Kink behavior in the sine-Gordon model under a variety of inhomogeneities

Jacek Gatlik[1], Tomasz Dobrowolski[2], and Panayotis G. Kevrekidis[3]

[1] Faculty of Physics and Applied Computer Science, AGH University of Krakow, Krakow, Poland
[2] Department of Computer Physics and Quantum Computing,
University of the National Education Commission in Krakow, Poland
[3] Department of Mathematics and Statistics, University of Massachusetts Amherst, USA

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Uniwersytet Komisji Edukacji Narodowej w Krakowie

AGH
Akademia Górniczo-Hutnicza im. Stanisława Staszica w Krakowie

University of Massachusetts 1863

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➢ Introduction

We consider sine-Gordon equation in the form

$$\partial_t^2\phi - \partial_x\left(\mathcal{F}(x)\,\partial_x\phi\right) + \sin\phi = 0$$

In the presence of the electromagnetic fields in the junction the electric current has the form

$$\vec{j} = \frac{e^*}{m^*}\left[\frac{1}{2}\imath\hbar\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) - \frac{e^*}{c}\vec{A}\,\psi\,\psi^*\right]$$

Which leads to the phase gradient

$$\nabla\varphi = \frac{2e}{\hbar c}\left[\frac{mc}{2|\psi|^2 e^2}\vec{j} + \vec{A}\right]$$

And in curved coordinates simplifies to

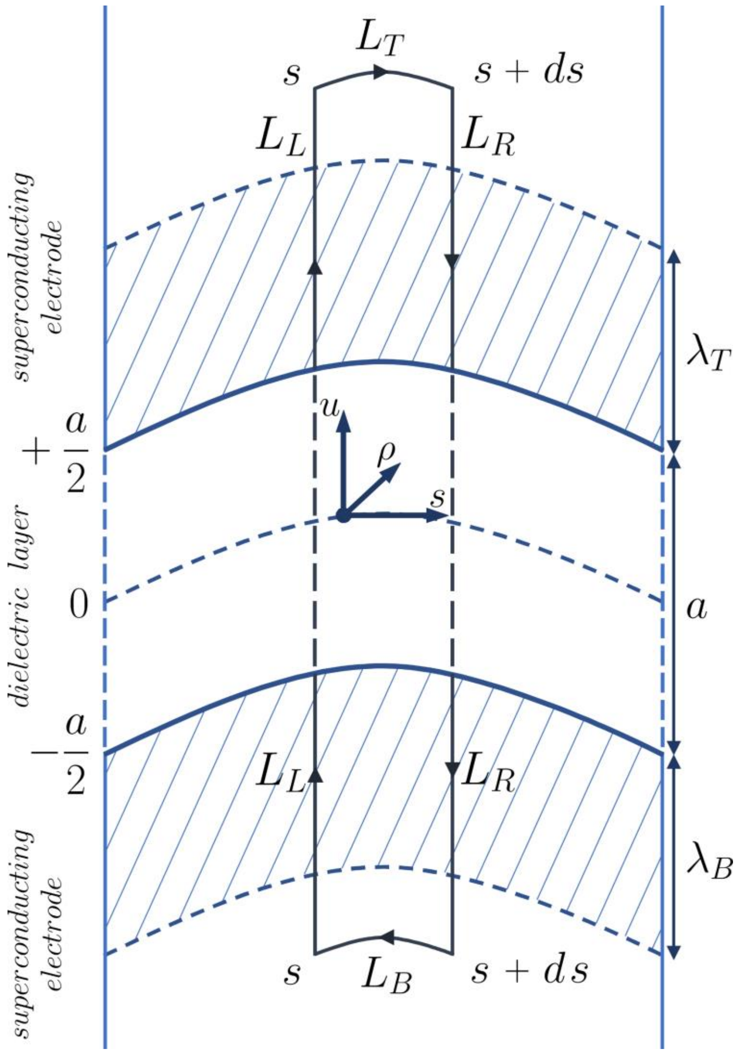
$$\frac{1}{G}\,\partial_s\varphi = (\mathbf{grad}\varphi)_s = \frac{2e}{\hbar c}\left[\frac{mc}{2|\psi|^2 e^2}j_s{}^{SH} + A_s\right] = \frac{2e}{\hbar c}A_s$$

For the selected contour we get

$$\frac{1}{G}\,\left(\phi(s+ds) - \phi(s)\right) = \frac{2e}{\hbar c}\oint d\vec{l}\vec{A}$$

By the Stokes theorem we can find the relationship between the field ϕ and the magnetic field H_ρ

$$\frac{1}{G}\,\partial_s\phi = \frac{1}{G}\left(\frac{\phi(s+ds) - \phi(s)}{ds}\right) = \frac{2e}{\hbar c}\,d_m H_\rho$$



Cross-section of the Josephson junction along the central line.

J. Gatlik and T. Dobrowolski, Physica D: Nonlinear Phenomena, **428**, 133061 (2021)

➢ Introduction

On the other hand, the Ampere's circular law with the Maxwell correction leads to

$$\frac{1}{G}\,\partial_s H_\rho = \frac{4\pi}{c}\,j_u + \frac{\varepsilon}{ac}\,\partial_t(\Delta V)$$

