Particle Physics for Specialists

Electron–Proton Scattering

Rafał Staszewski

16 January 2025

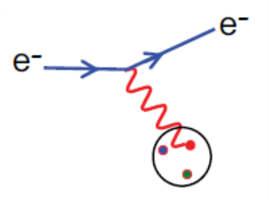
Introduction

Organizational matters

- 3x45 minutes for two topics
- split into two lectures
- today: ep scattering
- next week: forward physics

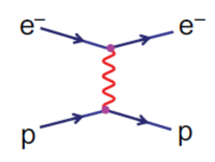
Motivation for interest in ep scattering

- proton is a composite particle
- probing proton structure with a small probe
- photon resolution ~ 1 / virtuality



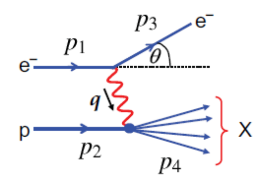
Types of ep scattering

Elastic ep scattering



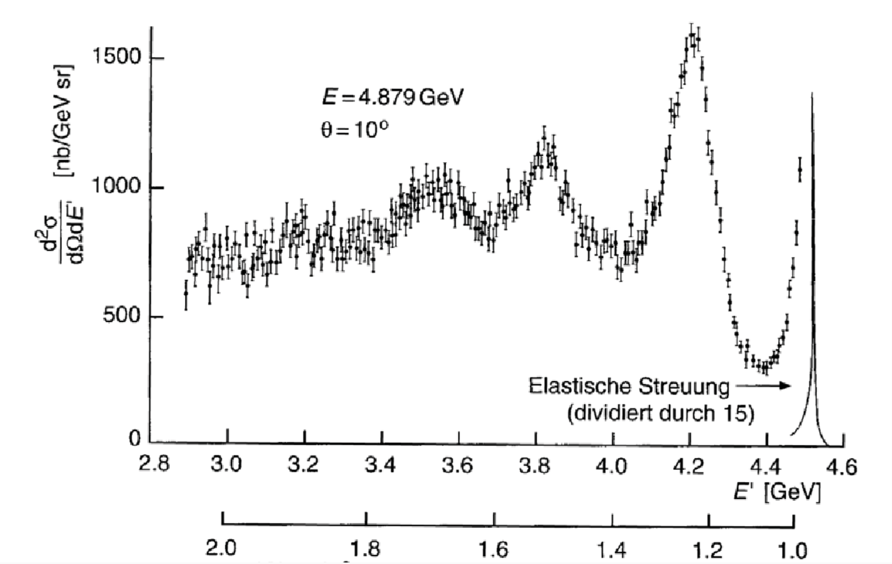
- 4 particles in the process:
  16 4-momentum components
- 3 DoF: boost of the reference frame
- 2 DoF: orientation of the ref. frame
- 4 constraints from fixed masses
- 4 constraints from $(E, \vec{p})$ conserv.
- 3 remaining DoF, for example:
  - centre-of-mass energy
  - azimuthal scattering angle – trivial distribution (when no polarization)
  - **polar scattering angle**

Inelastic ep scattering



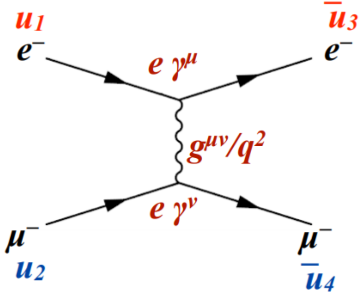
- Proton breaks up into state X
- $M_X > M_p$ (because of baryon number conservation)
- *Inclusive* analysis – state X only described by its invariant mass, the exact composition is not taken into account
- Then, one more DoF than in the elastic ep scattering
- Considering composition of the X state not covered in this lecture

Elastic and inelastic scattering



Contents

Electron–muon scattering



$$\mathcal{M} = e^2 \frac{g^{\mu\nu}}{q^2} J_{13}^{\mu} J_{24}^{\nu}$$

Vertex Couplings

$$\mathcal{M} = e^2 \frac{g^{\mu\nu}}{q^2} (\bar{u}_3 \gamma^\mu u_1)(\bar{u}_4 \gamma^\nu u_2)$$

