Volterra-Prabhakar function

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Agenda

The Prabhakar function $e^{\gamma}_{\alpha,\beta}$ is defined as

$$e^{\gamma}_{\alpha,\beta} : (0, \infty) \ni t \to t^{\beta-1} E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha}) \in \mathbb{C}, \tag{1}$$

where

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_r z^r}{r!\Gamma(\beta + \alpha r)} \tag{2}$$

is a value of the three parameters Mittag-Leffler function and

$$(\gamma)_r = \frac{\Gamma(\gamma + r)}{\Gamma(\gamma)} \tag{3}$$

is the Pochhammer (raising) symbol.

In general, the value range of the Prabhakar function parameters is as follows:

- $\Re(\alpha) > 0$
- $\Re(\beta) > 0$
- $\gamma > 0$
- $\lambda \geq 0$

The Volterra function $\mu_{\alpha,\beta}$ is defined as follows

$$\mu_{\alpha,\beta} : (0, \infty) \ni t \to \frac{1}{\Gamma(1+\beta)} \int_0^\infty \frac{t^{u+\alpha}\, u^\beta}{\Gamma(u+\alpha+1)}\, \mathrm{d}u \in \mathbb{C}, \tag{4}$$

where $\Re(\beta) > -1$.
Let the function $\epsilon^{\gamma}_{\alpha,\,p}$ whose value we define as

$$\epsilon^{\gamma}_{\alpha,\,p}(\lambda; t) = \int_0^\infty e^{\gamma}_{\alpha,u+p+1}(\lambda; t)\, \mathrm{d}u \tag{5}$$

be called the Volterra-Prabhakar function which for $\gamma = 0$ or $\lambda = 0$ reduces to $\mu_{p,0}(t)$.

Some selected properties of the Volterra-Prabhakar function:

- General expression for the nth-order derivative

$$\frac{d^n}{dt^n}[\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t)] = \epsilon^{\gamma}_{\alpha,\,p-n}(\lambda;t) \quad \text{for} \quad n \in \mathbb{N}_0,$$

- Convolution relation between the Volterra, Prabhakar, and the Volterra-Prabhakar functions

$$\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t) \star \mu_{\alpha-1,\beta-1}(t) = e^{\gamma}_{\alpha,\,p}(\lambda;t) \star \mu_{\alpha,\beta}(t),$$

- The Laplace transform of the Volterra-Prabhakar function is given by

$$\mathscr{L}\left[\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t);s\right] = \frac{s^{\alpha\gamma-p-1}}{(s^{\alpha}+\lambda)^{\gamma}\ln s}.$$

Definition (Bernstein–Hausdorff–Widder)

A function $f : (0, \infty) \to \mathbb{R}$ is a *completely monotone function* if f is of class C^∞ and

$$(-1)^n f^{(n)}(\lambda) \geq 0$$

for all $n \in \mathbb{N}$ and $\lambda > 0$.

Theorem (Bernstein)

Let $f : (0, \infty) \to \mathbb{R}$ be a completely monotone function. Then it is the Laplace transform of a unique measure μ on $[0, \infty)$ i.e. for all $\lambda > 0$

$$f(\lambda) = \mathscr{L}(\mu; \lambda) = \int_{[0,\infty)} \mathrm{e}^{-\lambda t} \, \mu(\mathrm{d}t).$$

Conversely, whenever $\mathscr{L}(\mu; \lambda) < \infty$ for every $\lambda > 0$, $\mathscr{L}(\mu; \lambda)$ is a completely monotone function.

Integral representation for the Volterra-Prabhakar function

The Laplace transform of the Volterra-Prabhakar function is given by

$$\mathscr{L}\left[\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t);s\right] = \frac{s^{\alpha\gamma-p-1}}{(s^{\alpha}+\lambda)^{\gamma}\ln s}. \tag{6}$$

