Small-x evolution and the proton spin puzzle

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Overview

Part 1: Proton spin puzzle

- 19 May 1988
- Theoretical prediction in the 70's
- The missing spin of the proton?

Part 2: Short intro to small-x physics

- DIS and DGLAP evolution
- Paradigm shift
- HE limit
- Simple dipole model

Part 3: Quark flavor-singlet helicity distribution

- Generalities
- Initial framework
- Let's do it again, new dipoles
- Revised and updated
- Recovering small-x pol DGLAP

Part 1: Proton spin puzzle

Proton spin puzzle / crisis.

Volume 206, number 2 PHYSICS LETTERS B 19 May 1988

The spin asymmetry in deep inelastic scattering of longitudinally polarised muons by longitudinally polarised protons has been measured over a large x range ($0.01<x<0.7$). The spin-dependent structure function $g_1(x)$ for the proton has been determined and its integral over x found to be $0.114 \pm 0.012 \pm 0.026$, in disagreement with the Ellis–Jaffe sum rule. Assuming the validity of the Bjorken sum rule, this result implies a significant negative value for the integral of g_1 for the neutron. These values for the integrals of g_1 lead to the conclusion that the total quark spin constitutes a rather small fraction of the spin of the nucleon.

Reminder

$$g_1^{\gamma} = \frac{1}{2} \sum_q e_q^2 \; (\Delta q + \Delta \bar{q}) \, , \qquad \Delta q = q^{\uparrow} - q^{\downarrow} \text{ w.r.t. the proton spin} \tag{1}$$

and they observed for the proton

$$\int_{0.01}^{0.7} g_1(x) \, \mathrm{d}x = 0.114 \pm 0.012(\text{stat.}) \pm 0.026(\text{syst.}) \tag{2}$$

Remarks

- In blue: finite range of integration. "... *the small x region is expected to make a large contribution to the integrals.*"
- In red: Ellis-Jaffe sum rule. → Theoretical understanding of the 70's.

→ How do we understand this value?

$\rightarrow$ How do we understand this value? $0.114 \pm 0.012(\text{stat.}) \pm 0.026(\text{syst.})$

Ellis-Jaffe sum rule, assumptions

- Sea $q\bar{q}$: $\lambda^+(x) \simeq \lambda^-(x) \simeq \bar{\lambda}^+(x) \simeq \bar{\lambda}^-(x)$
- Ansatz $\Delta s \sim 0$ (no intrinsic strangeness)
- Belief that valence quarks carry the proton spin.

we obtain[11]

$$\int_0^1 d\xi\, g_1^{ep}(\xi) = \frac{g_A}{12}(1.78), \quad (6)$$

$$\int_0^1 d\xi\, g_1^{en}(\xi) = \frac{g_A}{12}(-0.22), \quad (7)$$

where $g_A = 1.248 \pm .010$.

Ellis-Jaffe sum rule prediction (70's): $0.185 \pm 0.0015 \longrightarrow$ Not compatible with 0.114

Where is the missing spin ?

Old fundamental problem ($\sim 30y$) $\rightarrow$ looking at small number adding up to $1/2$.

- There are progresses $\rightarrow$ Still, we don't understand the spin of the proton in term of QCD dof.

A more recent picture of the proton spin.

Spin sum rule (Jaffe Manohar decomposition [Nucl. Phys. B337, 509 (1990)])

$$\frac{1}{2} = \frac{1}{2}\Sigma_q + \Sigma_g + L_q + L_g$$

Quark OAM

Gluon OAM

Gluon spin

Quark spin

Proton spin

Possibilities:

- Gluon spin
- Quark/Gluon angular orbital momentum

Large and low x region. Experiments only access a finite range of x...

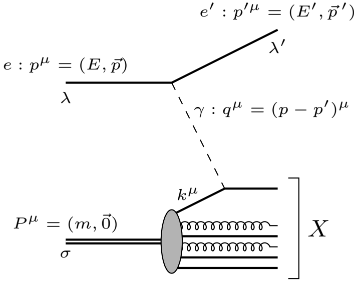
$$\Sigma_q = \int_0^1 \mathrm{d}x \left(q^{\uparrow}(x) - q^{\downarrow}(x)\right) \quad (3)$$

Possibilities

- Large-x?
- Small-x?

Part 2: Short intro to small-x physics

"The" Diagram:



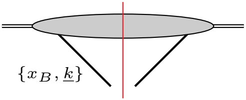
Kinematic invariants

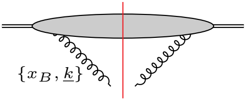
$$Q^2 \equiv -q^2, \qquad (\gamma\text{-virtuality})$$

$$x_B \equiv \frac{Q^2}{2P \cdot q} = \frac{Q^2}{\hat{s} + Q^2 - m^2} \qquad \text{(Bjorken-x)},$$

$$y \equiv \frac{E - E'}{E}, \qquad \text{(Inelasticity)}$$

Parton distribution functions (in the *infinite momentum frame*):



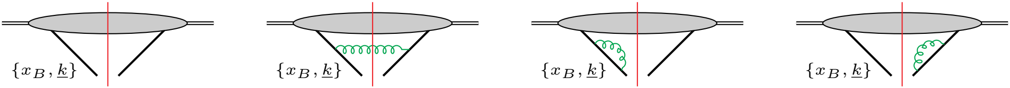


pdf noted $q_f(x_B, Q^2)$ and $g(x_B, Q^2)$

$\Rightarrow$ What are the lowest order QCD corrections ?

DIS and DGLAP evolution (2/2)

Lowest order QCD corrections (and similar for gluons)



- Slowly increase Q^2 open gluon emission phase-space.
- Each "loop" gives a large log: $\ln(Q^2/\Lambda^2) \gg 1$.
- Asymptotic freedom $\alpha_s(Q^2) \ll 1$
- $\alpha_s \ln(Q^2/\Lambda^2) \sim 1$. Need resummation. $\Rightarrow$ Resummation parameter $\alpha_s \ln(Q^2/\Lambda^2)$

At the leading-logarithmic-approximation (LLA)

- Look only at the large log and drop sub-leading terms
- Approach: relate the wave function Ψ_n to Ψ_{n-1}. $\Longrightarrow$ Splitting functions
- Virtual diagrams: use unitary trick $|\Psi\rangle \to |\Psi'\rangle = |\Psi\rangle + R|\Psi\rangle + V|\Psi\rangle$

Above diagrams gives:

$$Q^2 \frac{\partial q_f(x, Q^2)}{\partial Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z)\ q_f(x/z, Q^2) \tag{5}$$

$\Rightarrow$ With all possible diagrams: Matrix-equation with mixing between distributions.

