Confronting neutrino mixing schemes with correlations of neutrino oscillation data

Szymon Zięba

University of Silesia in Katowice

Based on arXiv:2310.20681, arXiv:2403.13966

and

arXiv:24xx.xxxxx
work in progress

2PiNTS, Kraków

September 12, 2024

Neutrino mixing: 3ν

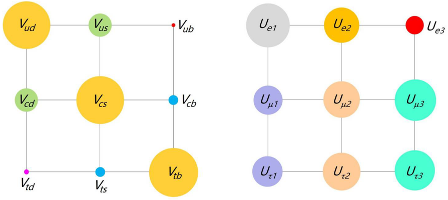
Neutrino flavor and mass eigenstates are related by

$$|\nu_\alpha\rangle = \mathrm{U}_{\alpha i}\,|\nu_i\rangle$$

Pontecorvo–Maki–Nakagawa–Sakata parametrization of mixing matrix

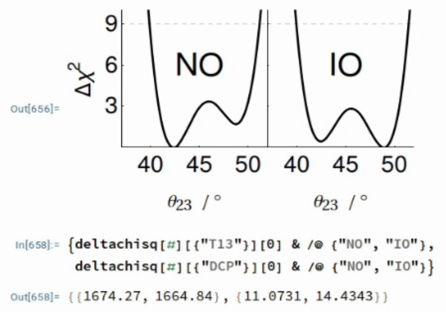
$$\mathrm{U}_{\mathrm{PMNS}} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta} \\ s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{s3} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix} \begin{bmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$, $\delta \equiv \delta_{\mathrm{CP}}$.



(a) CKM quark flavor mixing (b) PMNS lepton flavor mixing

Taken from arXiv:2210.11922, Figure 2.



Mathematica output

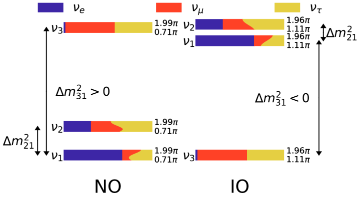
Mass ordering: $m_0 = m_{lightest}$

Normal mass ordering (NO)

$$m_1 = m_0,$$
$$m_2 = \sqrt{m_0^2 + \Delta m_{21}^2},$$
$$m_3 = \sqrt{m_0^2 + \Delta m_{31}^2},$$

Inverted mass ordering (IO)

$$m_1 = \sqrt{m_0^2 - \Delta m_{21}^2 - \Delta m_{32}^2},$$
$$m_2 = \sqrt{m_0^2 - \Delta m_{32}^2},$$
$$m_3 = m_0,$$



Taken from `https://globalfit.astroparticles.es`, updated arXiv:1806.11051, Figure 1

NO or IO ? $\delta_{\rm CP} \neq 0$? $\theta_{23} > \pi/4$ or $\theta_{23} < \pi/4$?

Dirac or Majorana ?

$\Delta m_{21}^2 > 0$ $|\Delta m_{31(32)}^2| > 0$ $\theta_{13} \neq 0$

BM, TB, GR, HG, $\theta_{13} = 0$, early 2010s, (pre)history

$$\mathrm{U_{PMNS}} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta} \\ s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{s3} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix}$$

$$\theta_{13} = 0^\circ \qquad \Downarrow \qquad \theta_{23} = 45^\circ$$

$$\mathrm{U_0} = \begin{bmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Downarrow$$

$\theta_{12} = 45^\circ$, $s_{12} = 1/\sqrt{2}$

Bimaximal Mixing

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\theta_{12} = 35,26^\circ$, $s_{12} = 1/\sqrt{3}$

Tribimaximal Mixing

$$\begin{bmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\theta_{12} = 31,7^\circ$

Golden Ratio Mixing

$$\begin{bmatrix} \frac{\varphi}{\sqrt{2+\varphi}} & \frac{1}{\sqrt{2+\varphi}} & 0 \\ \frac{-1}{\sqrt{4+2\varphi}} & \frac{\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{4+2\varphi}} & \frac{-\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\theta_{12} = 30^\circ$, $s_{12} = 1/2$

Hexagonal Mixing

$$\begin{bmatrix} \sqrt{\frac{3}{4}} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Golden Ratio Mixing: $\mathrm{tg}\,\theta_{12} = 1/\varphi$, $\varphi = (1+\sqrt{5})/2$ being the golden ratio.

Based on Biswajit Karmakar talk, link here.

