# *Time-ordering issue of TMD soft factors*

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# Outline

- Part I: Coordinate-space analyticity and time-ordering issue of TMD soft factors

a) General introduction to coordinate analyticity

b) Euclidean-type parametric representation in perturbation theory

c) Equalities between TMD soft factors

Based on JHEP09(2024)030

# Motivation

1. TMD soft factor is part of the TMDPDFs for inclusive process, for example small $q_T$ inclusive Drell-Yan process.
2. Contains information regarding rapidity evolution: Collins-Soper kernel.
3. Defined in non-Euclidean manner: sums over amplitudes-squares. But also, as Wilson-loop averages with specific Wilson-line directions.
4. Is it possible to find Euclidean-type representations? This point is confusing in literature, but crucial for lattice application.

# TMD soft factors: $S$ and $S_t$

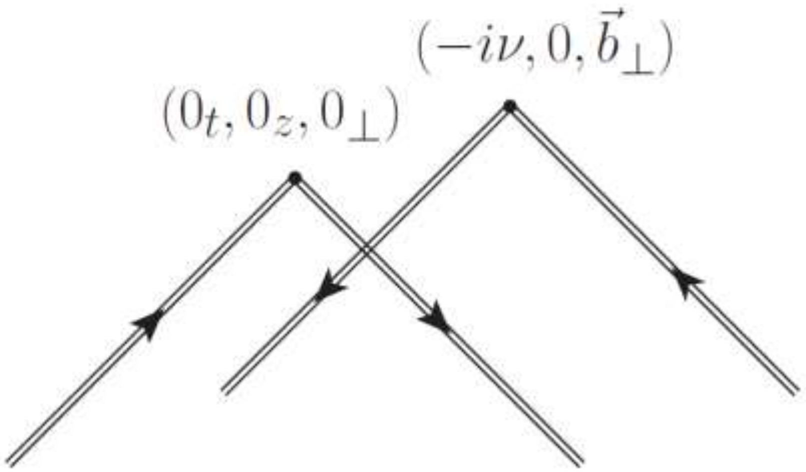



**Figure 4**: The $S$.

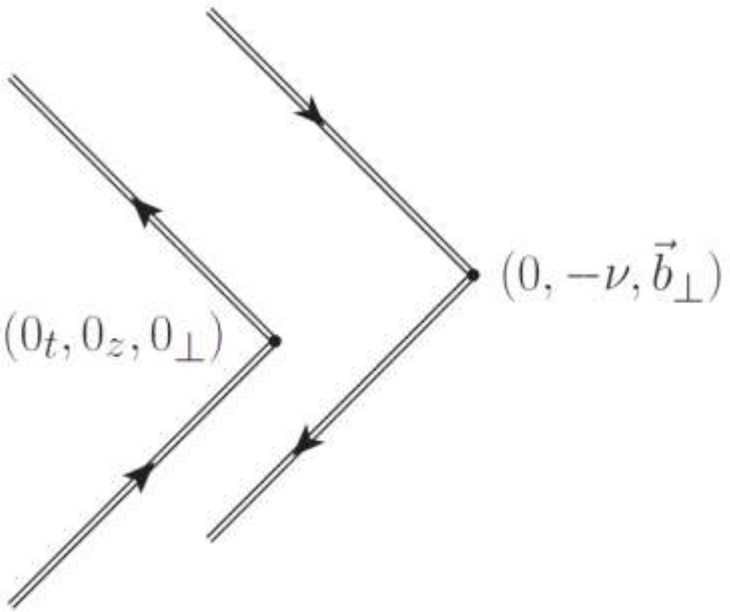



**Figure 5**: The $S_t$.

# General theory of coordinate space analyticity

- General theory of coordinate space analyticity

1. Axioms: spectral-condition && micro-locality && temperedness of *Wightman-Distributions.*

2. *Consequence: existence of analytic Wightman function in the ``permuted extended-tubes" and totally space-like region.*

3. *Established in three steps.*

- First: *Paley-Wiener type arguments* for analyticity in the *forward tubes* $T_n^P = \{(z_n, z_{n-1}, \ldots z_1); Im(z_{P_k} - z_{P_{k-1}}) \in -V_+\}$ *for* $\mathcal{W}^P(z_n, \ldots z_1) = \langle \phi(z_{P_n}) \ldots \phi(z_{P_1}) \rangle$. *Spectral condition is crucial.*

- Second: apply proper complex Lorentz transforms to analytic-continue $\mathcal{W}^P$ further into the *extended forward tubes.*

# General theory

- Finaly, local-commutativity implies that all the $n!$ analytic Wightman functions $\mathcal{W}^P$ can be combined to be a single-valued analytic Wightman function in the permuted extended-tubes: Union of all the extended tubes.