Using the Bromwich integral for the inverse Laplace transform of (5), one can find an integral representation for the Volterra-Prabhakar function with $\lambda = 1$ and $0 < \alpha \leq 1$. To do this, we need to calculate the following integral

$$\epsilon^{\gamma}_{\alpha,\,p}(1;t) \equiv \epsilon^{\gamma}_{\alpha,\,p}(t) = \frac{1}{2\pi \mathrm{i}} \lim_{a\to\infty} \int_{c-\mathrm{i}a}^{c+\mathrm{i}a} \mathrm{e}^{st} \frac{s^{\alpha\gamma-p}}{(s^{\alpha}+1)^{\gamma} s \ln s}\, \mathrm{d}s. \tag{7}$$

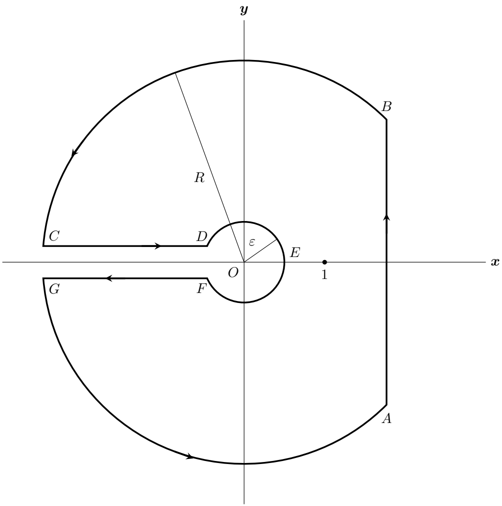

Figure: Bromwich contour

In the end, we obtain an integral representation of the Volterra-Prabhakar function in the form

$$\epsilon^{\gamma}_{\alpha,\,p}(t) = \frac{e^{t}}{2^{\gamma}} - \int_0^{\infty} e^{-rt}\, \tilde{K}^{\gamma}_{\alpha,p}(r)\, dr, \tag{8}$$

where

$$\widetilde{K}^{\gamma}_{\alpha,p}(r) = -\frac{r^{\alpha\gamma-p-1}}{\pi} \frac{(\ln r)\sin[\pi(\alpha\gamma-p)-\gamma\theta_{\alpha}] - \pi\cos[\pi(\alpha\gamma-p)-\gamma\theta_{\alpha}]}{[r^{2\alpha}+2r^{\alpha}\cos(\pi\alpha)+1]^{\gamma/2}(\pi^2+\ln^2 r)}, \tag{9}$$

$$\theta_{\alpha} = \arctan\left[\frac{\sin(\pi\alpha)}{\cos(\pi\alpha)+r^{-\alpha}}\right]. \tag{10}$$

It can be shown that $2^{\gamma}\widetilde{K}^{\gamma}_{\alpha,p}(r)$ is a probability density function when the conditions:

- $\alpha \in (0, \frac{1}{2}]$
- $\alpha\gamma - p = 2k,\ k \in \mathbb{Z}$
- $\gamma > 0$

are simultaneously satisfied so the modified Volterra-Prabhakar function

$$\mathfrak{e}^{\gamma}_{\alpha,\,p}(t) = \epsilon^{\gamma}_{\alpha,\,p}(t) - \frac{e^{t}}{2^{\gamma}} \tag{11}$$

is a completely monotone function.

Why are we concerned with the Volterra-Prabhakar function?

This function can be used as a kernel smearing the derivative in equations describing transport processes, for example:

- generalized Fokker-Planck equation

$$\int_0^t \gamma(t-\xi)\partial_\xi p(x,\xi)\,\mathrm{d}\xi = \partial_x^2[B(x,t)p(x,t)] - \partial_x[\lambda(x,t)p(x,t)],$$

- generalized Cattaneo-Vernotte equation

$$\int_0^t \eta(t-\xi)\partial_\xi^2 u(x,\xi)\,\mathrm{d}\xi + \frac{1}{\tau}\int_0^t \gamma(t-\xi)\partial_\xi u(x,\xi)\,\mathrm{d}\xi = a^2\partial_x^2 u(x,t)$$

Summary

- The Volterra-Prabhakar function or its generalization can be used as the kernel of an integral operator.
- The $\mathfrak{e}^{\gamma}_{\alpha, p}$ function is a completely monotone function.
- This function appears in the description of the anomalous diffusion process.

K. Gorska, T. Pietrzak, T. Sandev, Z. Tomovski
Volterra-Prabhakar function of distributed order and some applications
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Thank you for your attention!