$\theta_{13} \neq 0$, Daya Bay, RENO (2012)

BM, TB, GR, HG disfavored by non-zero θ_{13}.

$$\mathrm{U}_{\mathrm{PMNS}} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta} \\ s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{s3} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix}$$

$\theta_{13} \neq 0^\circ$ $\quad \Downarrow \quad$ $\theta_{23} = 45^\circ$

$$\mathrm{U}_0 = \begin{bmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Downarrow$$

$\theta_{12} = 45^\circ$, $s_{12} = 1/\sqrt{2}$ — Bimaximal Mixing

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\theta_{12} = 35,26^\circ$, $s_{12} = 1/\sqrt{3}$ — Tribimaximal Mixing

$$\begin{bmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\theta_{12} = 31,7^\circ$ — Golden Ratio Mixing

$$\begin{bmatrix} \frac{\varphi}{\sqrt{2+\varphi}} & \frac{1}{\sqrt{2+\varphi}} & 0 \\ \frac{-1}{\sqrt{4+2\varphi}} & \frac{\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{4+2\varphi}} & \frac{-\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\theta_{12} = 30^\circ$, $s_{12} = 1/2$ — Hexagonal Mixing

$$\begin{bmatrix} \sqrt{\frac{3}{4}} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Golden Ratio Mixing: $\mathrm{tg}\,\theta_{12} = 1/\varphi$, $\varphi = (1+\sqrt{5})/2$ being the golden ratio.

Based on Biswajit Karmakar talk, link here.

Non-zero θ_{13}: Successors of tribimaximal mixing

$$U_{\rm TBM} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & {\color{red}0} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad U_{\rm PMNS} \simeq \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & {\color{red}0.15} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

```
In[688]:= Sin[8.6*Pi/180]
          Sin[8.6*Pi/180]^2
Out[688]= 0.149535

Out[689]= 0.0223608
```

$$|U_{\rm TM_1}| = \begin{bmatrix} \frac{2}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \end{bmatrix}, \qquad |U_{\rm TM_2}| = \begin{bmatrix} * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \end{bmatrix},$$

$$U_{\rm TM_1} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}} & \frac{s_\theta}{\sqrt{3}} e^{-i\gamma} \\ -\frac{1}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}} - \frac{s_\theta}{\sqrt{2}} e^{i\gamma} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} - \frac{c_\theta}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}} - \frac{s}{\sqrt{2}} e^{i\gamma} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} + \frac{c_\theta}{\sqrt{2}} \end{bmatrix},$$

$$U_{\rm TM_2} = \begin{bmatrix} \frac{2c_\theta}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{2s_\theta}{\sqrt{6}} e^{-i\gamma} \\ -\frac{c_\theta}{\sqrt{6}} + \frac{s}{\sqrt{2}} e^{i\gamma} & \frac{1}{\sqrt{3}} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} - \frac{c_\theta}{\sqrt{2}} \\ -\frac{c_\theta}{\sqrt{6}} + \frac{s}{\sqrt{2}} e^{i\gamma} & \frac{1}{\sqrt{3}} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} + \frac{c_\theta}{\sqrt{2}} \end{bmatrix}.$$

Based on Biswajit Karmakar talk, link here.

TM_1 and TM_2 mixing schemes, partial $\mu-\tau$ symmetry

Comparing the corresponding elements of the first column of $\mathrm{U}_{\mathrm{PMNS}}$ and $\mathrm{U}_{\mathrm{TM}_1}$.

$$|\mathrm{U}_{\mathrm{e}1}|^2 = c_{12}^2 c_{13}^2 = 2/3 \quad : \quad \theta_{12}^{\mathrm{TM}_1}(\theta_{13}) \quad : \quad s_{12}^2 = \frac{1-3s_{13}^2}{3-3s_{13}^2},$$

$$|\mathrm{U}_{\mu1}|^2 = |\mathrm{U}_{\tau1}|^2 = 1/6 \quad : \quad \delta_{\mathrm{CP}}^{\mathrm{TM}_1}(\theta_{13},\theta_{23}) \quad : \quad \cos\delta_{\mathrm{CP}} = \frac{(1-5s_{13}^2)(2s_{23}^2-1)}{4s_{13}s_{23}\sqrt{2(1-3s_{13}^2)(1-s_{23}^2)}}.$$

Comparing the corresponding elements of the second column of $\mathrm{U}_{\mathrm{PMNS}}$ and $\mathrm{U}_{\mathrm{TM}_2}$.