Photon propagator, Electron current, Muon current

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} \frac{1}{(2S_1+1)(2S_2+1)} \sum_{S_3,S_4} (\bar{u}_3\gamma^\mu u_1)\,(\bar{u}_3\gamma^\nu u_1)^*(\bar{u}_4\gamma^\mu u_2)(\bar{u}_4\gamma^\nu u_2)^*$$

Electron–muon scattering

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{q^4}\frac{1}{(2S_1+1)(2S_2+1)}\sum_{S_3,S_4}(\bar{u}_3\gamma^\mu u_1)\,(\bar{u}_3\gamma^\nu u_1)^*(\bar{u}_4\gamma^\mu u_2)(\bar{u}_4\gamma^\nu u_2)^* \\
&= \frac{e^4}{q^4}\left(\frac{1}{(2S_1+1)}\sum_{S_3}(\bar{u}_3\gamma^\mu u_1)\,(\bar{u}_3\gamma^\nu u_1)^*\right)\left(\frac{1}{(2S_2+2)}\sum_{S_4}(\bar{u}_4\gamma^\mu u_2)(\bar{u}_4\gamma^\nu u_2)^*\right) \\
&= \frac{e^4}{q^4}\,L_e\,L_m
\end{aligned}
$$

$$
\begin{aligned}
L_e &= \frac{1}{(2S_1+1)}\sum_{S_3}(\bar{u}_3\gamma^\mu u_1)\,(\bar{u}_3\gamma^\nu u_1)^* \\
&= 2\left[p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_3\cdot p_1 - m_e^2)g^{\mu\nu}\right]
\end{aligned}
$$

$$
L_\mu = 2\left[p_4^\mu p_2^\nu + p_4^\nu p_2^\mu - (p_4\cdot p_2 - m_\mu^2)g^{\mu\nu}\right]
$$

Scattering cross sections

Rutherford scattering

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\mathsf{K}} \sin^4 \Theta/2}$$

- Low energy approximation ($E_{\mathsf{K}} \ll m_e$)
- E_{K} – electron kinetic energy
- Θ – scattering angle in the lab frame

Mott scattering

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E \sin^4 \Theta/2} \cos^2 \frac{\Theta}{2}$$

- High energy approximation ($m_e \approx 0$)
- E – electron energy (at high energy, equivalent to E_{K})
- $\cos^2 \Theta/2$ factor is related to the electron helicity

In both formulae:

- proton is treated as a point-like spin-half particle
- proton recoil is neglected

Proton recoil

Mott formula

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E\sin^4\Theta/2}\cos^2\frac{\Theta}{2}$$

Taking into account proton recoil

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1\sin^4\Theta/2}}_{\text{Mott}}\frac{E_3}{E_1}\left(\underbrace{\cos^2\frac{\Theta}{2}}_{\text{Mott}} + 2\tau\sin^2\frac{\Theta}{2}\right), \qquad \tau = -q^2/4M^2$$

- $\frac{E_3}{E_1}$ factor related to recoil
- $2\tau\sin^2\frac{\Theta}{2}$ term corresponds to spin–spin interaction

Form factors

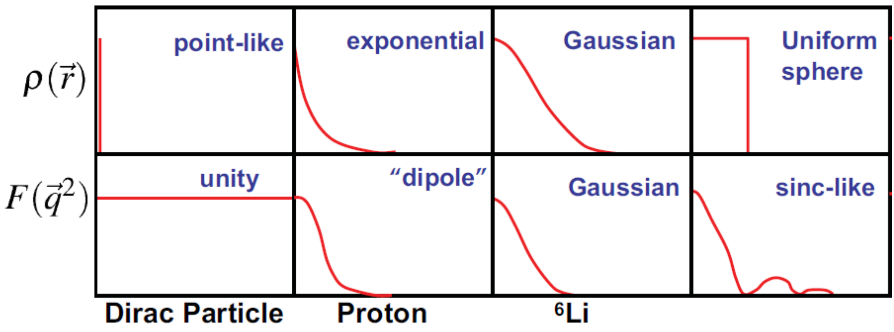
For an extended charge distribution:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} |F(\vec{q})|^2$$

- $\vec{q}$ – momentum of the virtual photon, related to electron: $q^2 = -4p^2 \sin^2 \Theta/2$
- $F(\vec{q})$ – form factor, Fourier transform of the charge distribution

$$F(\vec{q}) = \int \rho(\vec{r}) \exp(i\vec{q}\cdot\vec{r}) d^3\vec{r}$$

- $\rho(\vec{r})$ – charge distribution (normalised to unity: $\int \rho(\vec{r}) d^3\vec{r} = 1$)



