta_\psi \boldsymbol{P_e}\sin\theta \\ -(1+\alpha_\psi)\boldsymbol{P_e}\cos\theta & -\gamma_\psi \sin\theta\cos\theta & -\beta_\psi \boldsymbol{P_e}\sin\theta & -\alpha_\psi - \cos^2\theta \end{pmatrix}$$

$$\beta_\psi = \sqrt{1-\alpha_\psi{}^2}\,\sin(\Delta\Phi), \quad \gamma_\psi = \sqrt{1-\alpha_\psi{}^2}\,\cos(\Delta\Phi)$$

Production spin-density matrix

If e^- beam is **polarized** [NS, A. Kupść, V. Batozskaya et al., PRD 105, 116022 (2022)]:

$\bar{B}$ polarization

$$C_{\mu\nu}(\theta) \propto \begin{pmatrix} 1+\alpha_\psi \cos^2\theta & \gamma_\psi \boldsymbol{P_e} \sin\theta & \beta_\psi \sin\theta\cos\theta & (1+\alpha_\psi)\boldsymbol{P_e}\cos\theta \\ \gamma_\psi \boldsymbol{P_e} \sin\theta & \sin^2\theta & 0 & \gamma_\psi \sin\theta\cos\theta \\ -\beta_\psi \sin\theta\cos\theta & 0 & \alpha_\psi \sin^2\theta & -\beta_\psi \boldsymbol{P_e} \sin\theta \\ -(1+\alpha_\psi)\boldsymbol{P_e}\cos\theta & -\gamma_\psi \sin\theta\cos\theta & -\beta_\psi \boldsymbol{P_e} \sin\theta & -\alpha_\psi - \cos^2\theta \end{pmatrix}$$

B polarization

spin-correlation terms

$$\beta_\psi = \sqrt{1-\alpha_\psi{}^2}\,\sin(\Delta\Phi), \quad \gamma_\psi = \sqrt{1-\alpha_\psi{}^2}\,\cos(\Delta\Phi)$$

Production spin-density matrix

If e^- beam is **polarized** [NS, A. Kupść, V. Batozskaya et al., PRD 105, 116022 (2022)]:

$$C_{\mu\nu}(\theta) \propto \begin{pmatrix} 1+\alpha_\psi \cos^2\theta & \gamma_\psi P_e \sin\theta & \beta_\psi \sin\theta\cos\theta & (1+\alpha_\psi)P_e\cos\theta \\ \gamma_\psi P_e \sin\theta & \sin^2\theta & 0 & \gamma_\psi \sin\theta\cos\theta \\ -\beta_\psi \sin\theta\cos\theta & 0 & \alpha_\psi \sin^2\theta & -\beta_\psi P_e \sin\theta \\ -(1+\alpha_\psi)P_e\cos\theta & -\gamma_\psi\sin\theta\cos\theta & -\beta_\psi P_e\sin\theta & -\alpha_\psi - \cos^2\theta \end{pmatrix}$$

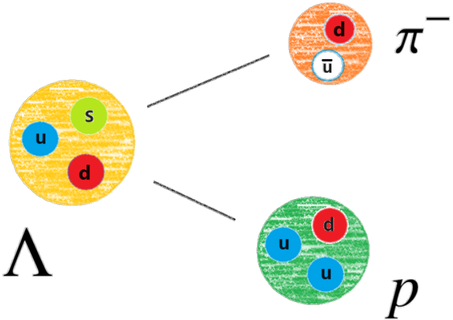
$\bar{B}$ polarization (first row, columns 2–4); B polarization (first column, rows 2–4); spin-correlation terms (lower-right 3×3 block).

$$\beta_\psi = \sqrt{1-\alpha_\psi{}^2}\,\sin(\Delta\Phi), \quad \gamma_\psi = \sqrt{1-\alpha_\psi{}^2}\,\cos(\Delta\Phi)$$

Beam polarization $P_e \neq 0$ introduces additional components in the $\mathbf{P}_{B,\bar{B}}$ polarization vectors and in the spin-correlation terms.