5. Important sub-regions of analyticity.

- Euclidean region $\mathcal{E}_n = \{(z_n, \dots z_1); z_i^0 \in -iR, Im(\vec{z}_i = 0), z_i \neq z_j\}$
- Totally space-like real separations $\{(x_n, \dots x_1); (x_i - x_j)^2 < 0\}$

7. $\mathcal{W}|_{\mathcal{E}_n}$ are called Schwinger functions && Euclidean correlation functions.

# Properties of analytic Wightman functions

8. Properties of analytic Wightman functions

- Covariance under proper complex Lorentz transformations.
- Permutation symmetry && anti-symmetry.
- Spin-statistics && CPT.

9. These properties can be non-trivial: for a complex scalar one has $\mathcal{W}_{\phi\phi^\dagger}(-iTe_t) = \mathcal{W}_{\phi\phi^\dagger}(iTe_t)$ $(T > 0)$,

$$\langle \phi^\dagger(0) e^{-HT} \phi(0) \rangle = \langle \phi(0) e^{-HT} \phi^\dagger(0) \rangle.$$

This is an operator relation that flips the operator ordering!

# Properties

10. Relation to real-time Wightman distributions. One can obtain real-time Wightman distribution as boundary-values of analytic Wightman functions.

11. The Wightman-prescription

$$\langle\phi(t_n)..\phi(t_1)\rangle = \lim_{\eta\to 0^+} \mathcal{W}(t_n - in\eta e_t, \ldots, t_1 - i\eta e_t)\,.$$

- One approaches the boundary point within the forward tube $T_n$.
- For invariant lengths, W-prescription means $-x_{ij}^2 + i\eta x_{ij}^0$.

12. The Feynman-prescription

$$\langle T\phi(t_n)..\phi(t_1)\rangle = \lim_{\eta\to 0^+} \mathcal{W}(t_n(1 - i\eta), \ldots, t_1(1 - i\eta))$$

- For invariant lengths, F-prescription means $-x_{ij}^2 + i\eta$.

# Schwinger functions from lattice models

- How to realize?

1. Non-perturbative level. Osterwalder-Schrader reconstruction theorem. Distributions in the Euclidean regions $\mathcal{E}_n$ that are rotational invariant, reflective-positive and grow moderately in $n$ are Schwinger functions of a Wightman QFT and can be uniquely continued back to real time.

2. Schwinger functions can be obtained as scaling limits of lattice models approaching critical points. $\langle \sigma(r\xi)\sigma(0)\rangle_{\xi\to\infty} \to Z(\xi) f(r)$. Many examples in 2D. Conjectured for QCD.

3. Short distance limit: $f(r) \to \frac{1}{r^{2d}}(1 + r \ln r + \cdots)$. Perturbation to the UV-CFT. UV of IR = IR of UV.

4. CFTs are the ``simplest" Wightman QFTs. Global (Hilbert space and operator algebra) from UV-asymptotics of local-correlators (OPE).

# Analyticity in Perturbation theory

- Momentum space analyticity in DR perturbation theory.

1. Analyticity in perturbation theory are again due to exponential decay in parametric representations.
2. In momentum space, *below-threshold* quantities allow Schwinger-parametrization of the form $\int_0^\infty D\alpha\, F(\alpha) e^{Q^2 P(\alpha)}$ where $P(\alpha) > 0$ are rational functions.
3. They can be continued to the region $Re(Q^2) < 0$.

- Similarly, in coordinate space for $n$-point function, one has parametric integrals of the forms $\mathrm{I} = \int_0^\infty U(\alpha) e^{\sum_{i<j} x_{ij}^2 P_{ij}(\alpha)}$ for Euclidean and totally-space-like separations.

# Analyticity in Perturbation theory

- Consider parametric integrals $\mathrm{I} = \int_0^\infty U(\alpha) e^{\sum_{i<j} x_{ij}^2 P_{ij}(\alpha)}$.