Paradigm shift

Why study the high energy behavior of QCD?

- DGLAP evolution in Q^2.
- Differential equation $\Rightarrow$ initial condition at fixed Q_0^2 for all x.
- For the proton spin puzzle: *we need the small-x asymptotic behavior of the distribution*.

Paradigm shift

- Recall $x_B = \frac{Q^2}{\hat{s}+Q^2-m^2}$, for fixed Q^2: "large $\hat{s}$" $\Longleftrightarrow$ "small-x",
- Drop the assumption $Q^2/Q_0^2 \gg 1$, and study the small-x behavior for the high energy behavior of QCD.

Approximation	Coupling	Transverse log	Longitudinal log
LLA in Q^2	$\alpha_s \ll 1$	$\alpha_s \ln(Q^2/Q_0^2) \sim 1$	$\alpha_s \ln(1/x) \ll 1$
LLA in $1/x$	$\alpha_s \ll 1$	$\alpha_s \ln(Q^2/Q_0^2) \ll 1$	$\alpha_s \ln(1/x) \sim 1$

$\Rightarrow$ Energy evolution and leading $\ln(1/x)$ approximation

Scattering in the high energy limit.

Taking the HE limit: consider the scattering of a probe on some (localized) external field,

- S-matrix $S_{\beta\alpha} = \langle \beta_{out} | \alpha_{in} \rangle$ (finite energy).
- Evolution operator U write $S_{\beta\alpha} = \langle \beta_{in} | U(\infty, -\infty) | \alpha_{in} \rangle = \langle \beta_{in} | T\, e^{i \int d^4x \mathcal{L}_{int}(x)} | \alpha_{in} \rangle$.
- Apply a boost in the longitudinal direction to the states α, β, and extract the limit of infinite boost of the amplitude.

Eikonal scattering: as a consequence to the infinite boost limit,

- Support of the external field is contracted, and its configuration is frozen.
- Component of the external gauge field are ordered in eikonality: $A^{\pm} \gg A_{\perp} \gg A^{\mp}$
- Wave function is evolved from $-\infty$ to 0, then the scattering is a phase rotation encoded into Wilson lines.

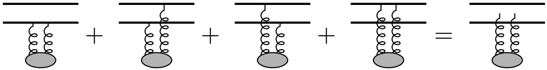
Beyond Eikonal, Sub-Eikonal,... : consists of relaxing the above limits.

"Simple" dipole model (1/2)

Multiple-rescatterings: GGM formula [Glauber-Gribov-Mueller]

Start with the dipole interaction with a single nucleon (cov. gauge):

$$\sigma^{q\bar{q}\text{ Nucleon}} \longleftarrow N(\underline{x}, \underline{b}, Y=0) =$$

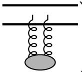


$\rightarrow$ increases with $\underline{x}$, violation of *black-disk limit* (i.e. $N \leq 1$).

Assume a large nucleus with well separated (dilute) nucleons.

The imaginary part of the forward amplitude N_A in (GGM) is:

$$\sigma^{q\bar{q}\text{ Nucleus}} \longleftarrow N_A(\underline{x}, \underline{b}, Y=0) = 1 - \exp\left(-K \quad\right) \tag{6}$$



$\rightarrow$ Obeys Froissart-Martin bound $N \leq 1$ (i.e. unitarity).

$\Longrightarrow$ No energy dependence (i.e. in the rapidity Y)

Rmks:

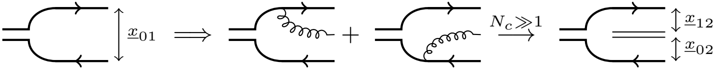
- Shock wave $\Longleftrightarrow$ GGM multiple-rescatterings.
- Wilson line $V \equiv \mathrm{P}e^{ig\int_C dx\cdot A}$

"Simple" dipole model (2/2)

Mueller's dipole model*(naive version)*: or "How do we generate Y-dependence?"

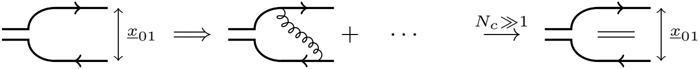
- Quantum evolution corrections!
- Write the dipole scattering amplitude in Light Cone Perturbation Theory (LCPT) [Similar to *old fashioned time ordered perturbation theory*, but on the light-cone]

Real diagrams:



(7)

Virt diagrams:



(8)

Brings an additional integral

$$\int^{\min(z_1,\,1-z_1)} \frac{dz_2}{z_2} \int \frac{\mathrm{d}^2\underline{x}_2}{4\pi} \tag{9}$$

$\Rightarrow$ Turns out to be proportional to $\alpha_s \ln 1/x$

$\Rightarrow$ Y-dependence!

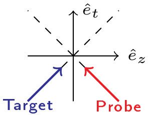
Rmks:

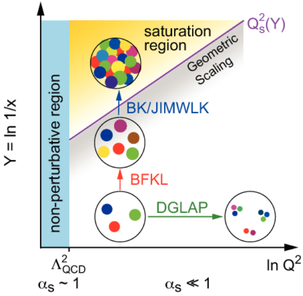
- The evolution equation of Mueller's dipole model has been shown to equivalent to the BFKL equation.
- To go further into the saturation region (very-low x), one needs BK/JIMWLK

Part 3: Quark flavor-singlet helicity distribution

Comments on the Light Cone Operator Treatment (1/3)

⚠ Frame choice: $\rightarrow$ Probe minus-moving, target plus-moving.





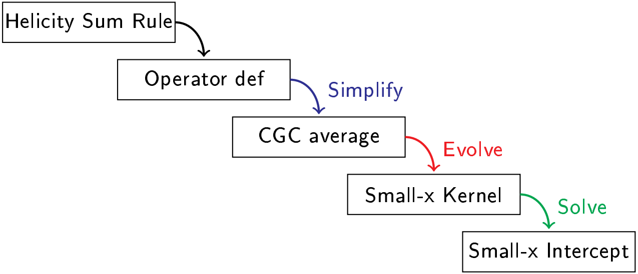
- Aim: Contribution to the spin using *small-x asymptotic*.
  $\longrightarrow$ Evolution in rapidity.
- Approach: Take a TMD,
  $\longrightarrow$ Simplify / Evolve / Solve.
- Equations in the spirit of BK-evolution.
  Initiated by [Kovchegov, Pitonyak, and Sievert].