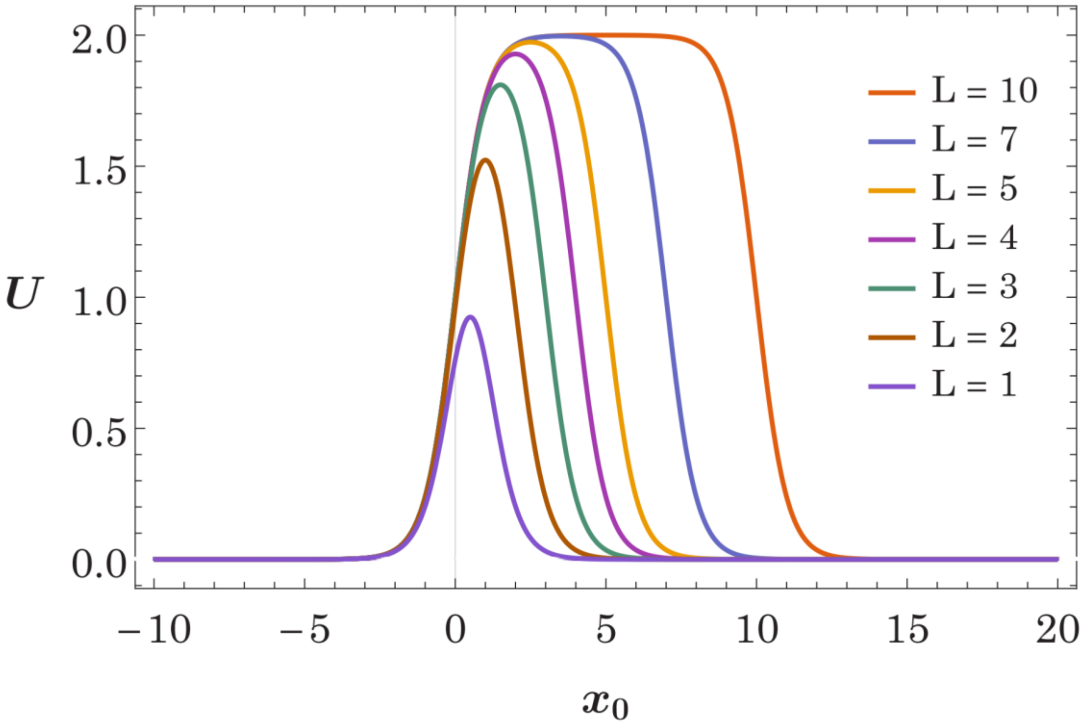
Adding the second Josephson law

$$\frac{\hbar c}{2ed_m}\,\partial_s\left(\frac{1}{G}\,\partial_s\phi\right) = G\frac{4\pi}{c}\,j_u + G\frac{\varepsilon}{ac}\,\frac{\hbar}{2e}\,\partial_t^2\phi$$

Averaging this formula with respect to the normal variable also denoting $\bar{c} = \sqrt{\frac{a}{\varepsilon d_m}}$ and $\lambda_J = \sqrt{\frac{\hbar c^2}{8\pi e d_m j_0}}$ we obtain

$$\frac{1}{\bar{c}^2}\,\partial_t^2\phi - \partial_s\left(\mathcal{F}\,\partial_s\phi\right) + \frac{1}{\lambda_J^2}\,\sin\phi = 0$$

$$\mathcal{F}(x) = 1 + \varepsilon g(x),\quad g(x) = \theta(x) - \theta(x-L)$$



The effective potential for small velocities (i.e. $\gamma \to 1$).

Finally, we consider sine-Gordon equation in the form

$$\partial_t^2\phi + \alpha\partial_t\phi - \partial_x(\mathcal{F}(x)\partial_x\phi) + \sin\phi = -\Gamma$$

Nonconservative Lagrangian

We study the perturbed sine-Gordon model in the form

$$\partial_t^2\phi + \alpha\partial_t\phi - \partial_x(\mathcal{F}(x)\partial_x\phi) + \sin\phi = -\Gamma$$

Due to the existence of dissipation in the studied system we used method based on a ***nonconservative Lagrangian*** *[1]* density

$$\mathcal{L}_N = \mathcal{L}(\phi_1, \partial_t\phi_1, \partial_x\phi_1) - \mathcal{L}(\phi_2, \partial_t\phi_2, \partial_x\phi_2) + \mathcal{R} \qquad \mathcal{R} = -\alpha\phi_-\partial_t\phi_+ - \Gamma\phi_-$$

We consider a model with two degrees of freedom based on the following ansatz $\phi_K(t,x) = 4\arctan e^{\gamma(t)(x-x_0(t))}$

Which leads to effective equations of motion

$$\partial_t\left(\frac{\partial L}{\partial \dot{x}_0}\right) - \frac{\partial L}{\partial x_0} = \left[\frac{\partial R}{\partial x_-} - \partial_t\left(\frac{\partial R}{\partial \dot{x}_-}\right)\right]_{PL}$$

$$\partial_t\left(\frac{\partial L}{\partial \dot{\gamma}}\right) - \frac{\partial L}{\partial \gamma} = \left[\frac{\partial R}{\partial \gamma_-} - \partial_t\left(\frac{\partial R}{\partial \dot{\gamma}_-}\right)\right]_{PL}$$

$$\ddot{x}_0 + \frac{\dot{\gamma}}{\gamma}\dot{x}_0 - \frac{1}{8\gamma}\frac{\partial L_\varepsilon}{\partial x_0} = \frac{1}{8\gamma}\left[\frac{\partial R}{\partial x_-} - \partial_t\left(\frac{\partial R}{\partial \dot{x}_-}\right)\right]_{PL}$$

$$\frac{2\pi^2}{3}\frac{\ddot{\gamma}}{\gamma} - \pi^2\frac{\dot{\gamma}^2}{\gamma^2} - 4\gamma^2\dot{x}_0^2 + 4(\gamma^2-1) - \gamma^2\frac{\partial L_\varepsilon}{\partial \gamma} = \gamma^2\left[\frac{\partial R}{\partial \gamma_-} - \partial_t\left(\frac{\partial R}{\partial \dot{\gamma}_-}\right)\right]_{PL}$$