Nonleptonic decays

The e^+e^--produced $Y\bar{Y}$ pairs decay further, e.g.



$$J = S_b + S_\pi + L_{b\pi}$$

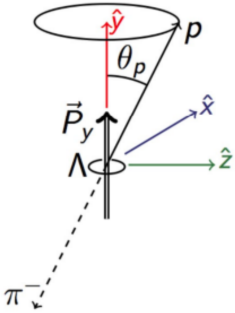
$$B(1/2) \to b(1/2)\, \pi(0)$$

$$L_{b\pi} = 0 \implies S\text{-wave, parity-violating}$$
$$L_{b\pi} = 1 \implies P\text{-wave, parity-conserving}$$

$$\mathcal{M} \propto \bar{u}_b\, (S + P\gamma_5)\, u_B$$

Nonleptonic decay parameters

From partial waves to observables:



$\Lambda \to p\pi^-$ decay

Nonleptonic decay parameters

From partial waves to observables:

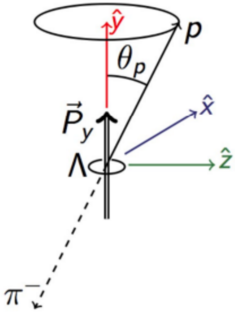
- Angular distribution $\frac{d\Gamma}{d\Omega} \propto 1 + \alpha\, \mathbf{P}_\Lambda \cdot \hat{\mathbf{n}}$

$$\alpha = \frac{2\Re(S^* P)}{|S|^2 + |P|^2}$$

- Spin $\mathbf{s}_\Lambda \to \mathbf{s}_p$ rotation

$$\beta = \frac{2\Im(S^* P)}{|S|^2 + |P|^2} = \sqrt{1 - \alpha^2} \sin\phi$$

measurable with $\mathbf{P}_\Lambda$, $\mathbf{P}_p$.



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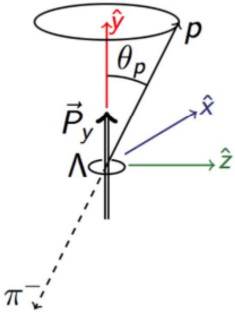

$\Lambda \to p\pi^-$ decay

CP tests [P. Adlarson, A. Kupść, PRD 100 (2019) 114005]

$$A_{CP} = \frac{\alpha + \bar{\alpha}}{\alpha - \bar{\alpha}}, \quad B_{CP} := \frac{\beta + \bar{\beta}}{\alpha - \bar{\alpha}}, \quad \Phi_{CP} = \frac{\phi + \bar{\phi}}{2}$$

Motivation: new data landscape

nature physics

LETTERS

https://doi.org/10.1038/s41567-019-0494-8

Polarization and entanglement in baryon–antibaryon pair production in electron–positron annihilation

The BESIII Collaboration*

[Nature Phys. 15 (2019) 631]

Article | Open Access | Published: 01 June 2022

Probing CP symmetry and weak phases with entangled double-strange baryons

The BESIII Collaboration

Nature **606**, 64–69 (2022) | Cite this article

11k Accesses | **7** Citations | **96** Altmetric | Metrics

[Nature 606, 64–69 (2022)]

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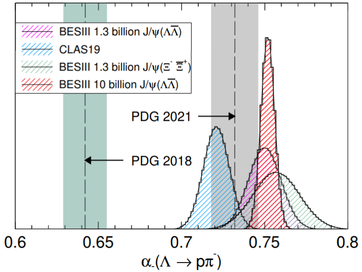
Precise Measurements of Decay Parameters and CP Asymmetry with Entangled Λ–$\bar{\Lambda}$ Pairs