Rmk: $\exists$ other frameworks for g_1 at small-x, such as Bartel, Ermolaev, and Ryskin [BER] - (1996).

Notation: we use V for fundamental WL, and U for adjoint WL.

Yuri's (and *al.*) approach "Simplify, Evolve, and Solve"

[Y.V Kovchegov, D. Pitonyak, and M. D. Sievert 2016 2017] [Y.V Kovchegov, and M. D. Sievert 2018]



Mixed space:

- $z \equiv k^+/P^+$
- $\underline{x} = \mathbf{x} = \vec{x}_\perp$

Fourier phase

- $e^{-i\underline{k}\cdot(\underline{\zeta}-\underline{y})}$

Dipole model

- Splitting factor $\underline{x}/\underline{x}^2$

Helicity distributions (flavor-singlet)

$$g_{1L}^S(x, k_T^2) = \frac{8N_c}{(2\pi)^6} \int d^2\underline{\zeta}\, d^2\underline{w}\, d^2\underline{y}\, e^{-i\underline{k}\cdot(\underline{\zeta}-\underline{y})} \int_{\Lambda^2/s}^{1} \frac{dz}{z} \frac{\underline{\zeta}-\underline{w}}{|\underline{\zeta}-\underline{w}|^2} \cdot \frac{\underline{y}-\underline{w}}{|\underline{y}-\underline{w}|^2} G_{\underline{w},\underline{\zeta}}(zs) \qquad (10)$$

where

$$G_{\underline{w},\underline{\zeta}}(zs) = \frac{k_1^- p^+}{N_c} \mathrm{Re} \left\langle \mathrm{T\,tr}\left[V_{\underline{\zeta}}\, V_{\underline{w}}^{pol\,\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{w}}^{pol}\, V_{\underline{\zeta}}^{\dagger}\right]\right\rangle \qquad (11)$$

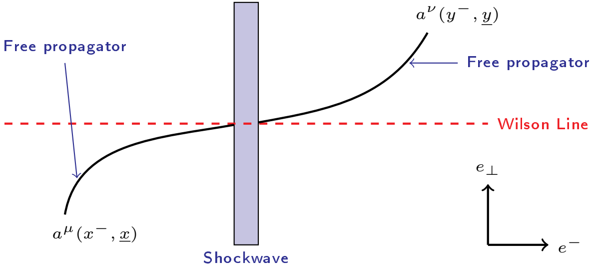
Think of it as a regular dipole amplitude (for the moment) $\longrightarrow$ *to be evolved.*

Comments on the Light Cone Operator Treatment (3/3) - Evolution

Recipe:

- Split the background field A^μ into a new background A^μ and a quantum field a^μ.
- Integrate out quantum fields a^μ.
- Require the propagator in the new background. Use shockwave approximation.
- Pull out the corresponding kernel for one step of evolution.

$$\overbracket{a^\mu(x^-, \underline{x}) a^\nu}(y^-, \underline{y}) \; =$$



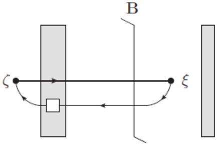
Remarks

- Introduce Wilson line and polarized Wilson lines, up to subeikonal level.
- Splitting field w.r.t. to their longitudinal momentum fraction.
  ⚠ Different from genuine Born-Oppenheimer, that would be w.r.t. frequency or Ioffe time.

Consider the quark helicity TMD [Kovchegov et al. 2018]

$$g_{1L}^q(x,k_T^2) = \frac{1}{(2\pi)^3}\frac{1}{2}\sum_{S_L} S_L \int \mathrm{d}^2\underline{r}dr^- \; e^{ik\cdot r}\langle p, S_L|\bar{\psi}(0)U[0,r]\frac{\gamma^+\gamma^5}{2}\psi(r)|p,S_L\rangle. \tag{12}$$

- Gauge-link $U[0,r]$ is process dependent, SIDIS $\to$ forward staple.
- Simplify at small-x, remaining diagram is B.



After some algebra...

$$\begin{aligned} g_{1L}^q(x,k_T^2) = &-\frac{2p^+}{(2\pi)^3}\int \mathrm{d}^2\zeta \mathrm{d}^2 w \frac{\mathrm{d}^2 k_1 dk_1^-}{(2\pi)^3} e^{i(\underline{k}_1+\underline{k})\cdot(\underline{w}-\underline{\zeta})}\theta(k_1^-)\sum_{\sigma_1,\sigma_2} \\ &\times \bar{v}_{\sigma_2}(k_2)\frac{1}{2}\gamma^+\gamma^5 v_{\sigma_1}(k_1) 2\sqrt{k_1^- k_2^-} \times \left\langle \mathrm{T} V_{\underline{\zeta}}^{ij}\left(\bar{v}_{\sigma_1}(k_1)\hat{V}_{\underline{w}}^{\dagger ji} v_{\sigma_2}(k_2)\right)\right\rangle \\ &\times \left.\frac{1}{[2k_1^- xP^+ + \underline{k}_1 - i\epsilon k_1^-][2k_1^- xP^+ + \underline{k}^2 + i\epsilon k_1^-]}\right|_{k_2^-=k_1^-,\underline{k}_2=-\underline{k}} + c.c. \end{aligned} \tag{13}$$

$\Rightarrow$ Focus on the green part!

Situation prior [2204.11898] (2/3)

The previous green operator reads

$$\left(\bar{v}_\sigma(p)\hat{V}_{\underline{x}}^{\dagger}v_{\sigma'}(p')\right) = 2\sqrt{p^- p'^-}\delta_{\sigma\sigma'}\left(V_{\underline{x}}^{\dagger} - \sigma V_{\underline{x}}^{pol\dagger} + \cdots\right). \tag{14}$$

Remarks

- Decompose into helicity-dependent and helicity-independent parts.
- Probing the helicity of the target: Expecting $\langle\hat{\mathcal{O}}^{pol}\rangle \propto \sigma\delta_{\sigma\sigma'}\Sigma\delta_{\Sigma\Sigma'}$
- Unpolariazed is simply $\langle\hat{\mathcal{O}}^{eik}\rangle \propto \delta_{\sigma\sigma'}\delta_{\Sigma\Sigma'}$