In the physical limit

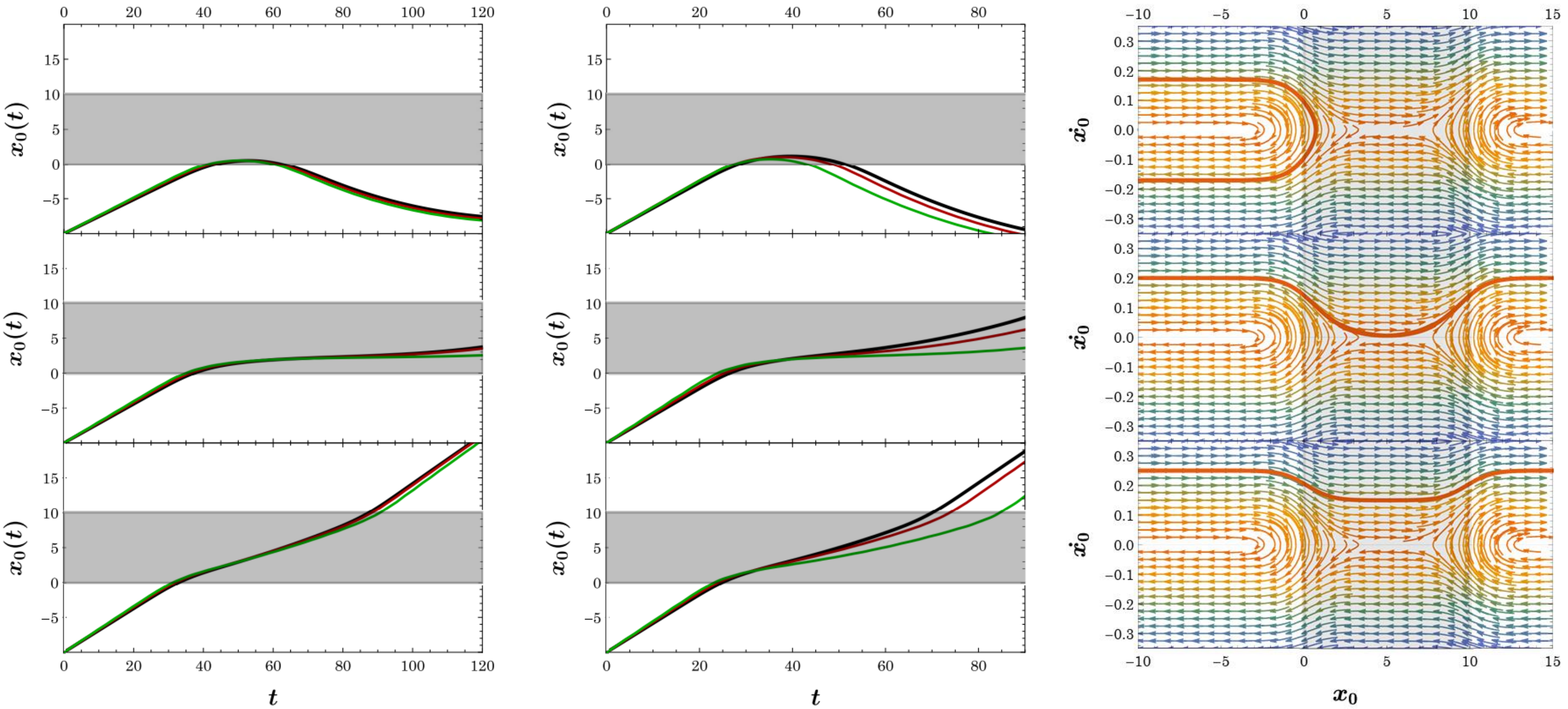
$$\ddot{x}_0 + \frac{\dot{\gamma}}{\gamma}\dot{x}_0 - \frac{1}{8\gamma}\frac{\partial L_\varepsilon}{\partial x_0} = -\alpha\dot{x}_0 + \frac{\pi}{4\gamma}\Gamma$$

$$\frac{2\pi^2}{3}\frac{\ddot{\gamma}}{\gamma} - \pi^2\frac{\dot{\gamma}^2}{\gamma^2} - 4\gamma^2\dot{x}_0^2 + 4(\gamma^2-1) - \gamma^2\frac{\partial L_\varepsilon}{\partial \gamma} = -\frac{2\pi^2}{3}\alpha\frac{\dot{\gamma}}{\gamma}$$

$$L_\varepsilon = -\frac{1}{2}\varepsilon\int_{-\infty}^{+\infty} g(x)(\partial_x\phi_K)^2,$$

[1] C. R. Galley, *Classical mechanics of nonconservative systems*, Phys. Rev. Lett. **110**, 174301 (2013).
C. R. Galley, D. Tsang, and L. C. Stein, *The principle of stationary nonconservative action for classical mechanics and field theories*, 2014.

➢ Nonconservative Lagrangian



Comparison of the position of the center of mass of the kink for the solution from the original field model (black line) and models with two degrees of freedom based on projecting onto the zero mode (green line) and on non-conservative Lagrangian (red line). On the right the phase diagrams corresponding to the same parameter values.

➢ The non-dissipative case

We insert the decomposition

$$\phi(t,x) = \phi_0(x) + \psi(t,x)$$

into sine-Gordon equation obtaining

$$\partial_t^2 \psi - \partial_x \left(\mathcal{F}(x) \partial_x \psi\right) + (\cos\phi_0)\psi = 0$$

$\phi_0(x)$ can be decomposed into static kink ϕ_K of the sine-Gordon model and a time independent correction χ