1. The rational functions $P_{ij}(\alpha)$ are positive.
2. The $P_{ij}(\alpha)$ allows explicit representations through spanning trees and connected paths between $i$ and $j$.
3. Only depends on invariant length-squares $x_{ij}^2 = \left(x_i - x_j\right)^2$.
4. Defines analytic function in the below-threshold region $\mathcal{E}_n' = \left\{(z_n, \dots z_1); Re\left(z_{ij}^2\right) < 0, \forall\, i \neq j\right\}$.
5. Agrees with spectral representation in $\mathcal{E}_n' \cap T_n^P$ for any $P$. This is because that $\mathcal{E}_n' \cap T_n^P$ is path-connected and contains $\mathcal{E}_n \cap T_n^P$.

# Analyticity in PQCD

- For QCD perturbation theory in covariant gauges (Feynman gauge for example). Spectral condition and local commutativity are satisfied for gluon fields, to all orders.
- Below threshold representation for gluonic correlators exist in Euclidean and totally space-like real points.
- Thus, one has below threshold representation for gluonic Wightman functions in the below-threshold region $\mathcal{E}'_n = \{(z_n, \dots z_1); Re(z_{ij}^2) < 0, \forall\, i \neq$

# DY TMD soft factor

- One application of the above is to establish the below-threshold representation for Drell-Yan TMD soft factor in the exponential regulator.
- $S(b_\perp, v, \epsilon) = \frac{1}{N_c}\langle Tr\, \bar{T}\, U_{\bar{n}n}(\vec{b}_\perp - ive_t) T\, U_{n\bar{n}}(0)\rangle$
- $U_{n\bar{n}}(x)$ is a Wilson-line cusp at $x$, formed by past-pointing gauge-links in light-like directions $n = \frac{1}{\sqrt{2}}(e_t + e_z)$ and $\bar{n} = \frac{1}{\sqrt{2}}(e_t - e_z)$. $v > 0$ is the exponential regulator. $\vec{b}_\perp$ is the transverse separation.
- The Wilson-loop can be expanded in terms of the gluonic Wightman functions picked-up from the Wilson-loops. Wightman prescriptions are used for the $T$ and $\bar{T}$ from analytic Wightman functions.