Proton recoil and proton structure

Point-like proton

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \Theta/2} \frac{E_3}{E_1} \left(\cos^2 \frac{\Theta}{2} + 2\tau \sin^2 \frac{\Theta}{2} \right), \qquad \tau = -q^2/4M^2$$

Finite-size proton

- In Feynman diagram calculation, proton vertex:

$$\gamma^\mu \quad \to \quad F_1(q^2)\gamma^\mu + \frac{\kappa}{2M} F_2(q^2) i\sigma^{\mu\nu} q_\nu, \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

- Rosenbluth formula

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \Theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2 \frac{\Theta}{2} - 2\tau G_M^2 \sin^2 \frac{\Theta}{2} \right)$$

$$G_E = F_1 + \frac{\kappa q^2}{4M^2} F_2 \qquad\qquad G_M = F_1 + \kappa F_2$$

- G_E and G_M can be interpreted as Fourier transform of charge and magnetic moment distributions *only* for very small scattering angles
- $G_E(0) = 1$, $G_M(0) = 2.79$
- Only one (relevant) DoF describing the final state: Θ, E_3 and q^2 are dependent

Experimental result

- Proton well described by a dipole form factor

$$G_E = \frac{1}{(1 + q^2/0.71\ \text{GeV}^2)^2}$$

$$G_M = 2.79 G_E$$

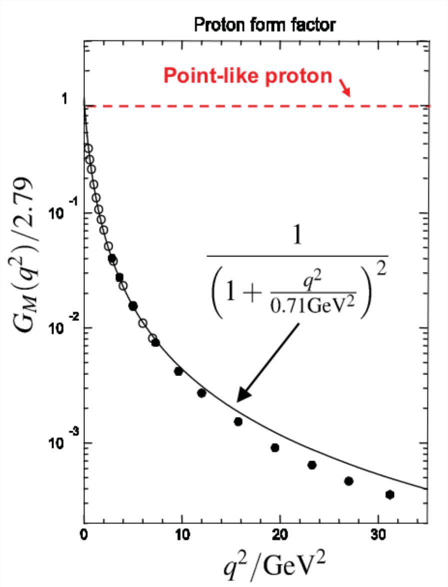
- Corresponds to exponential charge distribution

$$\rho(\vec{r}) = \rho_0 e^{-r/a}$$

with $a = 0.24$ fm

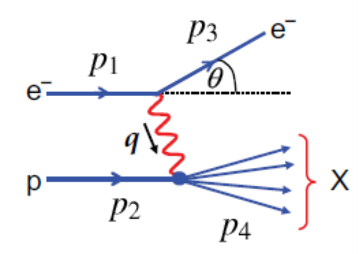
- This corresponds to proton radius

$$\langle r \rangle = 0.8\ \text{fm}$$



Contents

Kinematics



- Photon virtuality

$$Q^2 = -q^2$$

- Bjorken x

$$x = \frac{Q^2}{2p_2 \cdot q}$$

- Invariant mass of X:

$$W = M_X$$

- Electron energy loss

$$\nu = \frac{p_2 \cdot q}{m_p}$$

$$\nu = E_1 - E_3 \text{ (in the proton rest frame)}$$

- Inelasticity

$$y = \frac{p_2 \cdot q}{p_2 \cdot p_1}$$

$$y = \frac{\nu}{E_1} \text{ (in the proton rest frame)}$$

- Elastic scattering

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \Theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2 \frac{\Theta}{2} - 2\tau G_M^2 \sin^2 \frac{\Theta}{2} \right)$$

G_E and G_M are functions of q^2

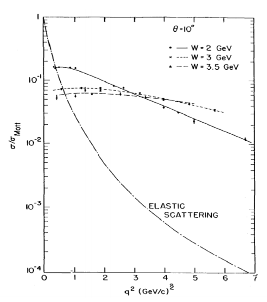
- Inelastic scattering

$$\frac{d\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \Theta/2} \frac{E_3}{E_1} \left(W_2(\nu, q^2) \cos^2 \frac{\Theta}{2} - 2W_1(\nu, q^2) \sin^2 \frac{\Theta}{2} \right)$$