$\Longrightarrow$ The flavor-singlet contribution simplified at small-x gives

$$g_{1L}^{S}(x,k_T^2) = \frac{8N_c i}{(2\pi)^5}\int \mathrm{d}^2\zeta \mathrm{d}^2\underline{w}\; e^{-i\underline{k}\cdot(\underline{\zeta}-\underline{w})}\int_{\Lambda^2/s}^{1}\frac{dz}{z}\frac{\underline{\zeta}-\underline{w}}{(\underline{\zeta}-\underline{w})^2)}\cdot\frac{\underline{k}}{\underline{k}^2}G_{\underline{w},\underline{y}}(zs). \tag{15}$$

The dipole operator $G_{\underline{w},\underline{y}}(zs)$ is

$$G_{\underline{w},\underline{y}}(zs) = \frac{k_1^- p^+}{N_c}\mathrm{Re}\left\langle \mathrm{T\,tr}\left[V_{\underline{x}}V_{\underline{w}}^{pol\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{w}}^{pol}V_{\underline{x}}^{\dagger}\right]\right\rangle, \tag{16}$$

where the polarized Wilson line reads

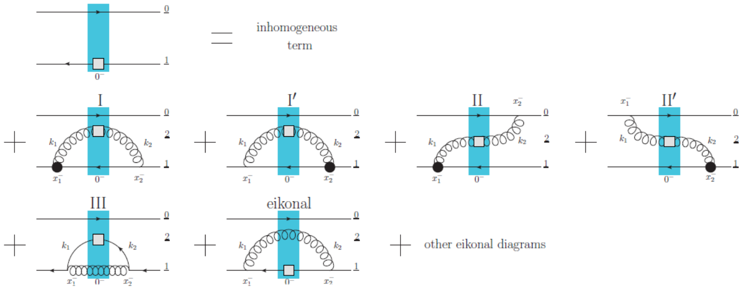
$$\begin{aligned} V_{\underline{x}}^{pol} &= ig\frac{p^+}{s}\int dx^- V_{\underline{x}}[\infty,x^-]F^{12}V_{\underline{x}}[x^-,-\infty] \\ &- g^2\frac{p^+}{s}\int dx_1^-\int_{x_1^-} dx_2^- V_{\underline{x}}[\infty,x_2^-]t^b\psi_\beta(x_2^-,\underline{x}U_{\underline{x}}^{ba}[x_2^-,x_1^-]\left[\tfrac{1}{2}\gamma^+\gamma^5\right]_{\alpha\beta}\bar{\psi}_\alpha(x_1^-,\underline{x})t^a V_{\underline{x}}[x_1^-,-\infty]. \end{aligned} \tag{17}$$

Situation prior to [2204.11898] (3/3)

Remarks

- "Dressed dipoles" involve polarized WL. Obtained as sub-eikonal corrections to the scattering of a quark on a target.
- Corrections are proportional to $\sigma\delta_{\sigma\sigma'}$ in helicity basis (Brodsky-Lepage spinors in the minus direction).

Evolution (DLA, Involves the same WL at different coordinates $\longrightarrow \sigma\delta_{\sigma\sigma'}$)



Solve

Intercept in the pure glue case is $\alpha_h^q \sim 2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}$.

✗ Disagreement with BER pure glue intercept $\alpha_h^q \sim 3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}$

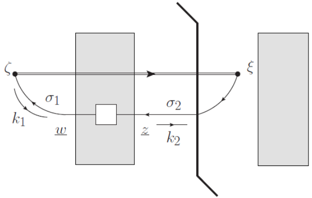
Quark flavor-singlet helicity TMD - New dipole (1/2)

Let us start again from the quark helicity TMD: [2204.11898]

$$g_{1L}^q(x,k_T^2) = -\frac{2p^+}{(2\pi)^3}\int \mathrm{d}^2\zeta \mathrm{d}^2 w \mathrm{d}^2 z \frac{\mathrm{d}^2 k dk^-}{(2\pi)^3} e^{i\underline{k}_1\cdot(\underline{w}-\underline{\zeta})+i\underline{k}\cdot(\underline{z}-\underline{\zeta})}\theta(k_1^-)\sum_{\sigma_1,\sigma_2}$$
$$\times \bar{v}_{\sigma_2}(k_2)\tfrac{1}{2}\gamma^+\gamma^5 v_{\sigma_1}(k_1) 2\sqrt{k_1^- k_2^-} \times \left\langle \mathrm{Ttr}\left[V_{\underline{\zeta}} V^{\dagger}_{\underline{z},\underline{w};\sigma_2,\sigma_1}\right]\right\rangle$$
$$\times \left.\frac{1}{[2k_1^- xP^+ + \underline{k}_1 - i\epsilon k_1^-][2k_1^- xP^+ + \underline{k}^2 + i\epsilon k_1^-]}\right|_{k_2^-=k_1^-,\underline{k}_2=-\underline{k}} + c.c. \qquad (18)$$

Remarks

- $V_{\underline{z},\underline{w};\sigma',\sigma}$ is the quark S-matrix for a quark-target scattering in helicity-basis.
- Allows for non locality before and after the shock wave.



At sub-eikonal order:

$$V_{\underline{x},\underline{y};\sigma',\sigma} = V_{\underline{x}}\,\delta^2(\underline{x}-\underline{y})\,\delta_{\sigma,\sigma'} \tag{19}$$
$$+\frac{i\,P^+}{s}\int_{-\infty}^{\infty} dz^-\,d^2z\;V_{\underline{x}}[\infty,z^-]\,\delta^2(\underline{x}-\underline{z})\left[-\delta_{\sigma,\sigma'}\overleftarrow{D}^i\,D^i+g\,\sigma\,\delta_{\sigma,\sigma'}\,F^{12}\right](z^-,\underline{z})\,V_{\underline{y}}[z^-,-\infty]\,\delta^2(\underline{y}-\underline{z})$$
$$-\frac{g^2P^+}{2\,s}\delta^2(\underline{x}-\underline{y})\int_{-\infty}^{\infty}dz_1^-\int_{z_1^-}^{\infty}dz_2^-\,V_{\underline{x}}[\infty,z_2^-]\,t^b\,\psi_\beta(z_2^-,\underline{x})\,U^{ba}_{\underline{x}}[z_2^-,z_1^-]\left[\delta_{\sigma,\sigma'}\,\gamma^+-\sigma\,\delta_{\sigma,\sigma'}\,\gamma^+\gamma^5\right]_{\alpha\beta}$$
$$\times\,\bar{\psi}_\alpha(z_1^-,\underline{x})\,t^a\,V_{\underline{x}}[z_1^-,-\infty],$$

Remarks

- Blue $\longrightarrow$ Already used in previous V^{pol}. Label *of the first kind*; notation $V^{pol\,[1]}$. Proportional to $\sigma\delta_{\sigma\sigma'}$.
- Red $\longrightarrow$ "NEW" (in our framework). Label *of the second kind*; notation $V^{pol\,[2]}$. Proportional to $\delta_{\sigma\sigma'}$.

