Butterflies revisited for asymptotically Lifshitz black hole

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Arxiv : 2406.18319 (In collaboration with Banashree Baishya and Nibedita Padhi)

2PiNTS
IFJ-PAN
12th September, 2024

The presentation is supported by the program "Excellence initiative – research university"

Outline:

- Holographic duality
- Holography in Lifshitz theories and asymptotic Lifshitz black hole
- Lyapunov exponent and butterfly velocity
- Chaos and its holographic understanding
- Quantum chaos
  - ➢ Entanglement Wedge method
  - ➢ Pole Skipping
  - ➢ OTOC
- Classical chaos
  - ➢ Eikonal phase
  - ➢ Lyapunov exponent
- Concluding remark

Holographic duality

- Perturbative gravity theory in d+1 dim ⟺ Nonperturbative QFT in d dim (In analogy with optical hologram)

- One-to-one correspondence between observables in both sides helps to understand nonperturbative QFTs.

- Best explored holography: AdS/CFT holography
  - Gravity side is on Anti-de-Sitter spacetime (Solution of Einstein's equation with negative curvature)
  - Dual QFT is conformally invariant (invariant under Lorentz, scaling and inversion symmetry)

- Candidate gravity theory: String theory on 5d AdS, Dual QFT: 4d Super Yang Mills theory on the boundary of AdS
- String coupling : g_s , YM Coupling: g_{YM} , Effective gauge coupling : $\lambda = g_{YM}^2 N$
- Holographic dictionary: $4\pi g_s = g_{YM}^2 = \frac{\lambda}{N}$.

- Perturbative string theory for g_s , $g_{YM} \ll 1$, Nonperturbative Yang-Mills theory occurs for $N \to \infty$.

Maldacena, 1998

Holographic duality for Lifshitz-like gauge/gravity theory

Gauge theory: d-dim Nonrelativistic QFT following Lifshitz scaling $t \to \Omega^{\xi} t, \quad z \to \Omega z, \quad \vec{x} \to \Omega \vec{x}$

Holographic dual : Gravity theory in d+1 dim Lifshitz spacetime $ds^2 = \frac{R^2}{z^2}\left[-\frac{R^{2(\xi-1)}}{z^{2(\xi-1)}}dt^2 + d\vec{x}^2_{d-1} + dz^2\right]$

Taylor, 2015

Scaling is different than the conformal one because of dynamical exponent ξ in the time scaling, causing a geometry different than AdS.

Anisotropy along time scaling causes breaking in the Lorentz symmetry.

Helpful for deeper understanding of strongly interacting nonrelativistic condensed matter systems.

Concrete holographic dictionary is yet to explore, bulk to boundary propagators, string and brane solutions, quantum entanglement properties are being studied in the context of Lifshitz holography.

Asymptotically Lifshitz Black Hole

Black hole geometries that asymptote nonrelativistic Lifshitz spacetime : $ds^2 = \frac{R^2}{z^2}\left[-\frac{R^{2(\xi-1)}}{z^{2(\xi-1)}}dt^2 + d\vec{x}^2_{d-1} + dz^2\right]$

(d+1)-dim Lifshitz black hole: $$ds^2 = \frac{R^2}{z^2}\left[-\frac{R^{2(\xi-1)}}{z^{2(\xi-1)}}f(z)dt^2 + d\vec{x}^2_{d-1} + \frac{dz^2}{f(z)}\right]$$

Danielsson, Thorlacius, 2009

$$f(z) = 1 - \left(\frac{z}{z_h}\right)^{d-1+\xi}.$$

z_h is the horizon

Satisfies the Lifshitz scaling: $t \to \Omega^\xi t, \quad z \to \Omega z, \quad \vec{x} \to \Omega\vec{x}$ (Anisotropic along t and Lorentz breaking)

Temperature : $T = \frac{1}{z_h^\xi}\frac{d-1+\xi}{4\pi} = \frac{1}{\beta^\xi} \equiv \frac{1}{\tilde{\beta}}.$

Legitimate holographic dual theory: d-dim nonrelativistic Lifshitz field theory with finite temperature.

Motif of this talk is to understand chaos in asymptotic Lifshitz black hole

We will use three different holography-based approach of deriving chaos, namely,

1. Entanglement Wedge Method
2. Pole-skipping
3. Measurement of Out-of-Time-Ordered correlators

Chaos: Lyapunov Exponent and Butterfly Velocity

Chaos in disordered system: Growing perturbations due to small changes in initial conditions.