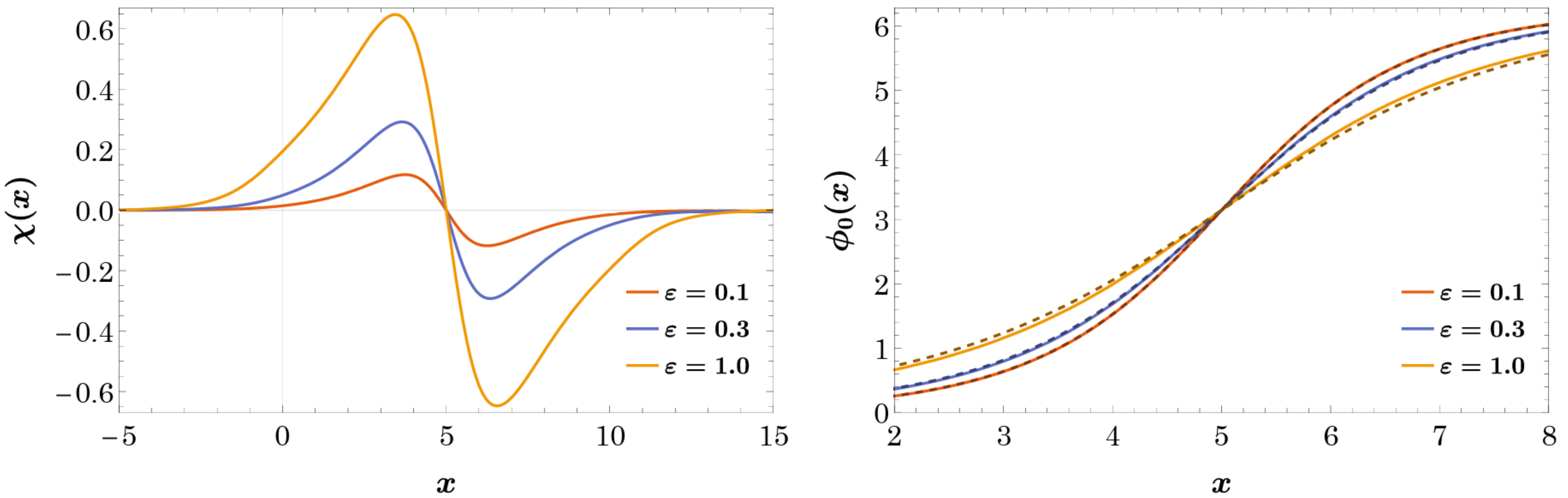
$$\phi_0(x) = \phi_K(x) + \chi(x)$$

$$-\partial_x \left(\mathcal{F}(x)\partial_x \chi\right) + (\cos\phi_K)\chi = \varepsilon \partial_x \left(g(x) \partial \phi_K\right)$$

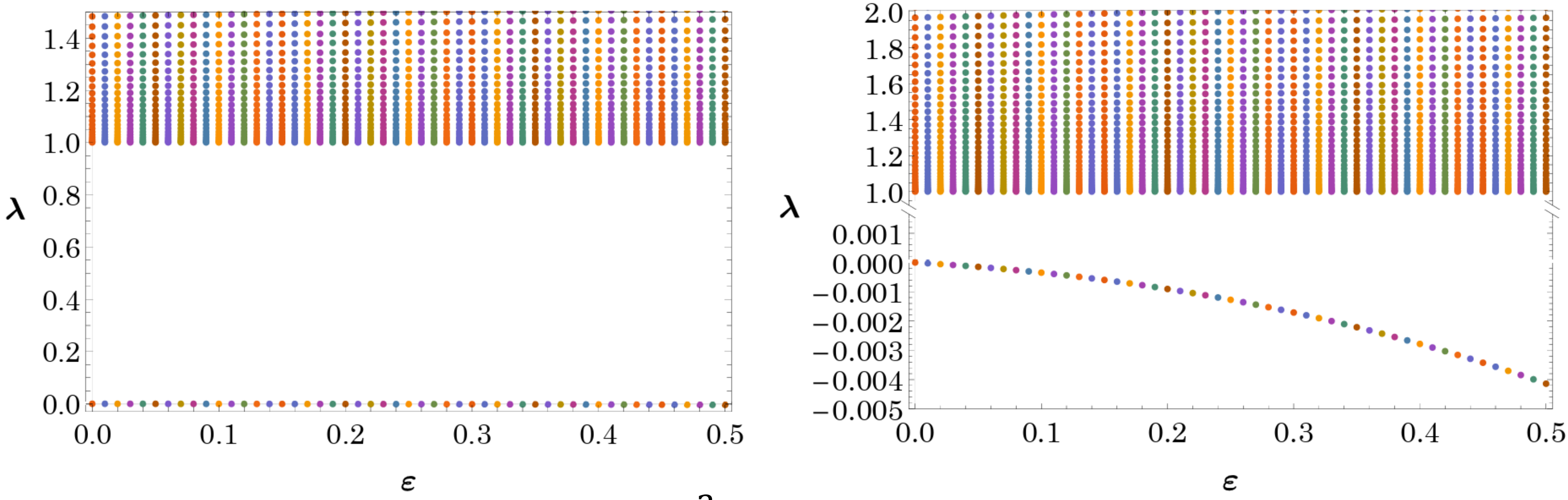
We can examine the spectral stability

$$-\partial_x \left(\mathcal{F}(x)\, \partial_x v(x)\right) + (\cos\phi_0)\, v(x) = \lambda v(x)$$