$$|\mathrm{U}_{\mathrm{e}2}|^2 = s_{12}^2 c_{13}^2 = 1/3 \quad : \quad \theta_{12}^{\mathrm{TM}_2}(\theta_{13}) \quad : \quad s_{12}^2 = \frac{1}{3-3s_{13}^2},$$

$$|\mathrm{U}_{\mu2}|^2 = |\mathrm{U}_{\tau2}|^2 = 1/3 \quad : \quad \delta_{\mathrm{CP}}^{\mathrm{TM}_2}(\theta_{13},\theta_{23}) \quad : \quad \cos\delta_{\mathrm{CP}} = -\frac{(2-4s_{13}^2)(2s_{23}^2-1)}{4s_{13}s_{23}\sqrt{(2-3s_{13}^2)(1-s_{23}^2)}}.$$

In partial μ-τ reflection symmetry, the mixing matrix symmetry is given by:

$$|U_{\mu1}| = |U_{\tau1}| \;\; (\mu1-\tau1) \quad : \quad \delta_{\mathrm{CP}}^{\mu1\text{-}\tau1}(\theta_{13},\theta_{12},\theta_{23}) \quad : \quad \cos\delta_{\mathrm{CP}} = \frac{(c_{23}^2-s_{23}^2)(c_{12}^2 s_{13}^2 - s_{12}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}},$$

$$|U_{\mu2}| = |U_{\tau2}| \;\; (\mu2-\tau2) \quad : \quad \delta_{\mathrm{CP}}^{\mu2\text{-}\tau2}(\theta_{13},\theta_{12},\theta_{23}) \quad : \quad \cos\delta_{\mathrm{CP}} = \frac{(c_{23}^2-s_{23}^2)(c_{12}^2 - s_{12}^2 s_{13}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}}.$$

How to confront the mixing schemes with experimental data ?

Progress and prospects for precision of oscillation data

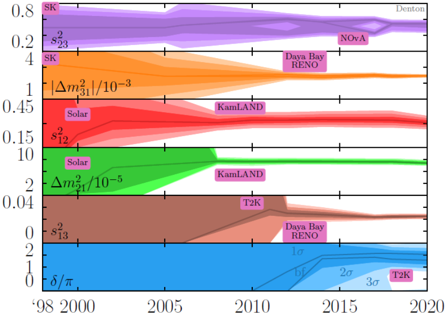


Figure taken form Peter B. Denton talk, link here.

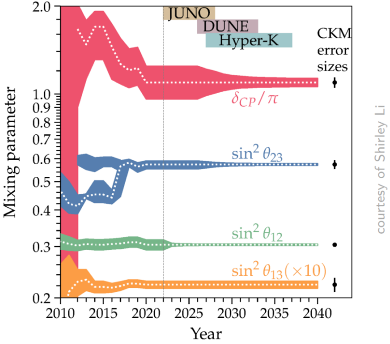


Figure taken form Biswajit Karmakar talk, link here.
See also arXiv:2012.12893, Figure 1 and arXiv:2204.08668, Figure 2.1

$$\theta_{13} \neq 0 \text{ (!)}, \qquad \delta_{\rm CP} \neq 0 \text{ (?)}, \qquad \theta_{23} > \pi/4 \text{ or } \theta_{23} < \pi/4 \text{ (??)}.$$

Prospects for progress in precision.

NuFIT 5.2 (2022) *vs* NuFIT 5.3 (2024)

NuFIT 5.2 (2022)

without SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 2.3$)	
	bfp $\pm1\sigma$	3σ range	bfp $\pm1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
$\sin^2\theta_{23}$	$0.572^{+0.018}_{-0.023}$	$0.406 \to 0.620$	$0.578^{+0.016}_{-0.021}$	$0.412 \to 0.623$
$\theta_{23}/^\circ$	$49.1^{+1.0}_{-1.3}$	$39.6 \to 51.9$	$49.5^{+0.9}_{-1.2}$	$39.9 \to 52.1$
$\sin^2\theta_{13}$	$0.02203^{+0.00056}_{-0.00059}$	$0.02029 \to 0.02391$	$0.02219^{+0.00060}_{-0.00057}$	$0.02047 \to 0.02396$
$\theta_{13}/^\circ$	$8.54^{+0.11}_{-0.12}$	$8.19 \to 8.89$	$8.57^{+0.12}_{-0.11}$	$8.23 \to 8.90$
$\delta_{CP}/^\circ$	197^{+42}_{-25}	$108 \to 404$	286^{+27}_{-32}	$192 \to 360$
$\frac{\Delta m^2_{21}}{10^{-5}\ eV^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ eV^2}$	$+2.511^{+0.028}_{-0.027}$	$+2.428 \to +2.597$	$-2.498^{+0.032}_{-0.025}$	$-2.581 \to -2.408$

with SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.4$)	
	bfp $\pm1\sigma$	3σ range	bfp $\pm1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \to 0.603$	$0.569^{+0.016}_{-0.021}$	$0.412 \to 0.613$
$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.5$
$\sin^2\theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02052 \to 0.02398$	$0.02223^{+0.00058}_{-0.00058}$	$0.02048 \to 0.02416$
$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$
$\delta_{CP}/^\circ$	232^{+36}_{-26}	$144 \to 350$	276^{+22}_{-29}	$194 \to 344$
$\frac{\Delta m^2_{21}}{10^{-5}\ eV^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ eV^2}$	$+2.507^{+0.026}_{-0.027}$	$+2.427 \to +2.590$	$-2.486^{+0.025}_{-0.028}$	$-2.570 \to -2.406$

NuFIT 5.3 (2024)

without SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 2.3$)	
	bfp $\pm1\sigma$	3σ range	bfp $\pm1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.307^{+0.012}_{-0.011}$	$0.275 \to 0.344$	$0.307^{+0.012}_{-0.011}$	$0.275 \to 0.344$
$\theta_{12}/^\circ$	$33.66^{+0.73}_{-0.70}$	$31.60 \to 35.94$	$33.67^{+0.73}_{-0.71}$	$31.61 \to 35.94$
$\sin^2\theta_{23}$	$0.572^{+0.018}_{-0.023}$	$0.407 \to 0.620$	$0.578^{+0.016}_{-0.021}$	$0.412 \to 0.623$
$\theta_{23}/^\circ$	$49.1^{+1.0}_{-1.3}$	$39.6 \to 51.9$	$49.5^{+0.9}_{-1.2}$	$39.9 \to 52.1$
$\sin^2\theta_{13}$	$0.02203^{+0.00056}_{-0.00058}$	$0.02029 \to 0.02391$	$0.02219^{+0.00059}_{-0.00057}$	$0.02047 \to 0.02396$
$\theta_{13}/^\circ$	$8.54^{+0.11}_{-0.11}$	$8.19 \to 8.89$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.90$
$\delta_{CP}/^\circ$	197^{+41}_{-25}	$108 \to 404$	286^{+27}_{-32}	$192 \to 360$
$\frac{\Delta m^2_{21}}{10^{-5}\ eV^2}$	$7.41^{+0.21}_{-0.20}$	$6.81 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.81 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ eV^2}$	$+2.511^{+0.027}_{-0.027}$	$+2.428 \to +2.597$	$-2.498^{+0.032}_{-0.024}$	$-2.581 \to -2.409$

with SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 9.1$)	
	bfp $\pm1\sigma$	3σ range	bfp $\pm1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.307^{+0.012}_{-0.011}$	$0.275 \to 0.344$	$0.307^{+0.012}_{-0.011}$	$0.275 \to 0.344$
$\theta_{12}/^\circ$	$33.67^{+0.73}_{-0.71}$	$31.61 \to 35.94$	$33.67^{+0.73}_{-0.71}$	$31.61 \to 35.94$
$\sin^2\theta_{23}$	$0.454^{+0.019}_{-0.016}$	$0.411 \to 0.606$	$0.568^{+0.016}_{-0.021}$	$0.412 \to 0.611$
$\theta_{23}/^\circ$	$42.3^{+1.1}_{-0.9}$	$39.9 \to 51.1$	$48.9^{+0.9}_{-1.2}$	$39.9 \to 51.4$
$\sin^2\theta_{13}$	$0.02224^{+0.00056}_{-0.00057}$	$0.02047 \to 0.02397$	$0.02222^{+0.00069}_{-0.00057}$	$0.02049 \to 0.02420$
$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.13}_{-0.11}$	$8.23 \to 8.95$
$\delta_{CP}/^\circ$	232^{+39}_{-25}	$139 \to 350$	273^{+24}_{-26}	$195 \to 342$
$\frac{\Delta m^2_{21}}{10^{-5}\ eV^2}$	$7.41^{+0.21}_{-0.20}$	$6.81 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.81 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ eV^2}$	$+2.505^{+0.024}_{-0.026}$	$+2.426 \to +2.586$	$-2.487^{+0.027}_{-0.024}$	$-2.566 \to -2.407$

Taken from http://www.nu-fit.org, updated arXiv:2007.14792, Table 3.

Since 2022, increase of NO preference over IO.

No noticeable progress in precision for 3σ ranges.