# TMD soft factors: $S$ and $S_t$

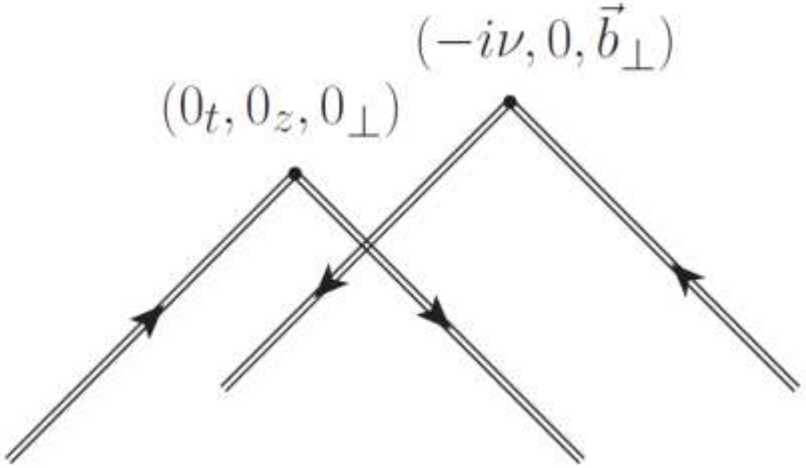



**Figure 4**: The $S$.

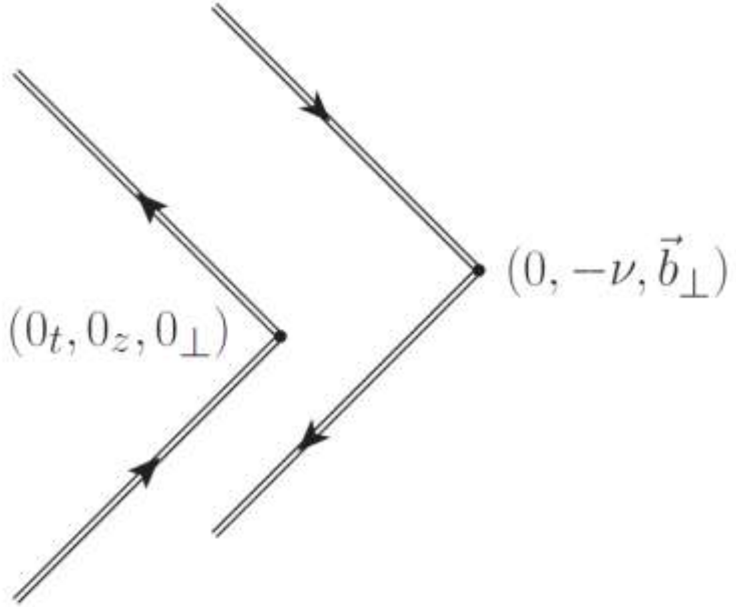



**Figure 5**: The $S_t$.

# Checking invariant lengths

- To see if these analytic Wightman functions allow below-threshold representations, one needs to check the invariant length-squares. There are four types.

1. Two points under the same $T$ or $\bar{T}$, on different Wilson-line. $x_{A,ij}^2 = -2\lambda_i\lambda_j < 0$. Space-like.

2. One point from $T$, another from $\bar{T}$, on same Wilson-line direction. $x_{B,ij}^2 = -\nu^2 - b_\perp^2 - \sqrt{2}i\nu(\lambda_i^L - \lambda_j^R)$. Below-threshold.

3. One point from $T$, another from $\bar{T}$, on different Wilson-line directions. $x_{C,ij}^2 = -\nu^2 - b_\perp^2 - 2\lambda_i^L\lambda_j^R - \sqrt{2}i\nu(\lambda_i^L - \lambda_j^R)$. Below-threshold.

# Null separation and below-threshold representation

4. Two points on the same Wilson-line. This is tricky since null-separation is encountered. But the $i\eta$ solves the problem.

5. $(\lambda_i n - \lambda_j n - i\eta e_t)^2 = -\eta^2 - \sqrt{2} i\eta(\lambda_i - \lambda_j)$. Approached within the blow-threshold region.

- Thus, below-threshold representation exists.
- Furthermore, the $\eta$s can be send to zero from the beginning for three reasons.

1. The $-\eta^2$ regulates UV-light-cone divergences, which are regulated by the DR already.
2. The $i\eta$ terms always have the same signs within the $T$ and $\bar{T}$ groups as the $i\nu$ terms. Thus, $i\eta s$ are replaced by the $i\nu$s.
3. The $-\eta^2, i\eta$ terms are added to terms with negative real parts that never vanish in the integration region.

# Below-threshold representation

- Thus, we conclude that the DY TMD soft-factor allows below-threshold representations in terms of three invariant lengths:

1. $x_{A,ij}^2 = -2\lambda_i\lambda_j$
2. $x_{B,ij}^2 = -\nu^2 - b_\perp^2 - \sqrt{2}i\nu(\lambda_i^L - \lambda_j^R)$.
3. $x_{C,ij}^2 = -\nu^2 - b_\perp^2 - 2\lambda_i^L\lambda_j^R - \sqrt{2}i\nu(\lambda_i^L - \lambda_j^R)$

- As far as $\nu \neq 0$ and $\epsilon \neq 0$, gluonic Wightman functions restricted to these separations are still covariant and permutation-symmetric.
- For $\nu = 0$, naïve invariant lengths for the DY-shape TMD soft factors. Can be used for the (non-gauge-invariant) $\delta$ regulator.

# Relationship between soft factors: $S = S_t$

- The existence of below-threshold representation can be used to establish certain identities.

- Consider $S_t(b_\perp, \nu, \epsilon) = \frac{1}{N_c}\langle Tr\ T\widetilde{U}^\dagger_{n\bar{n}}(\vec{b}_\perp - \nu e_z)\widetilde{U}_{n\bar{n}}(0)\rangle$ .

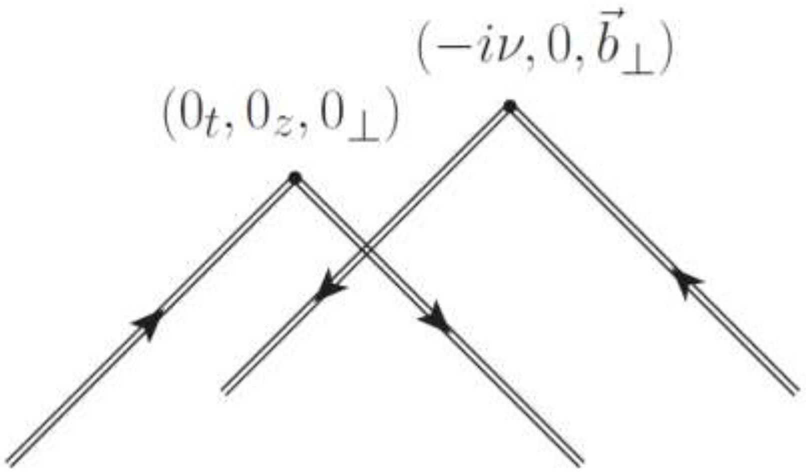
1. Here $\widetilde{U}_{n\bar{n}}(0)$ is a Wilson-line cusp with future-pointing light-like link in $\bar{n}$ directions.