$$\hat{\mathcal{L}}\, v(x) = \lambda v(x)$$

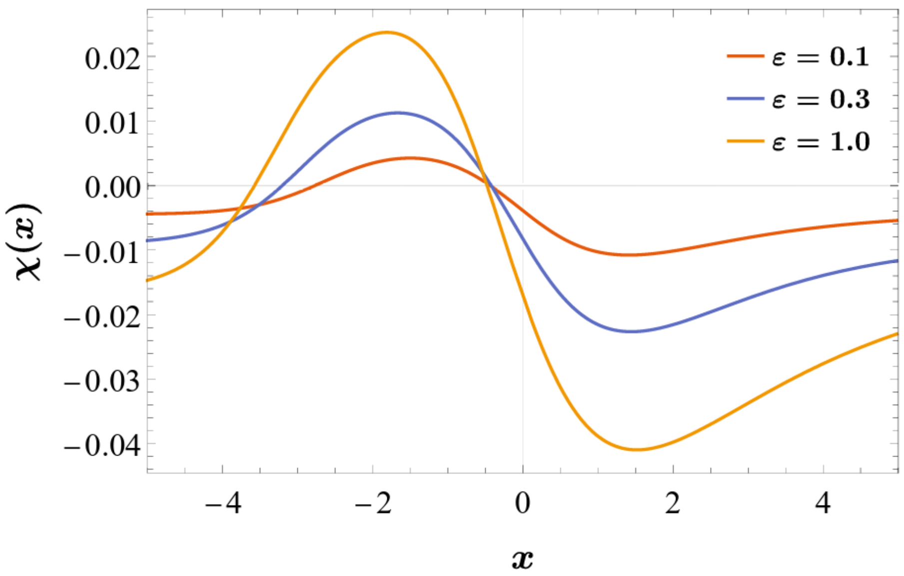


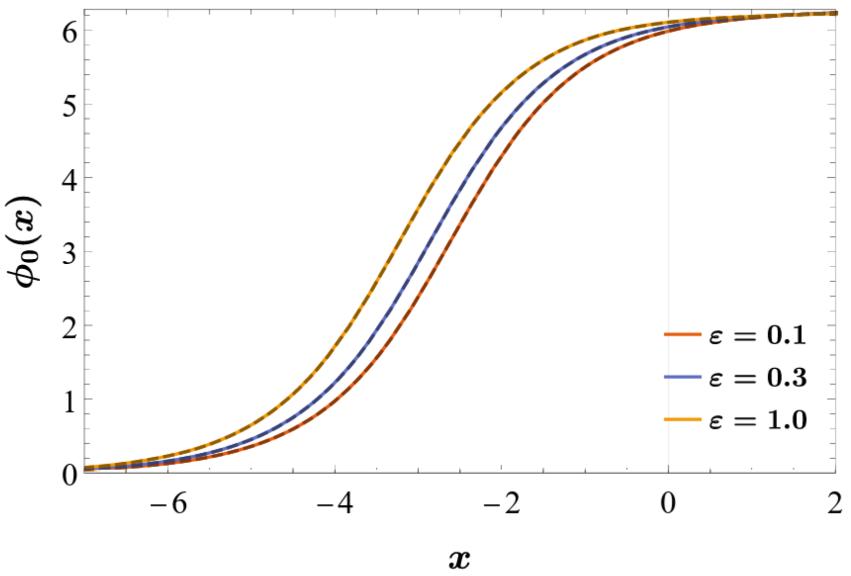
On the left panel, the $\chi(x)$ value for different values of ε. On the right panel, the solid line shows the sum of the kink ansatz $\phi_K(x)$ and $\chi(x)$ value.



Squared eigenfrequencies $\lambda = \omega^2$ calculated for the quasi-static configuration

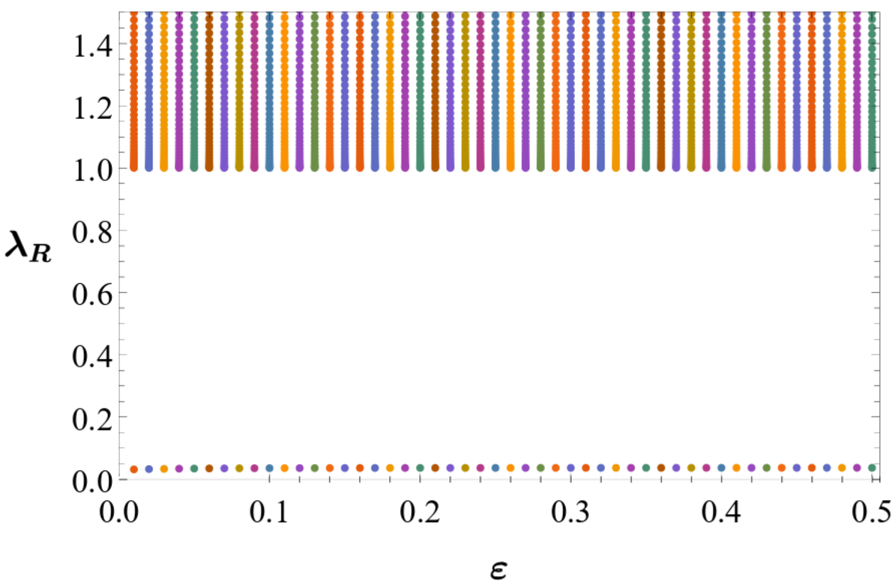
➢ The dissipative case

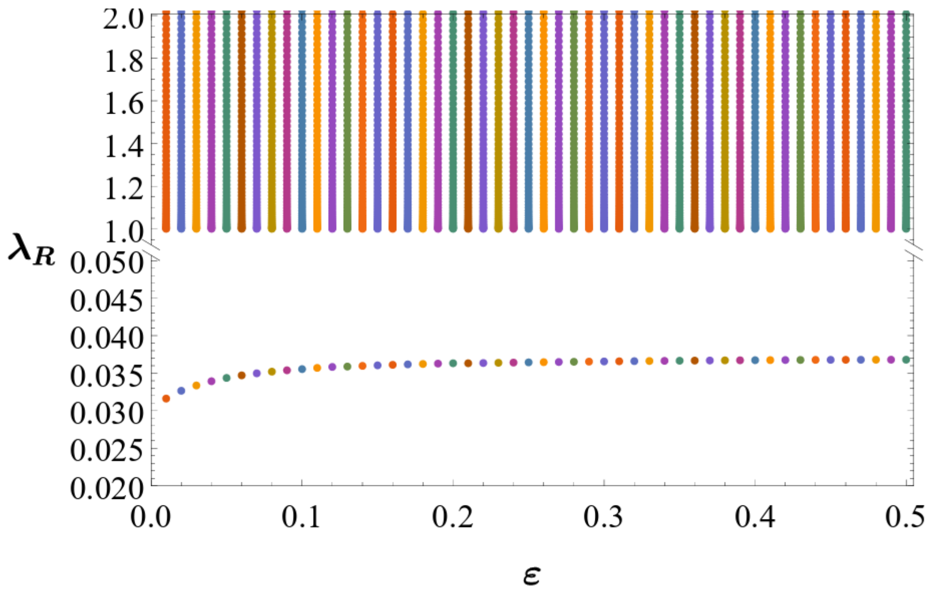




On the left panel, the $\chi(x)$ value for different ε. On the right panel, the solid line shows the sum of the kink ansatz $\phi_K(x)$ and $\chi(x)$ value for different ε, while the dashed line is the corresponding static solution determined from original field model.

Squared eigenfrequencies $\lambda = \omega^2$ calculated for the static configuration