M. Ablikim *et al.* (BESIII Collaboration)

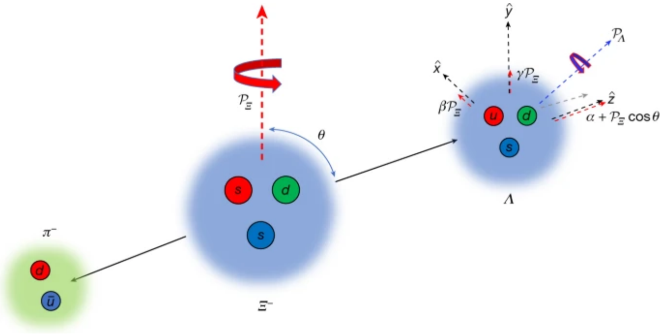
Phys. Rev. Lett. **129**, 131801 – Published 22 September 2022

[Phys.Rev.Lett. 129 (2022) 131801]



Two-step decays [NS, A. Kupść, V. Batozskaya et al., PRD 105, 116022 (2022)]

For $\Xi^- \to \Lambda(\to p\pi^-)\pi^-$, simultaneous A^{Ξ}_{CP}, Φ^{Ξ}_{CP} measurements are possible.



$$\mathbf{P}_\Lambda \cdot \hat{\mathbf{z}} = \frac{\alpha_\Xi + \mathbf{P}_\Xi \cdot \hat{\mathbf{z}}}{1 + \alpha_\Xi \mathbf{P}_\Xi \cdot \hat{\mathbf{z}}}$$

$$\mathbf{P}_\Lambda \times \hat{\mathbf{z}} = |\mathbf{P}_\Xi| \sqrt{1 - {\alpha_\Xi}^2}\, \frac{\sin\phi_\Xi \hat{\mathbf{x}} + \cos\phi_\Xi \hat{\mathbf{y}}}{1 + \alpha_\Xi \mathbf{P}_\Xi \cdot \hat{\mathbf{z}}}$$

$\Xi^- \to \Lambda(\to p\pi^-)\pi^-$ decay [Nature 606 (2022) 64–69]

Two-step decays [NS, A. Kupść, V. Batozskaya et al., PRD 105, 116022 (2022)]

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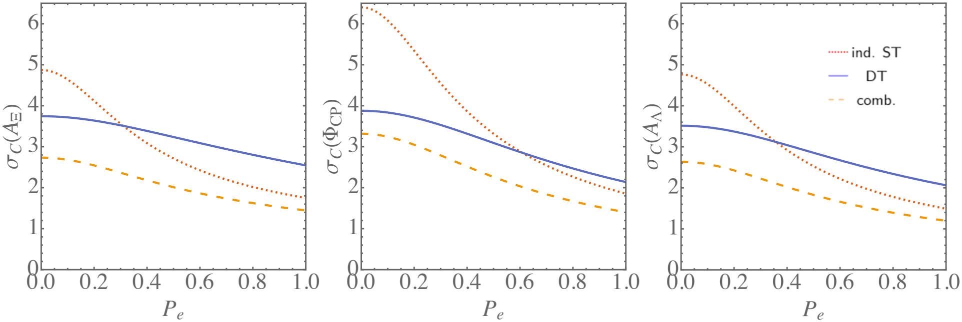
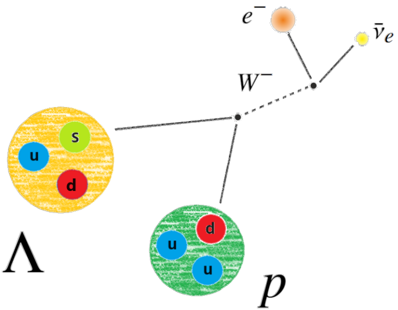


Results

- CP tests statistical uncertainties are reduced with a polarized e^- beam.
- Impact differs btw methods of event reconstruction: highest on Single Tag.