Picture?

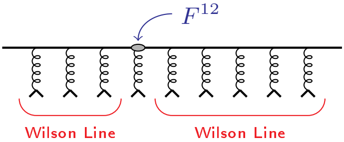
For the quark S-matrix at sub eikonal order, see also:

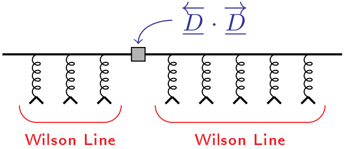
- Balitsky and Tarasov, e.g. [1505.02151]
- Chirilli, e.g. [1807.11435]
- Altinoluk et al., e.g. [2012.03886]
- Kovchegov et al., e.g. [1808.09010] [2108.03667]

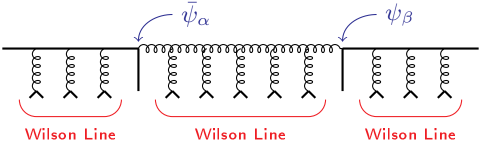
Polarized WL,

$$V_{\underline{x}}^{\mathrm{pol}[1]} = \underbrace{V_{\underline{x}}^{\mathrm{G}[1]} + V_{\underline{x}}^{\mathrm{q}[1]}}_{\sigma\,\delta_{\sigma\sigma'}}, \quad V_{\underline{x},\underline{y}}^{\mathrm{pol}[2]} = \underbrace{V_{\underline{x},\underline{y}}^{\mathrm{G}[2]} + V_{\underline{x}}^{\mathrm{q}[2]}\,\delta^2(\underline{x}-\underline{y})}_{\delta_{\sigma\sigma'}} .$$

can be represented as

$V_{\underline{x}}^{G[1]} =$ 

$V_{\underline{x}}^{G[2]} =$ 

$V_{\underline{x}}^{q[1/2]} =$ 

Contraction with $(\gamma^+\gamma^5)_{\alpha\beta} \times \sigma\delta_{\sigma\sigma'}$ or $\gamma^+_{\alpha\beta} \times \delta_{\sigma\sigma'}$

Simplified at small-x, the quark flavor-singlet helicity TMD reads

$$g_{1L}^S(x,k_T^2) = \frac{8\,N_c\,N_f}{(2\pi)^5} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int d^2x_{10}\, e^{i\underline{k}\cdot\underline{x}_{10}} \left[i\,\frac{\underline{x}_{10}}{x_{10}^2} \cdot \frac{\underline{k}}{\underline{k}^2} \left[Q(x_{10}^2,zs) + G_2(x_{10}^2,zs)\right] - \frac{(\underline{k}\times\underline{x}_{10})^2}{\underline{k}^2\, x_{10}^2}\, G_2(x_{10}^2,zs)\right], \tag{20}$$

The new dipole G_2 is defined with

$$G_{10}^{j}(zs) \equiv \frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{\zeta}}^{\dagger}\, V_{\underline{\xi}}^{j\ \mathrm{G}[2]} + \left(V_{\underline{\xi}}^{j\ \mathrm{G}[2]}\right)^{\dagger} V_{\underline{\zeta}}\right] \right\rangle\!\!\right\rangle \tag{21}$$

$$\int d^2\left(\frac{x_1+x_0}{2}\right)\, G_{10}^{i}(zs) = (x_{10})_{\perp}^{i}\, G_1(x_{10}^2,zs) + \epsilon^{ij}\, (x_{10})_{\perp}^{j}\, G_2(x_{10}^2,zs). \tag{22}$$

Remarks

- Dependence on previously used dipole $Q(x_{10}^2,zs)$.
- The previously missing dependence is proportional to $G_2(x_{10}^2,zs)$.
- New contribution depends on the sub-eikonal operator $\overleftrightarrow{\underline{D}}$, related to the Jaffe-Manohar polarized gluon distribution.

Evolution, revised and updated

One step of evolution reads the formal form

$$\hat{\mathcal{O}}_i = \hat{\mathcal{O}}_i^{(0)} + \sum_j \mathcal{K}_{ij} \otimes \hat{\mathcal{O}}_j \tag{23}$$

Features

- Kernel involves transverse and longitudinal logarithmic integrals.
  $\longrightarrow$**The evolution is DLA**, as opposed to the unpolarized one being SLA.
- **Lifetime ordering** is explicit $\theta(z\underline{x}_{10}^2 - z'\underline{x}_{21}^2)$.
- **Mixing** to operators involving Wilson lines of first and/or second kind.

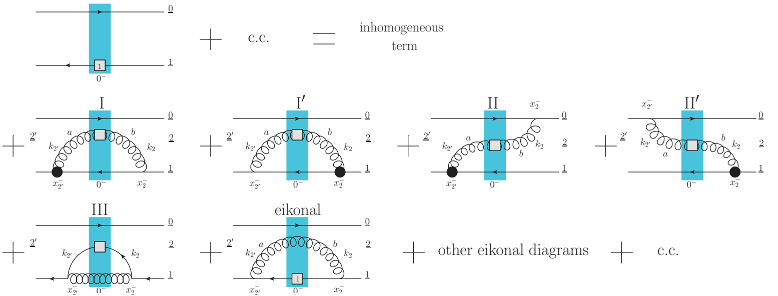
Remarks

- Similar to the Balitsky hierarchy, equation are not closed.
- Can be closed! In the 't Hooft large N_c-limit or Veneziano large $N_c \& N_f$-limit.

Results

$$\Delta G(x)|_{x\ll 1} \sim \left(\frac{1}{x}\right)^{\alpha} \tag{24}$$

- In the pure glue sector, the intercept becomes $\alpha_h^q \sim 3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}$. In complete agreement with BER result.
- Iterating this kernel, one recover the small-x spin-dependent DGLAP kernel.