2. Overall time-ordering.

3. A quark-anti-quark pair in $n$ direction moving from past to $t = 0$, then transits to another pair in $\bar{n}$ propagating to future. Space-like form factor.

4. $-\nu e_z$ in the $e_z$ direction.

# TMD soft factors: $S$ and $S_t$

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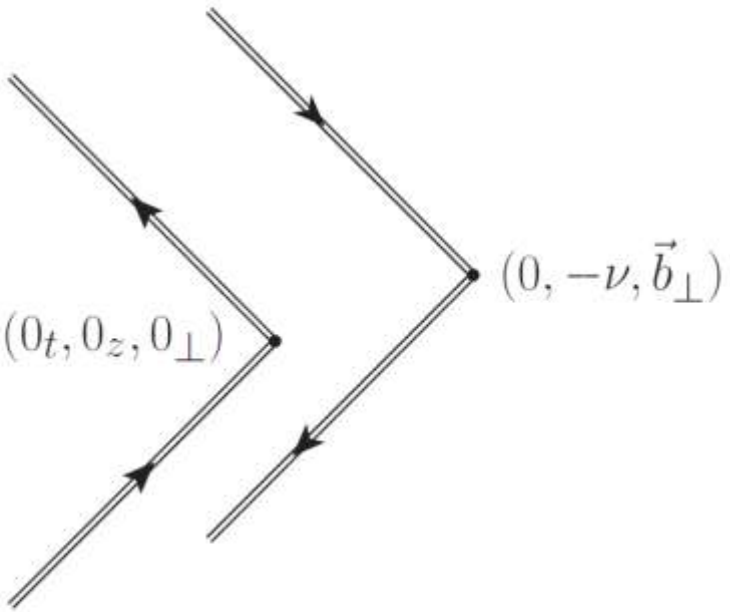


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# Relationship between soft factors: $S = S_t$

- One can show that $S = S_t$ based on Minkowskian parametric representations of $S_t$.
- The M-parametric representations of $S_t$, after Wick-rotation, become exactly the below-threshold representations for $S$.
- Thus, the DY-TMD soft factor can be represented as a space-like form factor.

# Generalization: Analytic Wilson-loop averages.

- We can conjecture the following:
- For any closed complex-space-time valued oriented loop $\mathcal{C}$, if

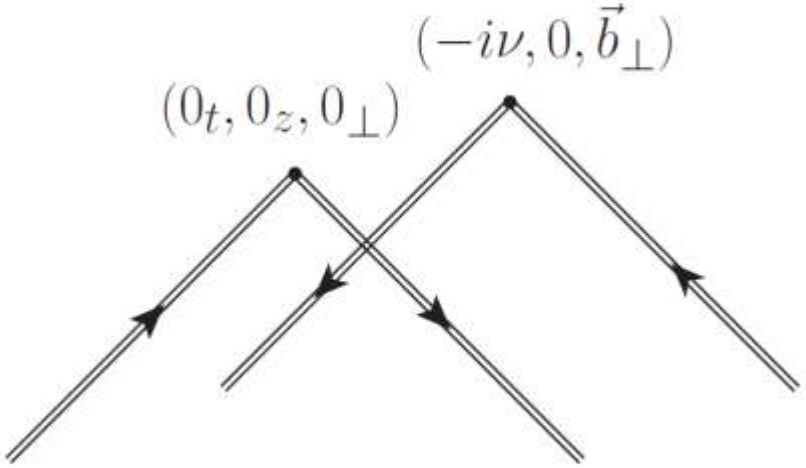
1. The loop is piece-wisely smooth with finite-numbers of cusp singularities with finite cusp angles.
2. An arbitrary non-coinciding set of points picked up from $\mathcal{C}$ always lives in the natural coordinate-space analyticity region (such as permuted – extended -tubes).