Semileptonic decays

The e^+e^--produced $Y\bar{Y}$ pairs can also decay **semileptonically**:



$$B(1/2) \to b(1/2)\, W^-(1)(\to l\bar{\nu}_l)$$

$$\langle b|J^V_\mu + J^A_\mu|B\rangle = \bar{u}(p_b)\Big[\gamma_\mu \left(F^V_1(q^2) - F^A_1(q^2)\gamma_5\right) - \frac{i\sigma_{\mu\nu}q^\nu}{M_B}\left(F^V_2(q^2) - F^A_2(q^2)\gamma_5\right) + \frac{q_\mu}{M_B}\left(F^V_3(q^2) - F^A_3(q^2)\gamma_5\right)\Big]u(p_B)$$

Form factors

- analytic functions of transferred momentum q^2
- related to baryon internal structure

Semileptonic decay matrix

The produced $B\bar{B}$ pair can decay via nonleptonically or semileptonically, e.g. $\Lambda\bar{\Lambda} \to pe^-\bar{\nu}_e\,\bar{p}\pi^+$.

How to extract the angular distributions of the $B\bar{B}$ chains, keeping track of **spin-correlation**?

Change of basis [E. Perotti et al., Phys. Rev. D 99, 056008 (2019)]:

$$\text{insert} \quad \sigma^m_\mu \to \sum_{\nu=0}^{3} a_{\mu\nu}\sigma^d_\nu \quad \text{into} \quad \rho_{B,\bar{B}} = \sum_{\mu,\bar{\nu}=0}^{3} \boxed{C_{\mu\bar{\nu}}(\theta)}\,\sigma^B_\mu \otimes \sigma^{\bar{B}}_{\bar{\nu}}$$

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$$\mathrm{Tr}\,\rho_{p\bar{p}} \propto \sum_{\mu,\bar{\nu}=0}^{3} \boxed{C^{\Lambda\bar{\Lambda}}_{\mu\bar{\nu}}(\theta)}\,\mathcal{B}^{\Lambda p}_{\mu 0}\, a^{\bar{\Lambda}\bar{p}}_{\bar{\nu}0}$$

Feature

This is a modular approach!

Semileptonic decay matrix

The produced $B\bar{B}$ pair can decay via nonleptonically or semileptonically, e.g. $\Lambda\bar{\Lambda} \to pe^-\bar{\nu}_e\,\bar{p}\pi^+$.

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$$\mathrm{Tr}\,\rho_{p\bar{p}} \propto \sum_{\mu,\bar{\nu}=0}^{3} C^{\Xi\bar{\Xi}}_{\mu\bar{\nu}}(\theta) \sum_{\mu'=0}^{3} a^{\Xi\Lambda}_{\mu\mu'}\mathcal{B}^{\Lambda p}_{\mu' 0} \sum_{\bar{\nu}'=0}^{3} a^{\bar{\Xi}\bar{\Lambda}}_{\bar{\nu}\bar{\nu}'} a^{\bar{\Lambda}\bar{p}}_{\bar{\nu}' 0}$$

SL decay matrix

Feature

This is a modular approach!

$$a_{\mu\nu} \longrightarrow \mathcal{B}_{\mu\nu}$$

Aligned decay matrix [V. Batozskaya, A. Kupść, NS et al., Phys. Rev. D 108, 016011 (2023)]

Full semileptonic transition amplitude can be factorized into

$m-d$ rotation dependence

$$\mathcal{A}(B \to b l \bar{\nu}_l) = \frac{1}{2\pi} \sum_{\lambda'=-1/2}^{1/2} \boxed{\mathcal{D}^{1/2*}_{\kappa,\lambda'}(\Omega_b)} \boxed{\mathcal{H}_{\lambda',\lambda_b}(\Omega_l, q^2, \lambda_l, \lambda_\nu)}$$

decay info

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decay info

SL decay matrix

$$\mathcal{B}_{\mu\nu} = \sum_{\kappa=0}^{3} \boxed{\mathcal{R}^{(4)}_{\mu\kappa}(\Omega_b)} \boxed{b_{\kappa\nu}(q^2, \Omega_l)} \quad \text{with} \quad \mathcal{R}^{(4)}(\Omega_b) = \text{diag}\,(1, \mathcal{R}(\Omega_b))$$