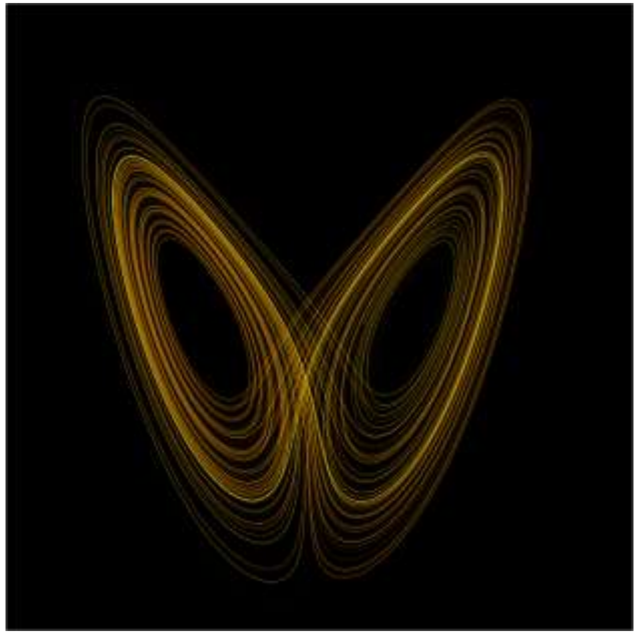
Salient measures of chaos : 1. Butterfly velocity, 2. Lyapunov exponent

Butterfly effect: Considerable change in the later state due to small deviation in the initial conditions.

Initial points arbitrarily close together separate over time at an exponential rate.

Quantum butterfly effect: Effect of small change in a Hamiltonian system with given position and velocity.

Butterfly velocity (v_B): Characterizes the propagation of chaos in a local system.



Lyapunov exponent ($\boldsymbol{\lambda}_L$): Rate of separation between the initial points.

Positive $\boldsymbol{\lambda}_L$ ⟹ Chaotic system

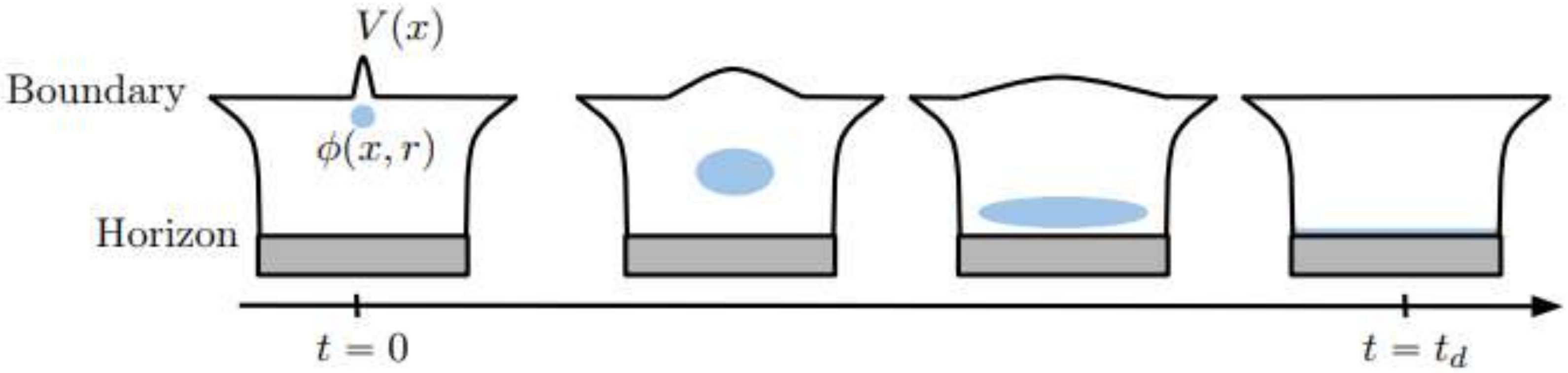
For two-derivative gravity:

$$\lambda_L \leq \frac{2\pi}{\beta}$$

Maldacena, Shenker, Stanford, 2015

Holographic understanding of chaos

- Operators in boundary field theory ⟷ Field in the gravitational system inside the bulk spacetime.
- Chaos in the thermal field theory: Perturbation of boundary operator and eventual scrambling in late time.
- Holographic bulk picture: Infalling quanta which is initially close to boundary gets absorbed in the BH interior.



Shenker, Stanford, 2013

- Spreading of trajectories in terms of 2-pt functions : $G(t) = \langle \mathbf{X}(t)\,\mathbf{X}(0)\rangle_\beta - \langle \mathbf{X}\rangle_\beta^2\,,$ Stanford, 2018
- Large G(t) → Significant chaos
- Holographically realised as exp [$\lambda_L t$] up to the scrambling time where chaos becomes maximum.
- In thermal Lifshitz black hole , chaos is characterized by its dynamical exponent ξ.