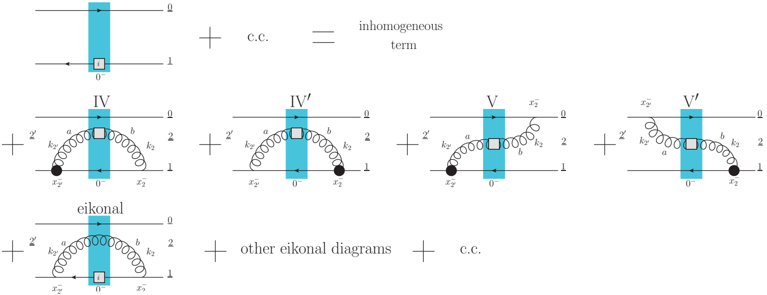
$$\frac{1}{2N_c}\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\right\rangle(zs) = \frac{1}{2N_c}\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\right\rangle_0(zs) \tag{95}$$

$$+ \frac{\alpha_s N_c}{2\pi^2}\int\limits_{\Lambda^2/s}^{z}\frac{dz'}{z'}\int d^2x_2 \Bigg\{ \left[\frac{1}{x_{21}^2} - \frac{\underline{x}_{21}}{x_{21}^2}\cdot\frac{\underline{x}_{20}}{x_{20}^2}\right]\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{\mathrm{pol}[1]}\right)^{ba} + \mathrm{c.c.}\right\rangle\!\right\rangle(z's)$$

$$+ \left[2\frac{\epsilon^{ij}\, x_{21}^j}{x_{21}^4} - \frac{\epsilon^{ij}\,(x_{20}^j + x_{21}^j)}{x_{20}^2\, x_{21}^2} - \frac{2\,\underline{x}_{20}\times\underline{x}_{21}}{x_{20}^2\, x_{21}^2}\left(\frac{x_{21}^i}{x_{21}^2} - \frac{x_{20}^i}{x_{20}^2}\right)\right]\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{i\,\mathrm{G}[2]}\right)^{ba} + \mathrm{c.c.}\right\rangle\!\right\rangle(z's)\Bigg\}$$

$$+ \frac{\alpha_s N_c}{4\pi^2}\int\limits_{\Lambda^2/s}^{z}\frac{dz'}{z'}\int \frac{d^2x_2}{x_{21}^2}\left\{\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}[1]\,\dagger}\right] U_{\underline{1}}^{ba}\right\rangle\!\right\rangle(z's) + 2\frac{\epsilon^{ij}\, x_{21}^j}{x_{21}^2}\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{i\,\mathrm{G}[2]\,\dagger}\right] U_{\underline{1}}^{ba}\right\rangle\!\right\rangle(z's) + \mathrm{c.c.}\right\}$$

$$+ \frac{\alpha_s N_c}{2\pi^2}\int\limits_{\Lambda^2/s}^{z}\frac{dz'}{z'}\int d^2x_2\, \frac{x_{10}^2}{x_{21}^2\, x_{20}^2}\left\{\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] U_{\underline{2}}^{ba}\right\rangle\!\right\rangle(z's) - \frac{C_F}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right]\right\rangle\!\right\rangle(z's) + \mathrm{c.c.}\right\}.$$



$$\frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\!\right\rangle(zs) = \frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\!\right\rangle_0(zs) \tag{106}$$

$$+ \frac{\alpha_s N_c}{4\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2 x_2 \left\{ \left[\frac{\epsilon^{ij} x_{21}^j}{x_{21}^2} - \frac{\epsilon^{ij} x_{20}^j}{x_{20}^2} + 2x_{21}^i\,\frac{\underline{x}_{21}\times\underline{x}_{20}}{x_{21}^2\,x_{20}^2}\right] \frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{\mathrm{pol}[1]}\right)^{ba} + \mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)\right.$$

$$+ \left[\delta^{ij}\left(\frac{3}{x_{21}^2} - 2\frac{\underline{x}_{20}\cdot\underline{x}_{21}}{x_{20}^2\,x_{21}^2} - \frac{1}{x_{20}^2}\right) - 2\frac{x_{21}^i\,x_{20}^j}{x_{21}^2\,x_{20}^2}\left(2\frac{\underline{x}_{20}\cdot\underline{x}_{21}}{x_{20}^2}+1\right) + 2\frac{x_{21}^i\,x_{21}^j}{x_{21}^2\,x_{20}^2}\left(2\frac{\underline{x}_{20}\cdot\underline{x}_{21}}{x_{21}^2}+1\right) + 2\frac{x_{20}^i\,x_{20}^j}{x_{20}^4} - 2\frac{x_{21}^i\,x_{21}^j}{x_{21}^4}\right]$$

$$\left.\times \frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{j\,\mathrm{G}[2]}\right)^{ba} + \mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)\right\}$$

$$+ \frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2 x_2\,\frac{x_{10}^2}{x_{21}^2\,x_{20}^2}\left\{\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right]\,\left(U_{\underline{2}}\right)^{ba}\right\rangle\!\!\right\rangle(z's) - \frac{C_F}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right]\right\rangle\!\!\right\rangle(z's) + \mathrm{c.c.}\right\}.$$

DLA Small-x evolution at large N_c

$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[\Gamma(x_{10}^2, x_{21}^2, z's) + 3\, G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) + 2\, \Gamma_2(x_{10}^2, x_{21}^2, z's) \Big], \tag{25a}$$

$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{32}^2}{x_{32}^2} \Big[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3\, G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) + 2\, \Gamma_2(x_{10}^2, x_{32}^2, z''s) \Big], \tag{25b}$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \Big[G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \Big], \tag{25c}$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \Big[G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \Big]. \tag{25d}$$

Recovering small-x pol DGLAP

- **Pol DGLAP** splitting function at small-x is

$$\Delta P_{gg}(z) \to \frac{\alpha_s}{2\pi} 4N_c + \left(\frac{\alpha_s}{2\pi}\right)^2 4N_c^2 \ln^2 z + \left(\frac{\alpha_s}{2\pi}\right)^3 \frac{7}{3} N_c^3 \ln^4 z \tag{26}$$

and DGLAP evolution is

$$\frac{\partial \Delta G(x, Q^2)}{\partial \ln Q^2} = \int_x^1 \frac{dz}{z} \Delta P_{gg} \Delta G\left(\frac{x}{z}, Q^2\right) \tag{27}$$