Aligned decay matrix [V. Batozskaya, A. Kupść, NS et al., Phys. Rev. D 108, 016011 (2023)]

Full semileptonic transition amplitude can be factorized into

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$$\mathcal{A}(B\to bl\bar{\nu}_l)=\frac{1}{2\pi}\sum_{\lambda'=-1/2}^{1/2}\boxed{\mathcal{D}^{1/2*}_{\kappa,\lambda'}(\Omega_b)}\boxed{\mathcal{H}_{\lambda',\lambda_b}(\Omega_l,q^2,\lambda_l,\lambda_\nu)}$$

decay info

SL decay matrix

$$\mathcal{B}_{\mu\nu}=\sum_{\kappa=0}^{3}\boxed{\mathcal{R}^{(4)}_{\mu\kappa}(\Omega_b)}\boxed{b_{\kappa\nu}(q^2,\Omega_l)}\quad\text{with}\quad \mathcal{R}^{(4)}(\Omega_b)=\text{diag}\,(1,\mathcal{R}(\Omega_b))$$

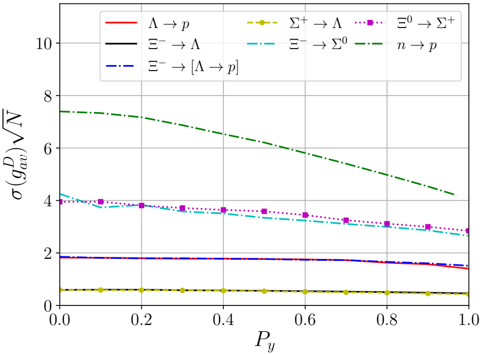
Features

- $b_{\kappa\nu}(q^2,\Omega_l)$ SL decay matrix when $m-d$ frames are aligned, i.e. $\Omega_b=\{0,0,0\}$.
- The decomposition applies to any transition between spin-1/2 baryons.

SL FF uncertainties [V. Batozskaya, A. Kupść, NS et al., PRD 108, 016011 (2023)]

$$g_{av} = \frac{F_1^A(0)}{F_1^V(0)}, \qquad g_w = \frac{F_2^V(0)}{F_1^V(0)}$$

Decay	$\sigma(g_{av})\sqrt{N}$	$\sigma(g_w)\sqrt{N}$
$\Lambda \to pe^-\bar{\nu}_e$	1.8	12
$\Xi^- \to [\Lambda \to pe^-\bar{\nu}_e]\pi^-$	1.8	12
$\Xi^- \to [\Lambda \to p\pi^-]e^-\bar{\nu}_e$	0.6	9
$\Xi^- \to [\Sigma^0 \to [\Lambda \to p\pi^-]\gamma]e^-\bar{\nu}_e$	5.0	29
$\Xi^0 \to [\Sigma^+ \to p\pi^0]e^-\bar{\nu}_e$	4.0	28
$\Sigma^+ \to [\Lambda \to p\pi^-]e^+\nu_e$	0.5	19



Conclusions and Outlook

Conclusions

1. $Y\bar{Y}$ pairs produced at e^+e^- colliders feature relevant spin-correlation and polarization properties. Modular framework to keep track of the spin-correlation properties within the produced pairs in their subsequent decays.
2. CP tests can be built on nonleptonic decay observables; significant improvement in their statistical uncertainties expected for a longitudinally polarized e^- beam.
3. Spin-correlation terms are also employed to improve statistical uncertainties of semileptonic form factors (unpolarized beam). Factorize and generalize the angular dependence on the mother-daughter helicity rotation.

Outlook

1. This modular framework describes general spin-1/2 baryons: **directly** applicable to e.g. charm baryons.
2. Extraction of the hyperon SL FFs from the data collected by the BESIII collaboration (currently).