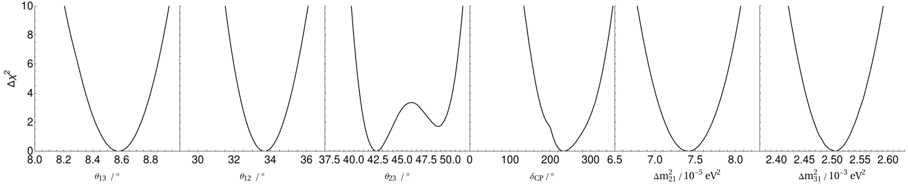
NuFIT tabularized $\Delta\chi^2$ data sets: 1D, 2D, 3D

http://www.nu-fit.org/?q=node/278

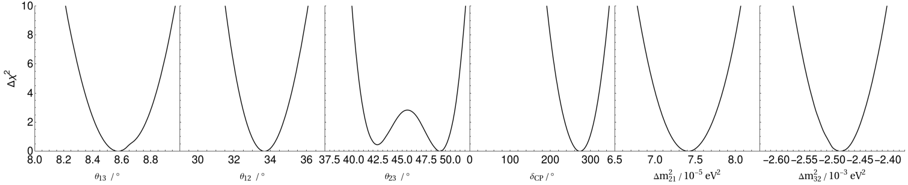
Available data files

We provide one-, two- and three-dimensional $\Delta\chi^2$ projections for both the analysis without (Normal and Inverted Ordering) and including (Normal and Inverted Ordering) Super-Kamiokande atmospheric data. A description of the content of these files and a summary of the data included in our analysis can be found here.

NO (with SK atm. data, 1D, minimized), $\theta_{23} = 48.7^\circ$ at 1.3σ (2nd local minimum, $\Delta\chi^2 = 1.71$)

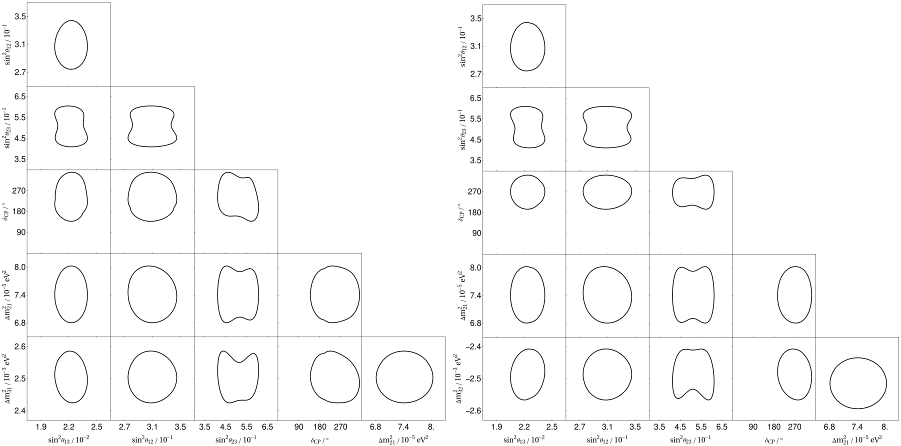


IO (with SK atm. data, 1D, minimized), $\theta_{23} = 42.4^\circ$ at 0.7σ (2nd local minimum, $\Delta\chi^2 = 0.45$)



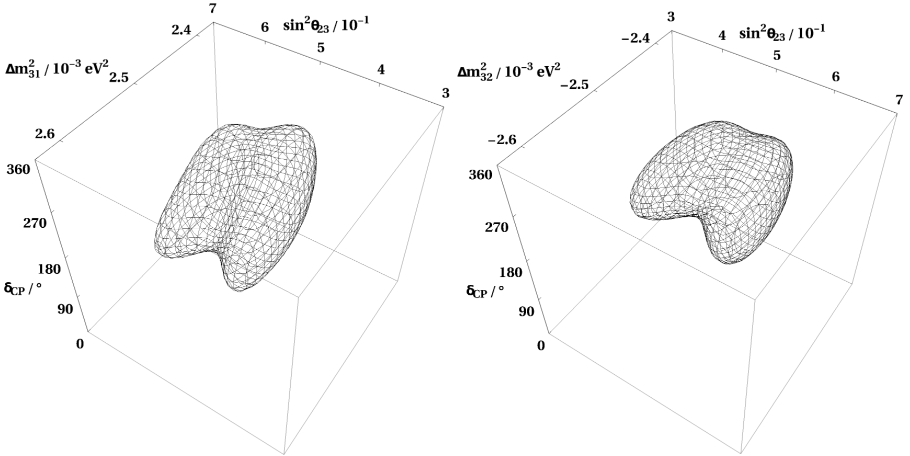
NuFIT tabularized $\Delta\chi^2$ data sets: 2D correlations

Presented for $\Delta\chi^2 \leqslant 9$, (with SK atm. data, 2D, minimized).
NO (left panel) and IO (right panel). Plots based on NuFIT 5.3 data sets.