New ansatz

We study the perturbed sine-Gordon model in the form

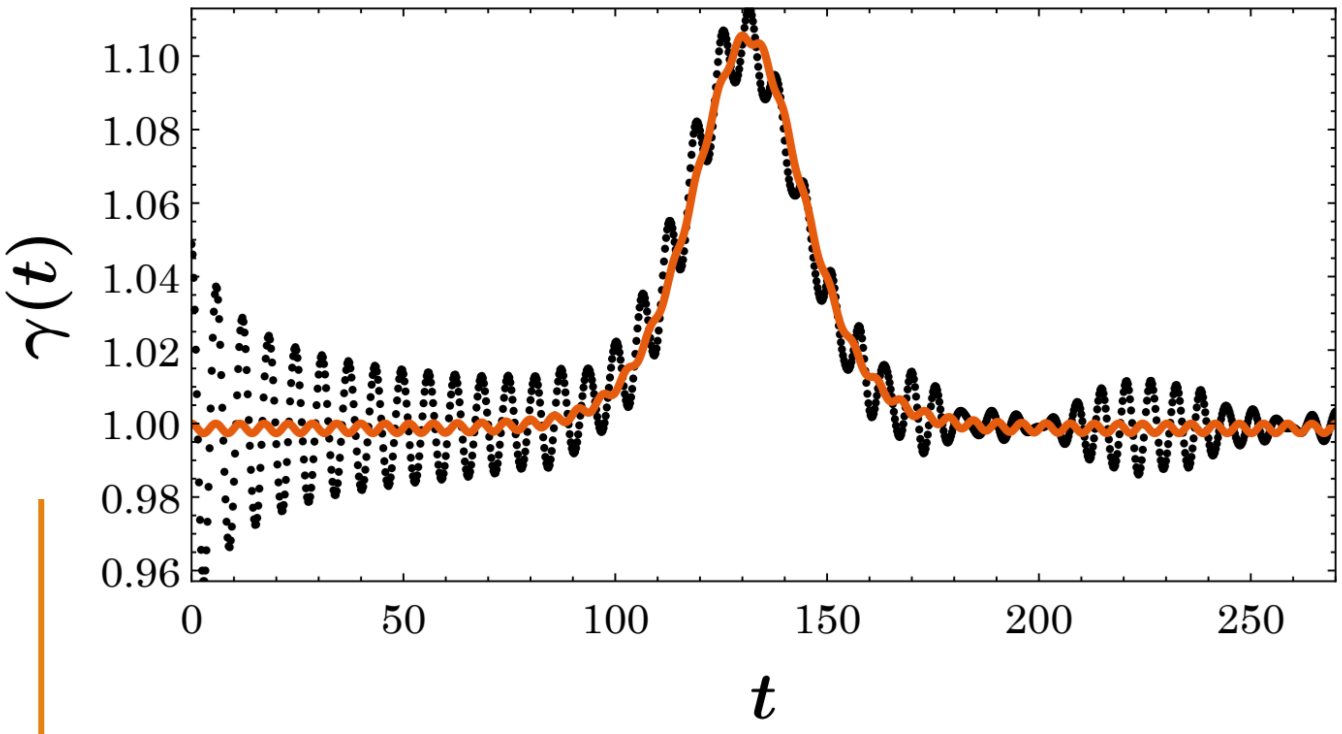
$$\partial_t^2 \phi + \alpha \partial_t \phi - \partial_x (\mathcal{F}(x) \partial_x \phi) + \sin \phi = -\Gamma$$

As an initial condition, we propose a new form of ansatz

$$\phi(0,x) = 4 \arctan \left[\exp \left(\frac{1}{\sqrt{\mathcal{F}(x_0)}} \gamma_0 (x - x_0). \right) \right]$$

$$\partial_t \phi(0,x) = -\frac{2v}{\sqrt{\mathcal{F}(x_0)}} \gamma_0 \, \mathrm{sech} \left[\frac{1}{\sqrt{\mathcal{F}(x_0)}} \gamma_0 (x - x_0) \right]$$

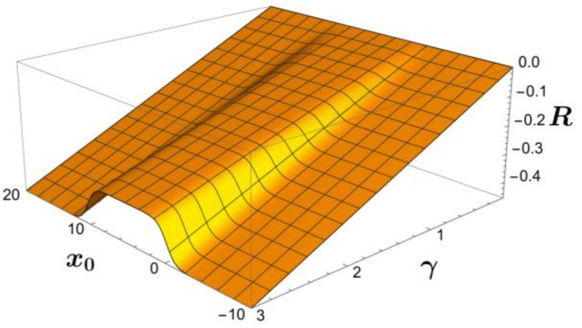
Where $\gamma_0 = \frac{1}{\sqrt{1-v^2}}$

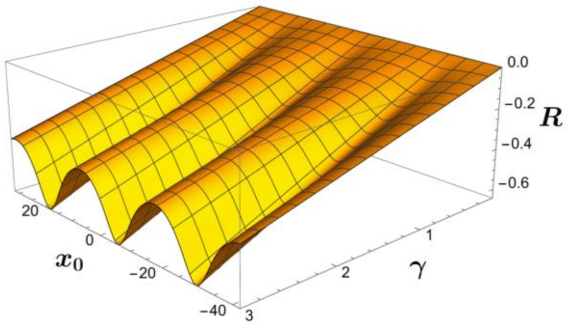


The kink width obtained from the field equation (black dots) at standard initial conditions shows significant oscillations especially at the beginning of the evolution. The $\gamma(t)$ variable is compared with the result of the effective model.

Moduli space of the system

Ricci scalar for inhomogeneity in the form of a barrier (left panel) and periodic inhomogeneity (right panel).





J. Gatlik, T. Dobrowolski, and Panayotis G. Kevrekidis, arXiv:2409.05436 [nlin.PS] (2024)

➢ Hamiltonian Case

In the absence of dissipation and forcing, the equation of motion simplifies significantly $\partial_t^2\phi - \partial_x(\mathcal{F}(x)\partial_x\phi) + \sin\phi = 0$

The Lagrangian for this system takes the form $L = \int_{-\infty}^{+\infty} dx\mathcal{L}(\phi) = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\ (\partial_t\phi)^2 - \frac{1}{2}\ \mathcal{F}(x)(\partial_x\phi)^2 - (1-\cos\phi)\right]$

By introducing $\phi(t,x) = 4\arctan e^{\xi(t,x)}$ where $\xi = \frac{1}{\sqrt{\mathcal{F}(x_0(t))}}\ \gamma(t)\,(x - x_0(t))$

The Lagrangian can be equivalently transformed to the form $L = 4\int_{-\infty}^{+\infty} dx\ \text{sech}^2\xi\left[\frac{1}{2}(\partial_t\xi)^2 - \frac{1}{2}\mathcal{F}(x)(\partial_x\xi)^2 - \frac{1}{2}\right]$

Effective Lagrangian is obtained by integrating over the spatial variable

$$L_{eff} = \tfrac{1}{2}M\dot{x}_0^2 + \tfrac{1}{2}m\dot{\gamma}^2 - \kappa\dot{x}_0\dot{\gamma} - V$$