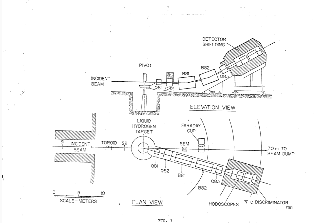
- Equivalent formulation in the limit of high Q^2:

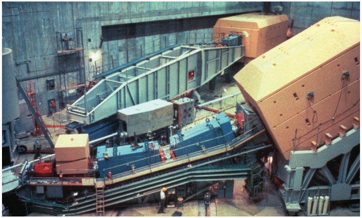
$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2 \mathrm{d}x} = \frac{4\pi\alpha^2}{Q^4} \left[\frac{1-y}{x} F_2(x, Q^2) + y^2 F_1(x, Q^2) \right]$$

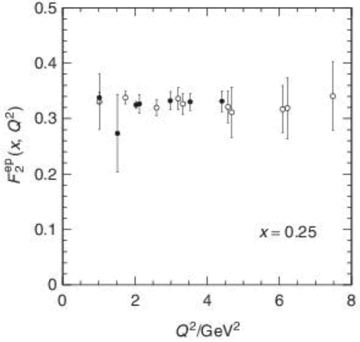
- At high Q^2, inelastic scattering dominates



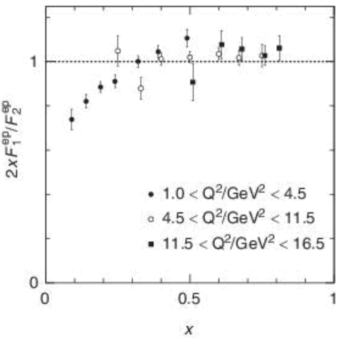
Experimental measurements at SLAC





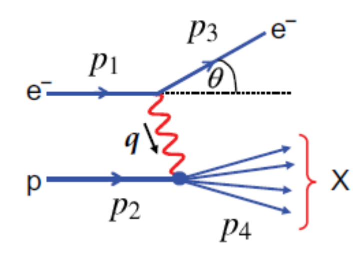


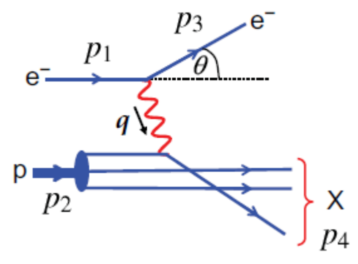
Bjorken scaling



Callan-Gross relation

Quark–Parton Model





Electron–quark scattering:

- quark momentum before scattering: p_{quark}
- quark momentum after scattering: $p'_{\text{quark}} = p_{\text{quark}} + q$
- negligible quark mass and no breaking up: $p^2_{\text{quark}} = p'^2_{\text{quark}} = m^2_{\text{quark}} \approx 0$

$$(p_{\text{quark}} + q)^2 = 0 \quad \rightarrow \quad Q^2 = 2\ p_{\text{quark}} \cdot q$$

$$x = \frac{Q^2}{2\ p_2 \cdot q} \quad \rightarrow \quad p_{\text{quark}} = x p_2$$

- x is the fraction of proton momentum carried by the quark (in a frame where the proton is very fast)

Parton Distribution Functions

- electron–quark scattering (neglecting masses):

$$\frac{\mathrm{d}\sigma_{eq}}{\mathrm{d}Q^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1 + (1-y)^2\right]$$

- $q(x)$ – probability density of scattering off a given quark flavour, then

$$\frac{\mathrm{d}\sigma_{ep}}{\mathrm{d}Q^2\mathrm{d}x} = q(x)\frac{\mathrm{d}\sigma_{eq}}{\mathrm{d}Q^2}$$

- Comparing with

$$\frac{\mathrm{d}\sigma_{eq}}{\mathrm{d}Q^2\mathrm{d}x} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{1-y}{x}F_2(x,Q^2) + y^2 F_1(x,Q^2)\right]$$