NuFIT tabularized $\Delta\chi^2$ data sets: 3D correlation

Presented for $\Delta\chi^2 \leqslant 9$, (with SK atm. data, 3D, minimized).
NO (left panel) and IO (right panel). Plots based on NuFIT 5.3 data sets.



The only (most correlated) 3D data sets provided by NuFIT.

Confronting Models with correlations at some $\Delta\chi^2$ level

First, construct $\Delta\chi^2$ function from available (1D, 2D, 3D) correlated data:

$$\begin{aligned} \text{NO}: &\quad \Delta\chi^2(\theta_{13}, \theta_{12}, \theta_{23}, \delta_{\rm CP}, \Delta m^2_{21}, \Delta m^2_{31}), \\ \text{IO}: &\quad \Delta\chi^2(\theta_{13}, \theta_{12}, \theta_{23}, \delta_{\rm CP}, \Delta m^2_{21}, \Delta m^2_{32}), \end{aligned}$$

Second, insert specific model formulas (model correlations, e.g. TM_1 or TM_2) :

$$\begin{aligned} &\theta_{12}^{\mathrm{TM}_{1(2)}}(\theta_{13}), \\ &\delta_{\rm CP}^{\mathrm{TM}_{1(2)}}(\theta_{13}, \theta_{12}^{\mathrm{TM}_{1(2)}}(\theta_{13}), \theta_{23}), \end{aligned}$$

Third, set $\Delta\chi^2$ level, ($\Delta\chi^2 \leqslant 9$ comparable with table data, can be any)

$$\begin{aligned} \text{NO}: &\quad \Delta\chi^2(\theta_{13}, \theta_{12}^{\mathrm{TM}_{1(2)}}, \theta_{23}, \delta_{\rm CP}^{\mathrm{TM}_{1(2)}}, \Delta m^2_{21}, \Delta m^2_{31}) \leqslant 9, \\ \text{IO}: &\quad \Delta\chi^2(\theta_{13}, \theta_{12}^{\mathrm{TM}_{1(2)}}, \theta_{23}, \delta_{\rm CP}^{\mathrm{TM}_{1(2)}}, \Delta m^2_{21}, \Delta m^2_{32}) \leqslant 9. \end{aligned}$$

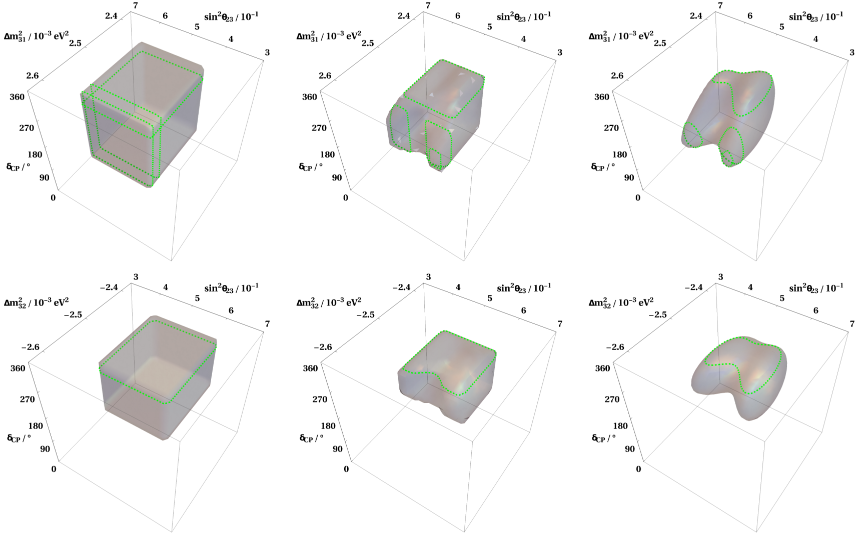
Complete NuFIT 5.3 (2024) with SK atmospheric 1D, 2D, 3D minimized data sets are used in this approach.

$\Delta\chi^2$ function in 3D from NuFIT data 1D *vs* 2D *vs* 3D

Here presented for $\Delta\chi^2 \leqslant 9$, NO (upper panel) and IO (lower panel).

Plots based on NuFIT 5.3 data sets with sample intersections.