Entanglement wedge and holographic entanglement entropy

Entanglement entropy: Measure of classical and quantum correlation between the states in entangled Hilbert spaces

*Ryu-Takayanagi prescription gives holographic notion of entanglement entropy. Ryu, Takayanagi, 2006

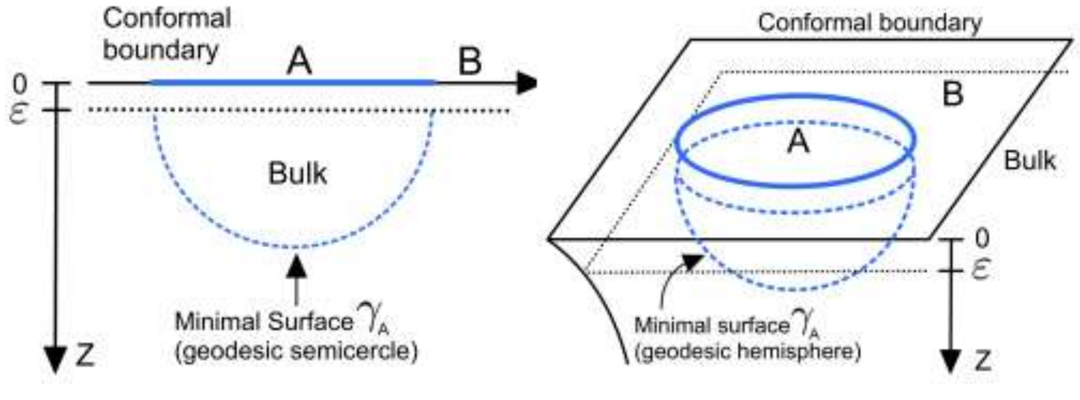
*RT prescription suggests: Entanglement entropy appears to be equivalent to the area of an extremal surface inside the bulk which has coinciding boundary as that of the bulk geometry.

$$S_A = \frac{\text{Area of } \gamma_A}{4G}$$

- Extremal surface area is given by the minimisation of $\int d^{d-1}\xi \, \frac{1}{z^{d-1}} \sqrt{\det\left(\frac{\partial z}{\partial \xi^i}\frac{\partial z}{\partial \xi^j} + \mathfrak{g}_{\mu\nu}(x,z)\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\right)}$

* Entanglement wedge: Slice in the bulk that bounded by the boundary and the extremal surface, captures all quantum information. (Generally, constant time slice)

Pennington, 2019



(a) (b)

Quantum chaos by entanglement wedge method:

✓ Local perturbation of a boundary state causes scrambling in bulk information after late time evolution.

✓ Trajectory of an infalling particle in the extremal RT surface changes.

✓ Constant rate of change: Butterfly velocity (v_B)

Extremal RT surface is determined from holographic Entanglement entropy: $S_{\mathrm{EE}} = 2\pi \int \mathrm{d}^{d-1} y \sqrt{\gamma}\,,$

γ is the induced metric.

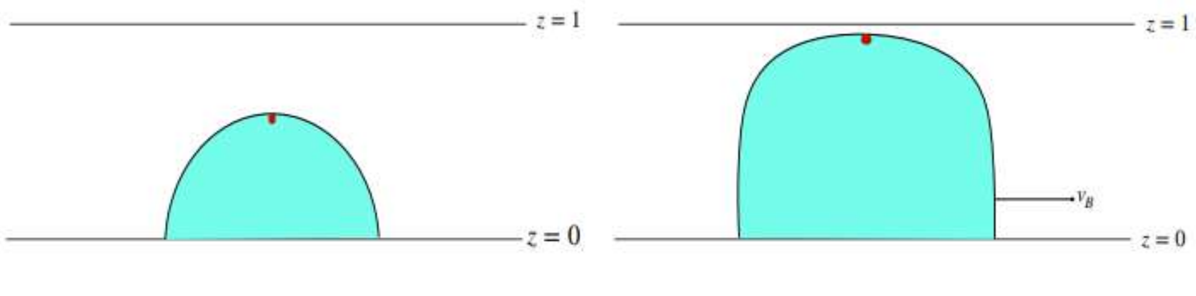


Figure 1. A particle (red dot) is enclosed in the RT surface (light teal)

Parametrization of RT profile for Lifshitz BH: $z(r) = 1 - \epsilon\, y(r)^2,$

$y(r)$ contains Bessel functions of first kind, $y(r) = r^n \left[\mathrm{J}_n(\mu r) + \mathrm{Y}_n(\mu r) \right]\,, \quad n = \frac{1}{2}(d-3), \quad \mu = \sqrt{\frac{(d-1)(d-1+\xi)}{2}}\,.$

$y(r=0,t) \sim e^{-\frac{2\pi}{\beta}t}$, for tip of the RT surface touching the boundary.

$$v_B = \sqrt{\frac{d-1+\xi}{2(d-1)}}\,.$$

in near-horizon limit. For $\xi \longrightarrow 1$, it retrieves v_B for AdS black hole.