The Euler-Lagrange equations obtained from the effective Lagrangian take the form of

$$\begin{aligned} &M\ddot{x}_0 - \kappa\ddot{\gamma} + \frac{1}{2}(\partial_{x_0}M)\dot{x}_0^2 - \frac{1}{2}(\partial_{x_0}m)\dot{\gamma}^2 - (\partial_\gamma\kappa)\dot{\gamma}^2 \\ &\quad +(\partial_\gamma M)\dot{\gamma}\dot{x}_0 + \partial_{x_0}V = 0, \\ &m\ddot{\gamma} - \kappa\ddot{x}_0 + \frac{1}{2}(\partial_\gamma m)\dot{\gamma}^2 - \frac{1}{2}(\partial_\gamma M)\dot{x}_0^2 - (\partial_{x_0}\kappa)\dot{x}_0^2 \\ &\quad +(\partial_{x_0}m)\dot{x}_0\dot{\gamma} + \partial_\gamma V = 0. \end{aligned}$$

$$M = 4\int_{-\infty}^{+\infty} dx\,\text{sech}^2(\xi)\,W(\xi)^2$$

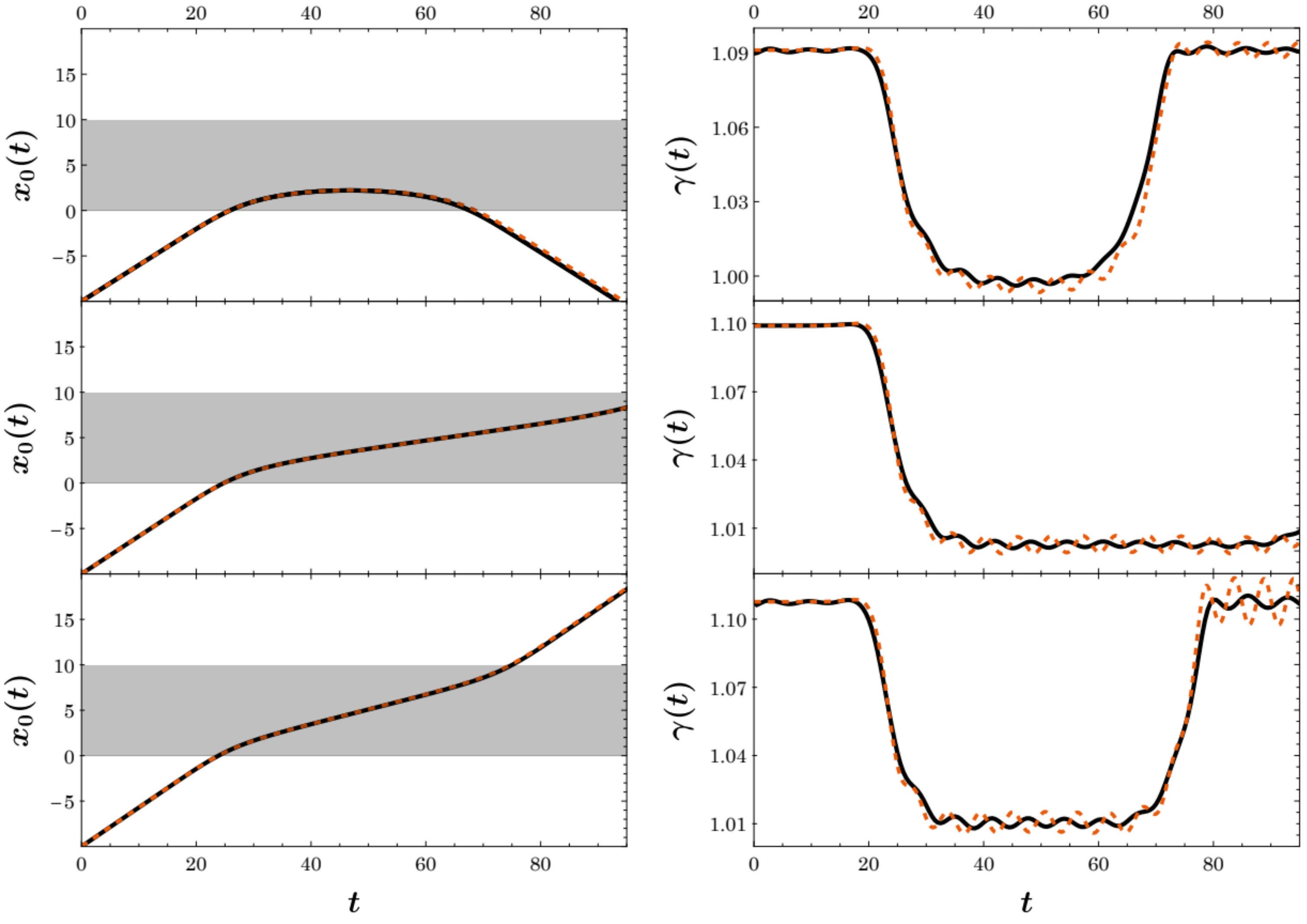
$$m = \frac{4}{\gamma^2}\int_{-\infty}^{+\infty} dx\,\text{sech}^2(\xi)\,\xi^2$$

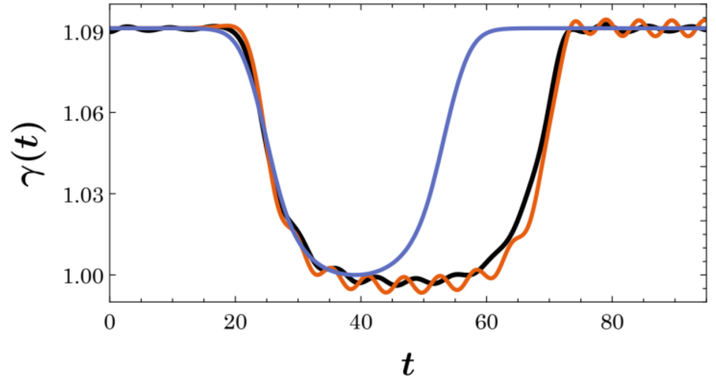
$$\kappa = \frac{4}{\gamma}\int_{-\infty}^{+\infty} dx\,\text{sech}^2(\xi)W(\xi)\,\xi$$