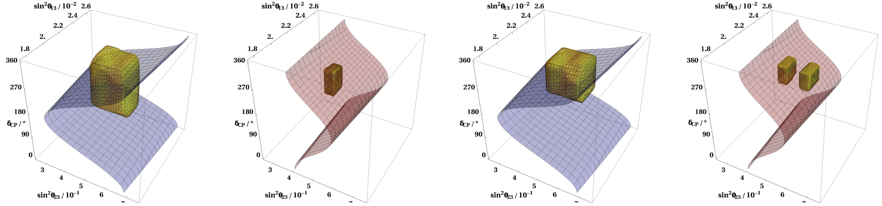
1D 2D 3D



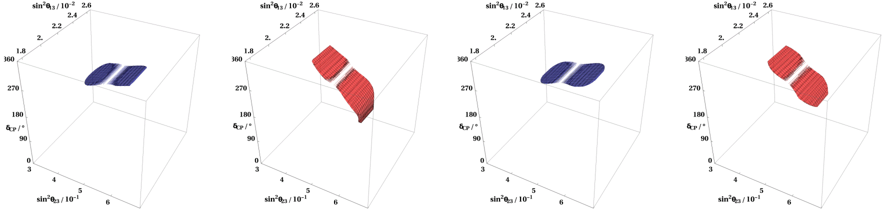
Intersections of experimental and model correlations

Here presented for $\Delta\chi^2 \leqslant 9$, NO (left panel) and IO (right panel):

TM_1 TM_2 TM_1 TM_2



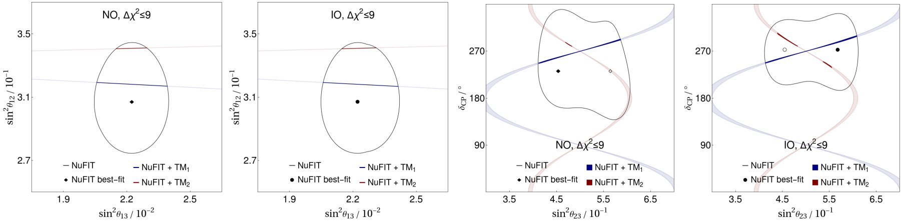
$\mu 1-\tau 1$ $\mu 2-\tau 2$ $\mu 1-\tau 1$ $\mu 2-\tau 2$



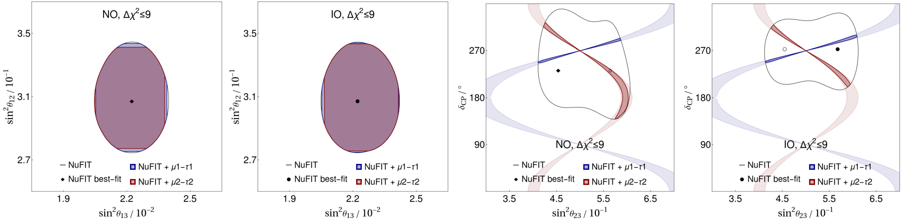
3D projections of 4D intersections for partial $\mu-\tau$ reflection symmetry.

TM_1 and TM_2 *vs* partial $\mu - \tau$ reflection symmetry

TM_1 and TM_2 - 2D projections.



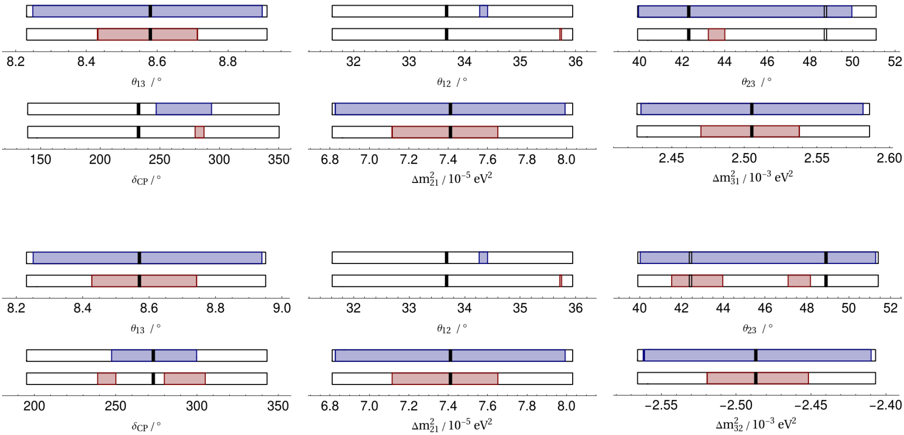
Partial $\mu - \tau$ reflection symmetry - 2D projections.