- Structure function in terms of parton distributions

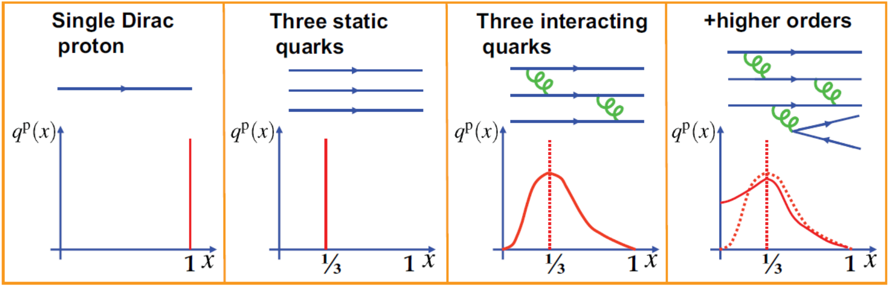
$$F_1(x,Q^2) = \frac{1}{2}\sum_q e_q^2 q(x)$$

$$F_2(x,Q^2) = 2xF_1(x,Q^2)$$

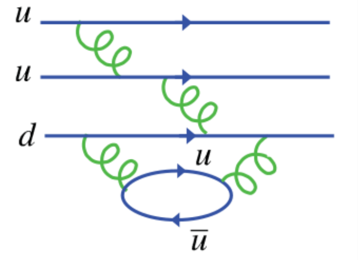
For example, including only u and d quarks:

$$\sum_q e_q^2 q(x) = \frac{4}{9}\left[u(x) + \bar{u}(x)\right] + \frac{1}{9}\left[d(x) + \bar{d}(x)\right]$$

Parton Distribution Functions



Valence and see quarks



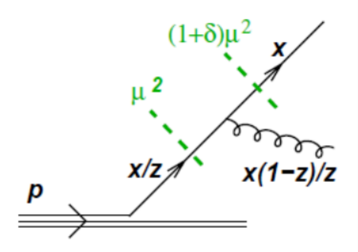
- proton $= uud \leftarrow$ valence quarks
- see quarks – quantum fluctuations
  eg. $g \to q\bar{q}$
- $u(x) = u_v(x) + u_s(x)$, $\bar{u}(x) = u_s(x)$
- $d(x) = d_v(x) + d_s(x)$, $\bar{d}(x) = d_s(x)$
- similar masses of u and d: $u_s(x) = d_s(x)$

Momentum fraction

- $q(x)$ – quark density
- $xq(x)$ – momentum density
- $\int xq(x)\mathrm{d}x = f_q$ – proton momentum fraction carried by q
- $F_2 = \sum_q xe_q^2 q(x) \to \int F_2(Q^2, x)\mathrm{d}x = \frac{4}{9}f_u + \frac{1}{9}f_d$
- neutron: $u \leftrightarrow d \to \int F_2^{\text{neutron}}(Q^2, x)\mathrm{d}x = \frac{4}{9}f_d + \frac{1}{9}f_u$
- $f_u \approx 0.36$, $f_d \approx 0.018$
- ∼50% of proton momentum carried by neutral partons (gluons)

DGLAP – evolution of PDFs

- DGLAP = Dokshitzer, Gribov, Lipatov, Altarelli, Parisi
- Ambiguity in splitting between proton structure and quark–electron process
- But F_2 is well defined and measurable
- Renormalization group equations



- Simplified case ($P_{qq}(z)$ – splitting function):

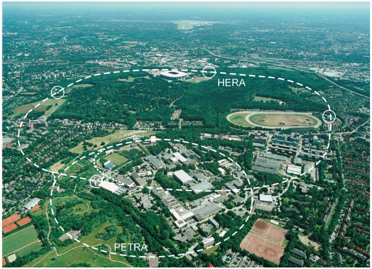
$$\frac{\mathrm{d}q(x,Q^2)}{\mathrm{d}\log Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 q(x,Q^2)P_{qq}\left(\frac{x}{y}\right)\frac{1}{y}\mathrm{d}y$$

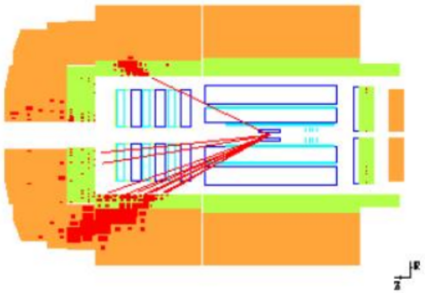
- In reality, also gluons are involved:

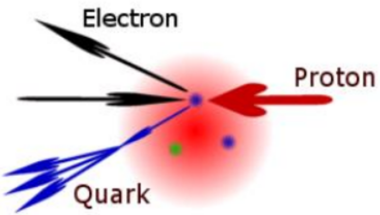
$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2}\begin{pmatrix} q \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{gq} \\ P_{qg} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q \\ g \end{pmatrix}$$