- For the **DLA small-x evolution equations**, start with

$$G^{(0)}(x_1 0^2, zs) = 0, \qquad G_2^{(0)}(x_1 0^2, zs) = 1 \tag{28}$$

iterate three times, one finds

$$\Delta G^{(3)}(x, Q^2) = \left(\frac{\alpha_s}{\pi}\right)^3 \left[\underbrace{\frac{7}{120} \ln^5\left(\frac{1}{x}\right) \ln\left(\frac{Q^2}{\Lambda^2}\right)}_{NNLO\ DGLAP_{gg}} + \frac{1}{6} \ln^4\left(\frac{1}{x}\right) \ln^2\left(\frac{Q^2}{\Lambda^2}\right) + \underbrace{\frac{2}{9} \ln^3\left(\frac{1}{x}\right) \ln^3\left(\frac{Q^2}{\Lambda^2}\right)}_{(LO)^3\ DGLAP_{gg}}\right]$$

using $1/x_{10}^2 \to Q^2, \qquad zsz_{10}^2 \to 1/x.$

A very last slide

A quick conclusion

- Small-x evolution equations for helicity distributions at DLA.
- Involve G_2 operator, which gives small-x DGLAP evolution to three loops.
- Numerical agreement with the intercept found by BER.

Some Prospects

- Going beyond the DLA limit. Resumming IR-log, and thus interfacing with full spin-dependent DGLAP.
- Fixing helicity-JIMWLK at DLA and Implementation.
- Phenomenology using the JAM framework.

Extra

- Gluon helicity and Lipatov vertex
- Large N_c limit
- TMD's
- g_1
- hJIMWLK slides

Gluon helicity

From the Jaffe-Manohar (JM) gluon helicity PDF

$$\Delta G(x,Q^2) = \int^{Q^2} d^2k\, g_{1L}^{G\,dip}(x,k_T^2) = \frac{-2i}{x\,P^+}\frac{1}{4\pi}\frac{1}{2}\sum_{S_L} S_L \int_{-\infty}^{\infty} d\xi^-\, e^{ixP^+\xi^-}$$

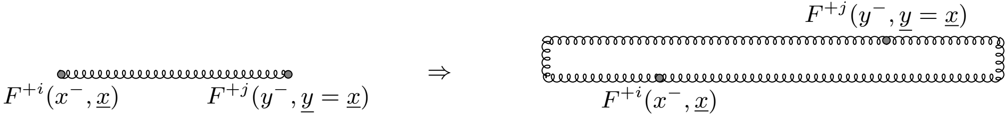
$$\times \langle P,S_L|\,\epsilon^{ij}\, F^{a+i}(0^+,0^-,\underline{0})\, U_{\underline{0}}^{ab}[0,\xi^-]\, F^{b+j}(0^+,\xi^-,\underline{0})\,|P,S_L\rangle\,, \qquad (29)$$

Identify after some algebra the dipole gluon helicity TMD

$$g_{1L}^{G\,dip}(x,k_T^2) = \frac{-2i}{x\,P^+\,V^-}\frac{1}{(2\pi)^3}\frac{1}{2}\sum_{S_L} S_L\ \langle P,S_L|\,\epsilon^{ij}\,\mathrm{tr}\Big[L^{i\,\dagger}(x,\underline{k})\,L^j(x,\underline{k})\Big]\,|P,S_L\rangle \qquad (30)$$

where we define the Lipatov vertex:

$$L^j(x,\underline{k}) \equiv \int_{-\infty}^{\infty} d\xi^-\, d^2\xi\, e^{ixP^+\xi^- - i\underline{k}\cdot\underline{\xi}}\ V_{\underline{\xi}}[\infty,\xi^-]\,(\partial^j A^+ + ixP^+ A^j)\, V_{\underline{\xi}}[\xi^-,-\infty] \qquad (31)$$



Gluon helicity

Expanding the Lipatov vertex in eikonality (i.e. Bjorken x)

$$L^j(x,\underline{k}) = \int_{-\infty}^{\infty} d\xi^- \, d^2\xi \, e^{-i\underline{k}\cdot\underline{\xi}} \, V_{\underline{\xi}}[\infty,\xi^-] \left[\partial^j A^+ + ixP^+ \left(\xi^- \partial^j A^+ + A^j\right) + \mathcal{O}(x^2)\right] V_{\underline{\xi}}[\xi^-,-\infty], \tag{32}$$

which we can write

$$L^j(x,\underline{k}) = -\frac{k^j}{g} \int d^2\xi \, e^{-i\underline{k}\cdot\underline{\xi}} \, V_{\underline{\xi}} - \frac{xP^+}{2g} \int d^2\xi \, e^{-i\underline{k}\cdot\underline{\xi}} \int_{-\infty}^{\infty} dz^- \, V_{\underline{\xi}}[\infty,z^-] \left[D^j - \overleftarrow{D}^j\right] V_{\underline{\xi}}[z^-,-\infty] \tag{33}$$

Performing the helicity dependent "CGC average"

$$g_{1L}^{G\,dip}(x,k_T^2) = \frac{-4i}{g^2\,(2\pi)^3}\,\epsilon^{ij}\,k^i \int d^2\zeta \, d^2\xi \, e^{-i\underline{k}\cdot(\underline{\xi}-\underline{\zeta})} \underbrace{\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{\zeta}}^\dagger \, V_{\underline{\xi}}^{j\,\mathrm{G}[2]} + \left(V_{\underline{\xi}}^{j\,\mathrm{G}[2]}\right)^\dagger V_{\underline{\zeta}}\right]\right\rangle\!\!\right\rangle}_{=2N_c G^j_{\underline{\xi},\underline{\zeta}}(zs)}, \tag{34}$$

with a polarized Wilson line of the second kind

$$V_{\underline{z}}^{i\,\mathrm{G}[2]} \equiv \frac{P^+}{2s} \int_{-\infty}^{\infty} dz^- \, V_{\underline{z}}[\infty,z^-] \left[D^i(z^-,\underline{z}) - \overleftarrow{D}^i(z^-,\underline{z})\right] V_{\underline{z}}[z^-,-\infty]. \tag{35}$$

$\Longrightarrow$ We call $G^j_{\underline{\xi},\underline{\zeta}}(zs)$ a Polarized dipole amplitude of the second kind.