AC, Baishya, Padhi, 2024

Quantum chaos from pole-skipping

- Pole-skipping: Lines of poles and zeros of retarded Green's function intersect----A would-be pole gets skipped !

Pole-skipping points of 2-point energy density correlation functions

↕

Blake, Lee, Liu, Davison, Grozdanov,2018; Ahn, Jahnke, Jeong, Kim, 2019

Holographic dual to perturbation in bulk metric

- At pole-skipping points, chaos is measured by frequency $\omega_* = i\lambda_L,$ and momentum $k_* = \frac{i\lambda_L}{v_B},$ of the 2-point correlators.

- 2+1D Lifshitz BH in Eddington-Finkelstein coordinates, perturbation of the metric: $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}(r)e^{-i\omega v+ikx}$.
- Retarded Green's function is governed by only $\delta g_{vv}, \delta g_{vx}$. due to $\delta g_{r\mu} = 0$ and $g^{\mu\nu}\delta g_{\mu\nu} = 0$
- Perturbed Einstein's equation in near horizon limit is vanishing only for δT^r_v
- This is obtained for pole skipping points
- For 2+1D Lifshitz black hole,

$$\omega_* = \frac{i}{2} r_h^\xi (1+\xi)\,, \qquad k_*^2 = -\frac{1}{2}(3r_h - 2r_h^\xi)(1+\xi)\,. \qquad r_h = \left(\frac{4\pi T}{1+\xi}\right)^{1/\xi}$$

$$\lambda_L = 2\pi T\,, \qquad v_B = \sqrt{\frac{1+\xi}{2}}\,.$$

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Out-of-time-ordered correlators and eikonal phase

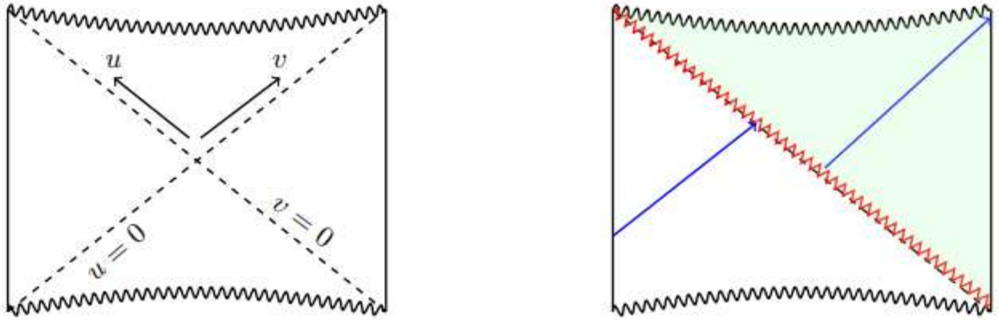
Out-of-Time-Ordered Correlators in a boundary field theory: $\langle \hat{W}(t_2,x_2)\hat{V}(t_1,x_1)\hat{W}(t_2,x_2)\hat{V}(t_1,x_1)\rangle_\beta \,,$

*Holographic realised as a consequence of a shockwave introduced in a 2-particle gravitational scattering between particles moving along the two horizons of double-sided black hole.

Maldacena, Shenker, Stanford, Roberts, Susskind

* Perturbation due to the shockwave is understood in field theo as the double commutator

$$C(t) = \langle V^\dagger(0)W^\dagger(t)W(t)V(0) + V^\dagger(0)W^\dagger(t)W(t)V(0) - \underbrace{W^\dagger(t)V^\dagger(0)W(t)V(0) - V^\dagger(0)W^\dagger(t)V(0)W(t)}_{\text{out-of time-ordered}}\rangle_\beta \,.$$



*Decrease in OTOC implies increase in C(t) causing significant chaos.

In holographic systems, $C(t,x) \approx \exp\left\{\lambda_L\left(t - t_* - \frac{|x|}{v_B}\right)\right\}$

*Gravitational scattering amplitude assumes exponential form $e^{i\delta}$, δ is termed as eikonal phase shift.