TM_1 and TM_2 - 1D projections

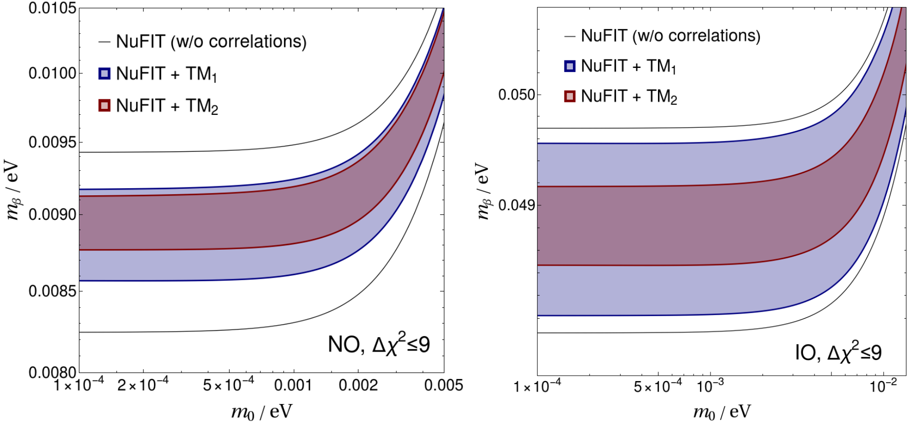
NuFIT, TM_1 and TM_2 - 1D projections.

Here presented for $\Delta\chi^2 \leqslant 9$ (3σ ranges), NO (upper panel) and IO (lower panel):



TM_1 and TM_2 - preliminary results for m_β

Here presented for $\Delta\chi^2 \leqslant 9$, (left panel) and IO (right panel):

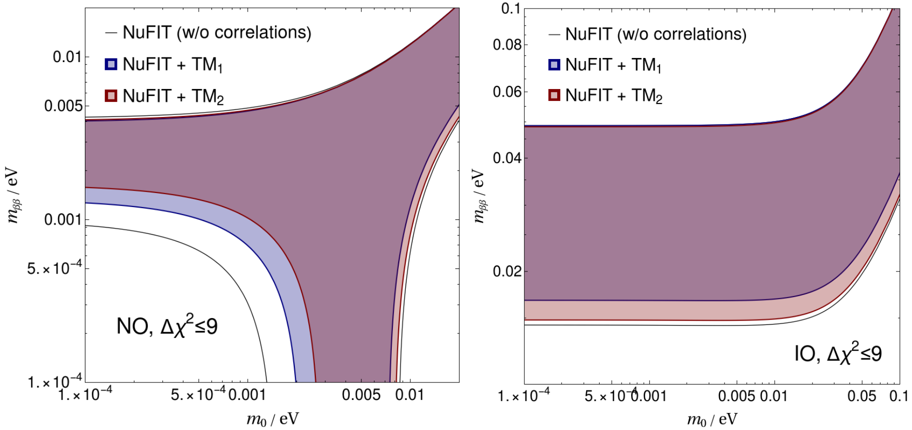


Effective electron neutrino mass:

$$m_\beta^2 = \frac{\sum_i m_i^2 |U_{ei}|^2}{\sum_i |U_{ei}|^2} = \sum_i m_i^2 |U_{ei}|^2 = c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2.$$

TM_1 and TM_2 - preliminary results for $m_{\beta\beta}$

Here presented for $\Delta\chi^2 \leqslant 9$, (left panel) and IO (right panel):



Effective Majorana mass:

$$m_{\beta\beta} = \left|\sum_i m_i U_{ei}^2\right| = \left|m_1 c_{13}^2 c_{12}^2 e^{i2\alpha_1} + m_2 c_{13}^2 s_{12}^2 e^{i2\alpha_2} + m_3 s_{13}^2 e^{-i2\delta_{\mathrm{CP}}}\right|.$$

Summary

Multidimensional oscillation parameter correlation data raise the construction and testing of neutrino mixing schemes to a level inaccessible to standard one-dimensional analysis.

With the presented approach, we are able to:

- test models over the full range of parameters at a given $\Delta\chi^2$ level,
- determine model-specific 3σ ranges (and compare with experimental best-fits),
- impose constraints on parameters that are not explicitly present in the mixing schemes (e.g. neutrino masses in TM_1, TM_2, partial $\mu - \tau$ reflection symmetry),
- find tighter, model-dependent constraints on the effective neutrino masses.

Challenges and limitations:

- NuFIT offers data sets up to 3D,
- the complexity of the problem increases with each additional parameter, and therefore the computational time increases,
- the size of the data sets increases with each additional parameter, which means that the memory requirements also increase.

The method is general and can be applied to any model providing analytical expressions involving the neutrino oscillation parameters.

Thank you
for your attention.

TM_1 and TM_2 - 2D projections