In the large N_c-limit (drop quarks t-channel exchanges)

$$U_{\underline{x}}^{\mathrm{pol}[1]} \to U_{\underline{x}}^{\mathrm{G}[1]} \tag{36}$$

Replace adjoint WL using:

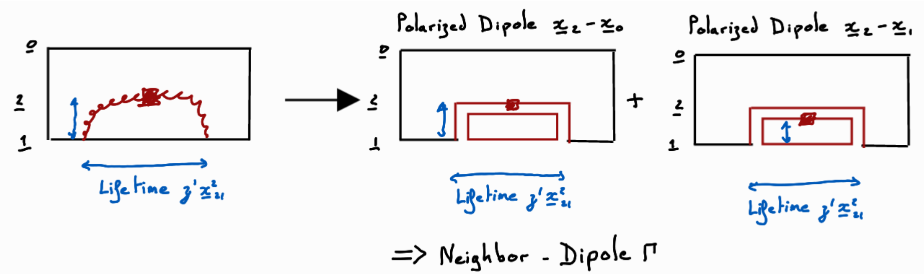
$$(U_{\underline{x}})^{ba} = 2\,\mathrm{tr}[t^b V_{\underline{x}} t^a V_{\underline{x}}^{\dagger}]. \tag{37}$$

and

$$\left(U_{\underline{x}}^{\mathrm{G}[1]}\right)^{ba} = 2 \times \left\{2\,\mathrm{tr}\left[t^b\, V_{\underline{x}}\, t^a\, V_{\underline{x}}^{\mathrm{G}[1]\,\dagger}\right] + 2\,\mathrm{tr}\left[t^b\, V_{\underline{x}}^{\mathrm{G}[1]}\, t^a\, V_{\underline{x}}^{\dagger}\right]\right\}. \tag{38}$$

$$\left(U_{\underline{x}}^{i\,\mathrm{G}[2]}\right)^{ba} = 2\,\mathrm{tr}\left[t^b\, V_{\underline{x}}\, t^a\, V_{\underline{x}}^{i\,\mathrm{G}[2]\,\dagger}\right] + 2\,\mathrm{tr}\left[t^b\, V_{\underline{x}}^{i\,\mathrm{G}[2]}\, t^a\, V_{\underline{x}}^{\dagger}\right] \tag{39}$$

Notice the factor 2 in the former. A gluon has twice the spin of a quark.



After Fiertzing arround, introduce neighbor dipole amplitude Γ to enforce lifetime ordering at each step of the evolution.

Solving, 't Hooft limit - Equation and intercept

$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{sx_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[\Gamma(x_{10}^2, x_{21}^2, z's) + 3\, G(x_{21}^2, z's)$$

$$+ 2\, G_2(x_{21}^2, z's) + 2\, \Gamma_2(x_{10}^2, x_{21}^2, z's) \Big], \tag{40a}$$

$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{sx_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{32}^2}{x_{32}^2} \Big[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3\, G(x_{32}^2, z''s)$$

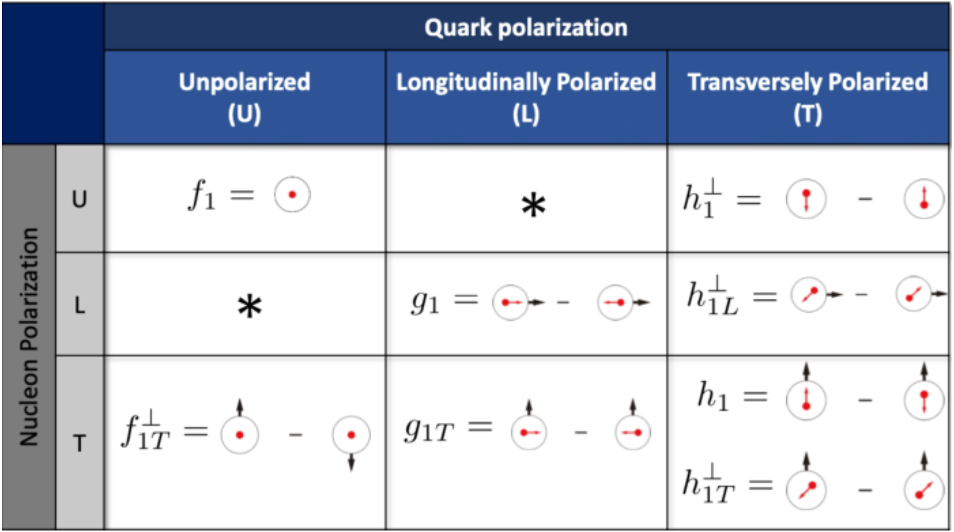
$$+ 2\, G_2(x_{32}^2, z''s) + 2\, \Gamma_2(x_{10}^2, x_{32}^2, z''s) \Big], \tag{40b}$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \right], \tag{40c}$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \left[G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \right]. \tag{40d}$$

Numerical solution for the intercept:

$$\Delta\Sigma(x, Q^2) \sim \Delta G(x, Q^2) \sim g_1(x, Q^2) \sim (1/x)^{3.66 \sqrt{\frac{\alpha_s N_c}{2\pi}}}. \tag{41}$$



From "QCD2019 Workshop Summary"

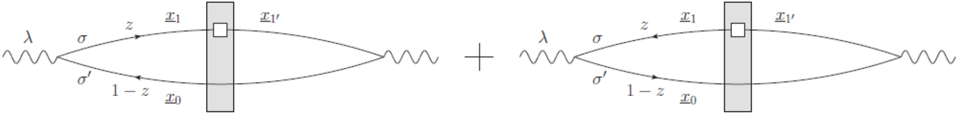
Getting g_1 - short recap

From the antisym hadronic tensor (e.g. [PDG] [Lampe and Reya 2000])

$$W^{[\mu\nu]} \sim i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho}{M_p P \cdot q} \left[S^\sigma g_1(x, Q^2) + \left(S^\sigma - \frac{Q \cdot q}{P \cdot q} P^\sigma \right) g_2(x, Q^2) \right] \tag{42}$$

DIS pol Scattering cross section is

$$\sigma^{\gamma^* p}(\lambda, \Sigma) = -\frac{8\pi^2 \alpha_{EM} x}{Q^2} \lambda\Sigma \left[g_1(x, Q^2) - \frac{4x^2 M_p^2}{Q^2} g_2(x, Q^2) \right] \tag{43}$$



One finally obtain (take the DLA limit)

$$g_1(x, Q^2) = -\sum_f \frac{Z_f^2}{2} \frac{N_c}{2\pi^3} \int_{\Lambda^2/s}^{1} \frac{dz}{z} \int_{1/zs}^{min\{1/zQ^2, 1/\Lambda^2\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2G_2(x_{10}^2, zs) \right] \tag{44}$$