- Then the analytic Wilson-loop average $\langle W(\mathcal{C})\rangle$ exists and behave like the analytic Wightman functions in the analyticity region.

# Generalization: Analytic Wilson-loop averages.

- The analytic Wilson-loop average $\langle TrW(\mathcal{C})\rangle$ depends only on the $\mathcal{C}$ and the orientation.
- If $\mathcal{C} = C_1 \cup C_2 \cup C_3 .. C_n$ with $C_i \cap C_j = \emptyset$. Then $\langle TrW(\mathcal{C})\rangle = \langle TrW(\mathcal{C}^P)\rangle$ where $\mathcal{C}^P = C_{P_1} \cup C_{P_2} \cup C_{P_3} .. C_{P_n}$. This plays the role of local-commutativity.
- For small Wilson-loop sizes, $\langle TrW(\mathcal{C})\rangle$ allows perturbative expansion in terms of the perturbative gluonic Wightman functions.
- Analytic Wilson-loops leads to analytic Wightman functions of gauge-invariant operators such as $tr\ F^2$, if one performs small size OPE for the Wilson-loops.

# TMD soft factors: $S$ and $S_t$

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The Wilson-lines for $S$ can be deformed to space-like directions without changing the below-threshold property

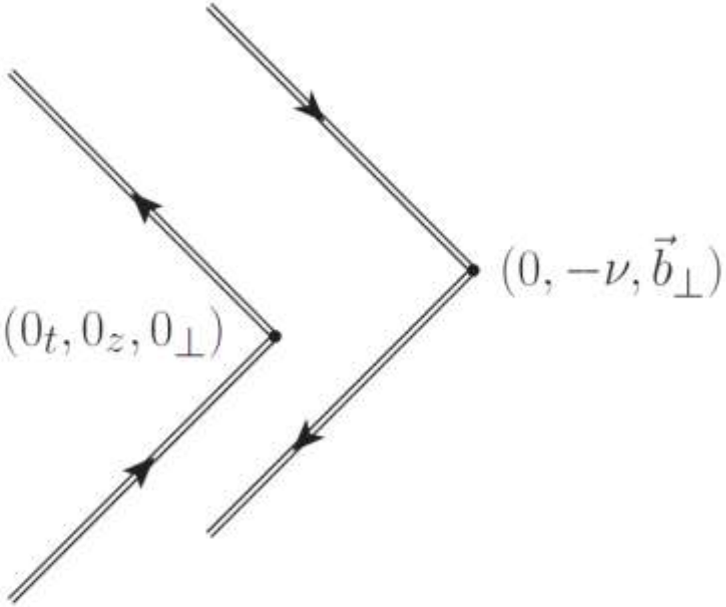


**Figure 5**: The $S_t$.

The Wilson-lines for $S_t$ can be deformed to time-like directions without changing the below-threshold property

# Soft factor relations for three rapidity regulators.

- Given the above, one can define TMD soft factors that contains three regulators at once: off-light-cone, finite LF-length and exponential.

1. $S_t(T_1, T_2, b_\perp, v, Y, \epsilon)$: still a ``real-time" Wilson-loop with ``transverse" gauge links in $\vec{b}_\perp - v e_z$ directions.

- $S_t$ is defined with time-like links with $v = n + e^{-Y}\bar{n}$ and $v' = \bar{n} + e^{-Y} n$. Resembles the heavy-quark form factor in the 2019 Ji-Liu-Liu paper.

- Time-ordering: $T_1 = T_1(1 - i\eta)$ and $T_2 = T_2(1 - i\eta)$. Can be analytically continued smoothly to Euclidean times $T_1 = -iL^-$ and $T_2 = -iL^+$ where $L^\pm > 0$.

2. $S(L^+, L^-, b_\perp, v, Y, \epsilon)$: a complex-valued Wilson-loop with ``transverse" gauge-links in $\vec{b}_\perp - i v e_z$ direction.

# Soft factor relations for three rapidity regulators.

- S is defined with space-like links in $n_Y = n - e^{-Y}\,\bar{n}$ and $\bar{n}_Y = \bar{n} - e^{-Y} n$. Resembles the Collins off-light-cone TMD-soft factor.
- All underlying separations for $S(L^+, L^-, b_\perp, \nu, Y, \epsilon)$ are below-threshold. No null separations at all.