Quantum chaos from out-of-time-ordered correlators

We used a doubled sided version of 2+1D Lifshitz black hole by using Kruskal-Szek $u = e^{\frac{\alpha}{2}(r_*-t)}, \quad v = e^{\frac{\alpha}{2}(r_*+t)}$

Perturbation in the metric by a shock wave along one of the horizons in the Kruskal form of 2+1D Lifshitz BH.

- Back-reacted Einstein's equation : $E_{\mu\nu} = \kappa \left(T^{\text{matter}}_{\mu\nu} + T^{\text{shock}}_{\mu\nu}\right)$. gives C(t, x) = $\frac{1}{\sqrt{2(1+\xi)}} e^{\frac{2\pi}{\beta}\left(t-t_*-\frac{\beta\mathcal{M}}{2\pi}x\right)}$

$\mathcal{M}(u,v)|_{v=0} = \sqrt{\frac{1+\xi}{2}}$ in near-horizon limit. This yields $\lambda_L = 2\pi T\,,$ $v_B = \sqrt{\frac{1+\xi}{2}}\,.$

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All the above methods are equivalent and equally competent to study chaos in asymptotically Lifshitz black hole

Eikonal phase and Lyapunov exponent from classical approach

- Eikonal phase: $\delta \equiv p_t(\Delta t) - p_x(\Delta x)$
- Probes the amplitude of gravitational high energy scattering. Parnachev, Sen, Ng, Kulaxizi, 2019, 2021
- In near-boundary zone:

$$\delta \sim i\gamma z_0\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2\xi-2}; \frac{2\xi-1}{2\xi-2}; \frac{z_0^{2\xi-2}}{\gamma^2}\right). \Longleftrightarrow \text{inelastic}$$

- In near-horizon zone:

$$\delta \sim \frac{z_0^{\xi}}{\xi(\xi+1)}\, {}_2F_1(1,\xi;\xi+1;z_0) \quad \text{for } p_x \leq p_t \Longleftrightarrow \text{elastic}$$

$$\delta \sim -\frac{4\gamma\left(\sqrt{z_0-1}-i\right)p_t}{\sqrt{\xi+1}} \quad \text{for } p_x \geq p_t \Longleftrightarrow \text{inelastic}$$

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- z_0 ⟶ Turning point of null geodesic
- γ ⟶ Impact parameter

•Lyapunov exponent :

$$\lambda_L = \sqrt{-\frac{V''_{eff}}{2\dot{t}^2}}.$$

Cardoso, Miranda, Berti, Witek, Zachin, 2009

V_{eff} ⟶ Effective potential for radial motion

•For circular motion, $V_{eff}(z_0) = 0$ and $V'_{eff}(z_0) = 0$

$$\lambda_L = \sqrt{\frac{f_0\left(z_0^2 f_0'' - 2f_0\left(2\xi^2 - 5\xi + 3\right)\right)}{2z_0^{2\xi}}}.$$

•For any unstable orbit, Lyapunov exponent is real.

Limitations on choice of turning point of null geodesic

- Null geodesic equation: $$z_0^{2\xi-2} + \frac{\gamma^2}{z_h^{\xi+1}} z_0^{\xi+1} - \gamma^2 = 0 \,.$$
- Valid roots of this polynomial are real, lie within the singularity and the boundary.
- Conditions on the turning point:

✓ $\gamma = \sqrt{\frac{z_h^{\xi+1}}{z_h^{\xi+1} - 1}}$. For $\gamma \gg 1$ and $z_0 = 1$ ⟶ Not possible for any arbitrary positive real z_h

✓ For, $\gamma \leq 1$, $z_0 \neq 1$. Also, $z_0 \neq 1$ for $z_h = 1$ for any arbitrary γ.

✓ $z_0 \neq 0$ for any arbitrary γ.

✓ Turning point should not be in the vicinity of horizon and boundary.

✓ Lyapunov exponent is also inconsistent around horizon and boundary.

✓ Suitable polynomial distribution may help to find z_0.

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Summary :

- Butterfly effects in asymptotically 2+1D Lifshitz BH using three different methods---EW, PS, OTOC
- All of these give exactly similar results for v_B and $\boldsymbol{\lambda}_L$, hence equivalent for studying chaos.
- Classical chaos is studied by deriving eikonal phase and Lyapunov exponent.
- All these quantities are nontrivial functions of anisotropy index.
- Different scattering scenarios in the vicinity of boundary and horizon.
- Potential limitations on the choices of turning point of the null geodesic.

Future scope:

- Phase factor is exponential in quantum chaos, hypergeometric in classical chaos. To find their connection and see possible classical/quantum correspondence.
- Construction of OTOCs for dual thermofield double states of finite temperature LFTs.

Thank you