Dziękuję za uwagę!
Jakieś pytania?

Approximate maximum likelihood method

Fisher information matrix

$$\mathcal{I}(\omega_k,\omega_l) := N\int \frac{1}{\mathcal{P}}\frac{\partial\mathcal{P}}{\partial\omega_k}\frac{\partial\mathcal{P}}{\partial\omega_l}\mathrm{d}\boldsymbol{\xi}$$

To compute e.g. $\mathcal{I}_0(A_{\mathrm{CP}})$ assume

$$\frac{1}{\mathcal{P}} = \frac{\mathcal{V}}{C_{00}}\frac{1}{1+\mathcal{G}} = \frac{\mathcal{V}}{C_{00}}\sum_{i=0}^{\infty}(-\mathcal{G})^i \text{ with } \int \mathcal{G}\mathrm{d}\boldsymbol{\xi} = 0\ ,\ \mathcal{G} \geq -1$$

$$\mathcal{I}_0(A_{\mathrm{CP}}) = \frac{2N}{3}\alpha^2\left\langle \mathbf{P}_B^2\right\rangle \implies \sigma(A_{\mathrm{CP}}) \approx \sqrt{\frac{3}{2}}\frac{1}{\alpha\sqrt{N\left\langle \mathbf{P}_B^2\right\rangle}}\ .$$

CPV in hyperon decays

Introducing CP-odd and final-state interaction phases [PRD 105 (2022) 116022], [PRD 34 (1986) 833]:

$$S = |S| \exp(i\xi_S + i\delta_S)$$
$$P = |P| \exp(i\xi_P + i\delta_P)$$

$$A_{\mathrm{CP}} = -\tan(\delta_P - \delta_S)\tan(\xi_P - \xi_S)$$
$$\Phi_{\mathrm{CP}} = \frac{\alpha}{\sqrt{1-\alpha^2}}\cos\phi\tan(\xi_P - \xi_S)$$

New results in Ξ decays:

- CP-odd phase difference [Nature 606, 64–69 (2022)]

$$\xi_P - \xi_S = (1.2 \pm 3.4 \pm 0.8) \times 10^{-2}\mathrm{rad} \qquad \mathrm{SM}: \ \xi_P - \xi_S = (-2.1 \pm 1.7) \times 10^{-4}\mathrm{rad}$$

- Ξ^- polarization and decay parameters

$$\alpha_\Xi = -0.376 \pm 0.007 \pm 0.003, \ \phi_\Xi = 0.011 \pm 0.019 \pm 0.009 \ \mathrm{rad}$$

$$A^{\Xi}_{\mathrm{CP}} = (6 \pm 13 \pm 6) \times 10^{-3} \qquad \mathrm{SM}: \ A^{\Xi}_{\mathrm{CP}} = (-0.6 \pm 1.6) \times 10^{-5}$$

Nonleptonic decay matrix

P.d.f. for two-step decay $\Xi \to \Lambda(\to N\pi)\pi$

$$\mathcal{P}^{\Xi\bar{\Xi}}(\xi_{\Xi\bar{\Xi}};\omega_{\Xi}) = \frac{1}{(4\pi)^5}\sum_{\mu,\nu=0}^{3} C_{\mu\nu}(\theta)\left(\sum_{\mu'=0}^{3} a^{\Xi}_{\mu\mu'}a^{\Lambda}_{\mu'0}\right)\left(\sum_{\nu'=0}^{3} a^{\bar{\Xi}}_{\nu\nu'}a^{\bar{\Lambda}}_{\nu'0}\right)$$