- DGLAP $\rightarrow$ evolution of PDFs in Q^2 predicted by QCD

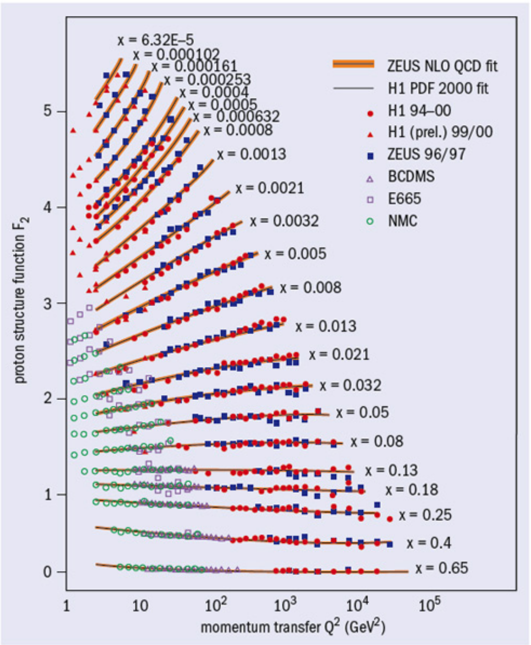
HERA Accelerator at DESY





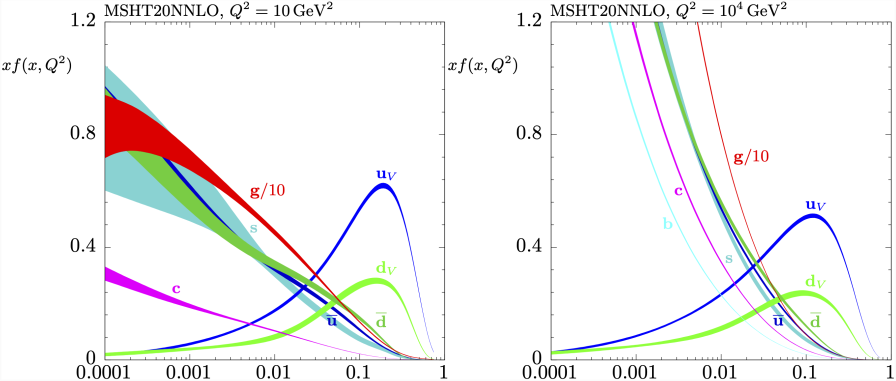


F_2 measurement at HERA

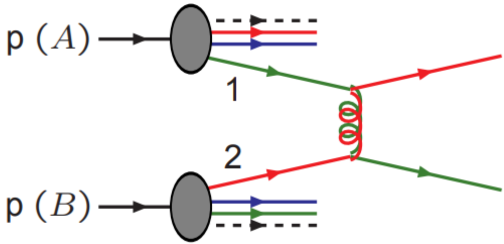


Global PDF fits

- Assume functional form of the x dependence of PDFs at a fixed Q_0
- A simplistic example: $u(x, Q_0^2 = 1 \text{ GeV}^2) = A \cdot x^B \cdot (1-x)^C$
- Evolve PDFs to the needed Q^2 using DGLAP
- Compute the observable and compare with the experimental measurement
- Build a global χ^2 function, find parameters $(A, B, C, \ldots)$ minimising the χ^2



Proton–proton collisions



$$s = (p_A + p_B)^2$$

$$x_1 \approx E_1/E_A$$

$$x_2 \approx E_2/E_B$$

$$\hat{s} = x_1 x_2 s$$

$$\sigma = \sum_{i,j} \iiint \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}\hat{t}\, f_i^{(A)}(x_1, Q^2)\, f_j^{(B)}(x_2, Q^2)\, \frac{\mathrm{d}\hat{\sigma}_{ij}}{\mathrm{d}\hat{t}}$$

Outlook

Related topics, not covered in this lecture

- NLO, NNLO
- higher twist effects
- BFKL
- saturation
- Other QCD factorisations
- More general parton distributions: GPD, TMD, . . .
- Spin-dependent proton structure
- Nuclear PDFs
- Multi-parton structure
- PDFs from lattice

Ackwnoladgements

This lecture was inspired by:

– Halzen and Martin *Quarks and leptons*

– Prof. Mark Thomson's lectures (available online)