3. Complex Lorentz transform : $\Lambda(t, z) = (iz, it)$, or $\Lambda(e_t, e_z) = (ie_z, ie_t)$.

- Under $\Lambda$, $v \to in_Y$, $v' \to -i\,\bar{n}_Y$ and $-ve_z \to -ive_t$.
- The Wilson-loop for $S_t(-iL^-, -iL^+, b_\perp, \nu, Y, \epsilon)$ maps exactly to the Wilson-loop for $S(L^+, L^-, b_\perp, \nu, Y, \epsilon)$ under the $\Lambda$.

4. Thus, one has the master equality $S_t(-iL^-, -iL^+, b_\perp, \nu, Y, \epsilon) = S(L^+, L^-, b_\perp, \nu, Y, \epsilon)$.

# Comments

- The relations above implies that the rapidity evolution kernel for TMDPDFs and for LFWFs are same : $S_t$ is the natural soft factor for LFWFs.
- The renormalization are multiplicative.
- Three standard orders of limits

1. $Y \to \infty$ first, $L^{\pm} \to \infty$ second gives the exponential regulator.
2. $Y \to \infty$ first, $\nu \to 0$ second gives the finite LF length regulator.
3. $\nu \to 0, L^{\pm} \to \infty$ first at finite $Y$ gives the off-light-cone regulator.

- Another possibility, keep $L^{\pm}$ and $\nu$ finite, is it possible that $Y \to \infty$ and $\epsilon \to 0$ are related to each-other perturbatively ?

# Outline

- Part I: Coordinate-space analyticity and time-ordering issue of TMD soft factors
- Part II: Bjorken limit of 2D large N Gross Neveu

# Exact results for space-like structure function in 2D Gross-Neveu

- Consider the 2D Gross Neveu in large $N$. $\mathcal{L} = \bar{\psi} i\gamma \cdot \partial\psi - \sigma\,\bar{\psi}\psi - \frac{\sigma^2}{2g_0^2}$

1. Large $N$ expansion. Condensate $\sigma_0 = m$ (fermion mass). Running coupling $\frac{1}{g^2(\mu)} = \frac{N}{2\pi} \ln \frac{\mu^2}{m^2}$.
2. Large $N$ expansion can be performed systematically using effective coupling $g^2(k) = \frac{2\pi}{N} \int_0^\infty \int_{c-i\infty}^{c+i\infty} \frac{dt\,ds}{2\pi i} \frac{\Gamma(1-2s)\Gamma(s+t)}{\Gamma(1-s+t)} \left(\frac{m^2}{-k^2}\right)^{-s}$. The propagator for $\sigma - m$.
3. Large $k^2$ expansion: shifts to $s = -n - t$. Borel-integrals at power $\left(\frac{m^2}{-k^2}\right)^n$. Marginality manifest.

# Space-like structure function

- Define the ``twist-three-type" correlator $\mathcal{E}(z^2m^2,\lambda)\bar{u}(p)u(p) = \langle p,i|\bar{\psi}^i(x)\psi^i(x)|p,i\rangle - \langle p,i|\bar{\psi}^j(x)\psi^j(x)|p,i\rangle$.

1. $z^2 = -x^2 > 0$ space-like and $\lambda = -p\cdot x$. Analyticity in $\lambda$ in whole complex plane.
2. We calculate $\mathcal{E}(z^2\mathrm{m}^2,\lambda)$ to NLO in $\frac{1}{N}$. One-bubble-chain diagrams.
3. $\mathcal{E}^{(1)}(z^2\mathrm{m}^2,\lambda) = \frac{2\pi}{N}e^{-i\lambda}(-F_1+F_2-F_3)(z^2m^2,\lambda)$.
4. Bjorken limit $z^2\to 0$ at fixed $\lambda$. Exact twist-expansion.

# Twist-expansion.

- Hard functions and non-perturbative functions.

1. $F_1(z^2m^2,\lambda) = \sum_{l=0}^{\infty}\left(\frac{z^2m^2}{4}\right)^l \int_0^\infty dt\, q_1^{(l)}(t,\lambda,\mu) +$
$\sum_{l=0}^{\infty}\left(\frac{z^2m^2}{4}\right)^l \int_0^\infty dt\, \sum_{p=0}^{\infty}\left(\frac{z^2m^2}{4}\right)^p \left(\left(\frac{z^2m^2}{4}\right)^t \mathcal{H}_1^{l,p}(t,\lambda,\mu) + q_1^{l,p}(t,\lambda,\mu)\right)$

2. Borel integrands $\mathcal{H}_1^{l,p}(t,\lambda,\mu)$ contains renormalon singularity at $t=n$ that cancels with the singularity of $q_1^{l,p+n}(t,\lambda,\mu)$.