Nonleptonic decay matrix: [E. Perotti et al, Phys. Rev. D 99 (2019) 056008]

d polarization P_i^d for $P_i^m = 0$

$$a_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ \alpha\sin\theta\cos\varphi & \gamma\cos\theta\cos\varphi - \beta\sin\varphi & -\beta\cos\theta\cos\varphi - \gamma\sin\varphi & \sin\theta\cos\varphi \\ \alpha\sin\theta\sin\varphi & \beta\cos\varphi + \gamma\cos\theta\sin\varphi & \gamma\cos\varphi - \beta\cos\theta\sin\varphi & \sin\theta\sin\varphi \\ \alpha\cos\theta & -\gamma\sin\theta & \beta\sin\theta & \cos\theta \end{pmatrix}$$

m polarization P_i^m for $P_i^d = 0$ — spin-correlation $m - d$

$$\alpha = \frac{2\Re(S^*P)}{|S|^2 + |P|^2}, \quad \beta = \sqrt{1-\alpha^2}\sin\phi, \quad \alpha^2 + \beta^2 + \gamma^2 = 1$$

Examples of aligned matrices [V. Batozskaya, A. Kupść, NS et al., Phys. Rev. D 108, 016011 (2023)]

$$B \to b\pi :$$

$$a_{\mu\nu} \propto \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & \gamma & -\beta & 0 \\ 0 & \beta & \gamma & 0 \\ \alpha & 0 & 0 & 1 \end{pmatrix}$$

$$B \to bW^-_{\text{off-shell}}(\to l^- \bar{\nu}_l) : \qquad b_{\mu\nu} = b^{\text{nf}}_{\mu\nu} + \epsilon b^{\text{f}}_{\mu\nu}$$

$$b^{\text{nf}}_{\mu\nu} = \begin{pmatrix} b^{\text{nf}}_{00} & -\Re(\mathcal{I}^{\text{nf}}_{01}) & \Im(\mathcal{I}^{\text{nf}}_{10}) & b^{\text{nf}}_{03} \\ -\Re(\mathcal{I}^{\text{nf}}_{10}) & \Re(\mathcal{E}^{\text{nf}}_{00} + \mathcal{E}^{\text{nf}}_{11}) & -\Im(\mathcal{E}^{\text{nf}}_{00} + \mathcal{E}^{\text{nf}}_{11}) & \Re(\mathcal{I}^{\text{nf}}_{13}) \\ \Im(\mathcal{I}^{\text{nf}}_{10}) & \Im(\mathcal{E}^{\text{nf}}_{00} - \mathcal{E}^{\text{nf}}_{11}) & \Re(\mathcal{E}^{\text{nf}}_{00} - \mathcal{E}^{\text{nf}}_{11}) & -\Im(\mathcal{I}^{\text{nf}}_{13}) \\ b^{\text{nf}}_{30} & -\Re(\mathcal{I}^{\text{nf}}_{31}) & \Im(\mathcal{I}^{\text{nf}}_{31}) & b^{\text{nf}}_{33} \end{pmatrix}$$

$$b^{\text{f}}_{\mu\nu} = \begin{pmatrix} b^{\text{f}}_{00} & -\Re(\mathcal{I}^{\text{f}}_{01}) & \Im(\mathcal{I}^{\text{f}}_{10}) & b^{\text{f}}_{03} \\ -\Re(\mathcal{I}^{\text{f}}_{10}) & \Re(\mathcal{E}^{\text{f}}_{00} - \mathcal{E}^{\text{f}}_{11}) & -\Im(\mathcal{E}^{\text{f}}_{00} - \mathcal{E}^{\text{f}}_{11}) & \Re(\mathcal{I}^{\text{f}}_{13}) \\ \Im(\mathcal{I}^{\text{f}}_{10}) & \Im(\mathcal{E}^{\text{f}}_{00} + \mathcal{E}^{\text{f}}_{11}) & \Re(\mathcal{E}^{\text{f}}_{00} + \mathcal{E}^{\text{f}}_{11}) & -\Im(\mathcal{I}^{\text{f}}_{13}) \\ b^{\text{f}}_{30} & -\Re(\mathcal{I}^{\text{f}}_{31}) & \Im(\mathcal{I}^{\text{f}}_{31}) & b^{\text{f}}_{33} \end{pmatrix}$$