3. The $\mu$ dependency cancels between $\mathcal{H}^{l,p}$ and $q_1^{l,p}$ for $p \geq 1$.

4. For $p=0$, $\mu$ dependency cancels between $q_1^{(l)}(t,\lambda,\mu)$ and $\mathcal{H}_1^{l,0}$. $q_1^{l,0} \equiv 0$.

5. $q^{(l)}(t,\lambda,\mu)$ contains no Borel singularity at $t \geq 0$.

# Operator content at LP

- Operator content. There are four quark operator even at LP.

1. The ``Hard function” at LP reads

$$\mathcal{H}^{(0)}(t,\alpha(z),\lambda) = \frac{1}{4\pi}\left(\frac{1}{t}\,{}_1\tilde{F}_1(2,1,-\lambda) + \left(\frac{z^2m^2}{4}\right)^t \Gamma(-t)\,{}_1\tilde{F}_1(2,1+t,-\lambda)\right)$$
$$+\frac{\lambda}{2\pi}\left(\frac{1}{t}\,{}_1\tilde{F}_1(2,2,-\lambda) + \left(\frac{z^2m^2}{4}\right)^t \Gamma(-t)\,{}_1\tilde{F}_1(2,2+t,-\lambda)\right).$$

2. The first-line: explained by the operators $\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{H}_n(\alpha(z))x_{\mu_1}...x_{\mu_n}\bar{\psi}_i\overleftrightarrow{\partial}^{\{\mu_1}...\overleftrightarrow{\partial}^{\mu_n\}}\psi_i$
3. Second line: explained by the four-quark operators

$$\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\mathcal{H}}_n(\alpha(z))x_{\mu_1}x_{\mu_2}...x_{\mu_{n+1}}\bar{\psi}_i\gamma^{\{\mu_1}\overleftrightarrow{\partial}^{\mu_2}...\overleftrightarrow{\partial}^{\mu_{n+1}\}}\psi_i\bar{\psi}\psi$$

# Operator content at LP : condensate contributes

4. Due to the fact that $g_0^2\langle\bar{\psi}\psi\rangle = -m(1+O\left(\frac{1}{N}\right))$. The contributions from

$$\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\mathcal{H}}_n(\alpha(z))x_{\mu_1}x_{\mu_2}...x_{\mu_{n+1}}\bar{\psi}_i\gamma^{\{\mu_1}\overleftrightarrow{\partial}^{\mu_2}...\overleftrightarrow{\partial}^{\mu_{n+1}\}}\psi_i\bar{\psi}\psi$$

are non-vanishing at the order $\frac{1}{N}$. Namely, $\tilde{\mathcal{H}}_n$ is of order $g_0^2\frac{1}{N}$, while $\langle\bar{\psi}\psi\rangle$ is of order $N$.

- Thus, vacuum condensates start to contribute even at the leading power.
- At NLP $(O(z^2m^2))$, there are up to eight quark operators $\bar{\psi}\gamma^+(\partial^+)^n\psi\,(\bar{\psi}\psi)^3$.
- Parton picture is non-longer convenient.

# Twist-expansion and threshold expansion

- The small-$z^2$ expansion, in terms of the Borel-resumed hard and ``collinear" functions, converges absolutely for any $z^2 < 0$.
- No instanton-like contributions in the coefficient functions.
- The threshold expansion $\lambda \to +i\infty$ can also be performed exactly.

1. Threshold expansion in $\frac{1}{-i\lambda}$ commute with small $z^2$ expansion.
2. Threshold expansion is asymptotic. Resurgence analysis can be performed.
3. ``Conspiracy" between Borel singularity of threshold expansion and branch-singularity of $\frac{1}{\ln x}$ for small-$x$ expansion.

# Conclusion

1. Introduction to coordinate-space analyticity in local QFT.
2. Relation ships between TMD soft factors as an application.
3. Generalizable to three rapidity regulators implemented simultaneously.
4. Space-like structure function in 2D large N Gross-Neveu carefully investigated. Convergence of small $z^2$ expansion.
5. Vacuum condensate contribute even at LP.