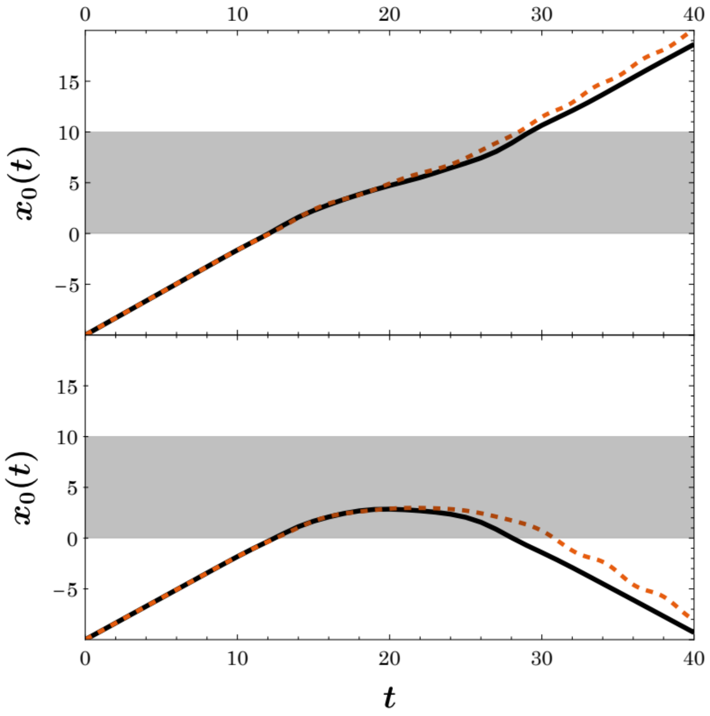
$$V = 2\int_{-\infty}^{+\infty} dx\,\text{sech}^2(\xi)\left(1 + \frac{\mathcal{F}(x)}{\mathcal{F}(x_0)}\gamma^2\right)$$

$$W(\xi) = \frac{1}{2\mathcal{F}(x_0)}(\partial_{x_0}\mathcal{F}(x_0))\,\xi + \frac{1}{\sqrt{\mathcal{F}(x_0)}}\,\gamma$$

➢ Hamiltonian Case







Comparison of the position of the center of mass and the $\gamma(t)$ variable of the kink for the solution from the original field model (black line) and the approximate model.

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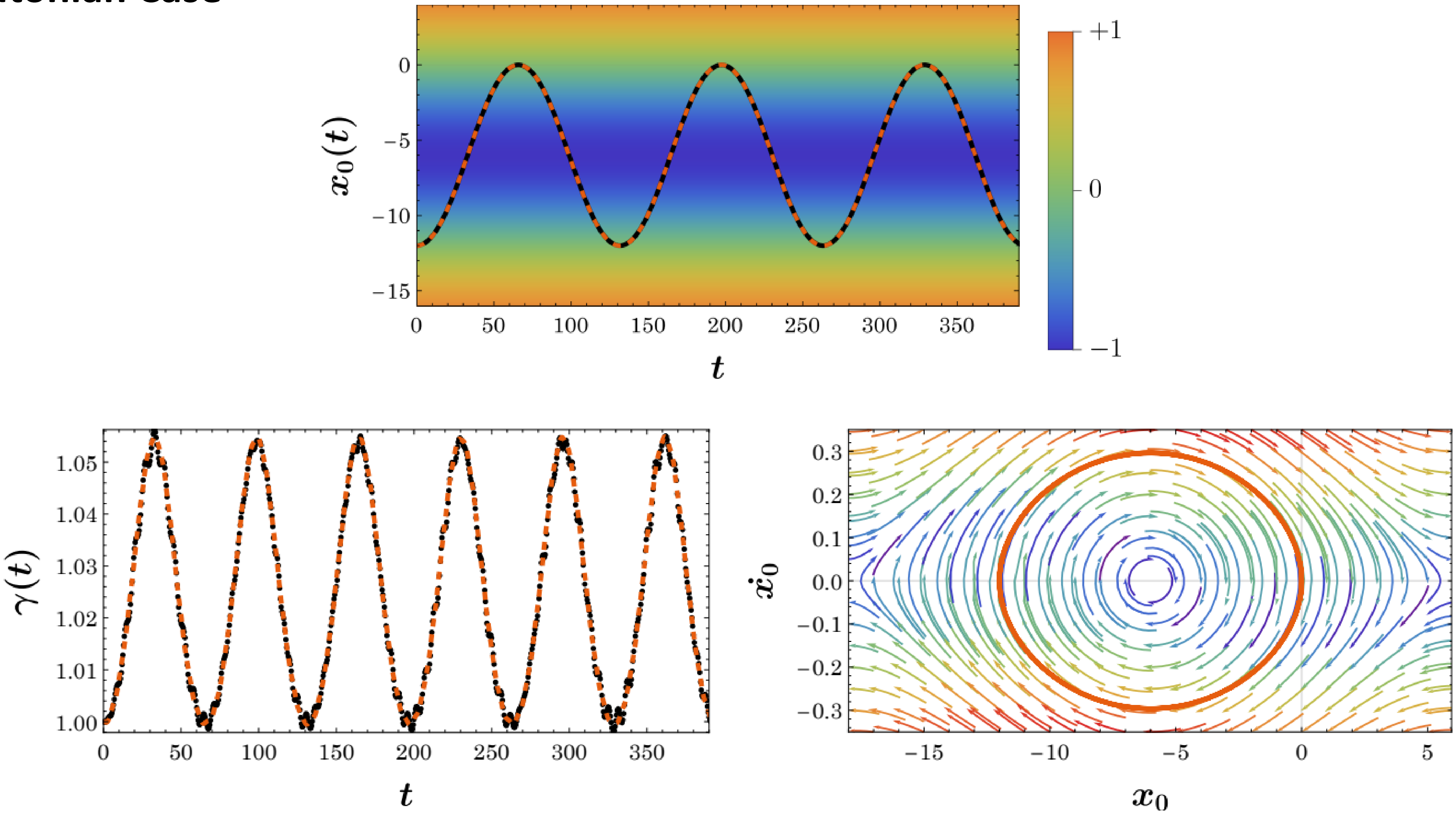
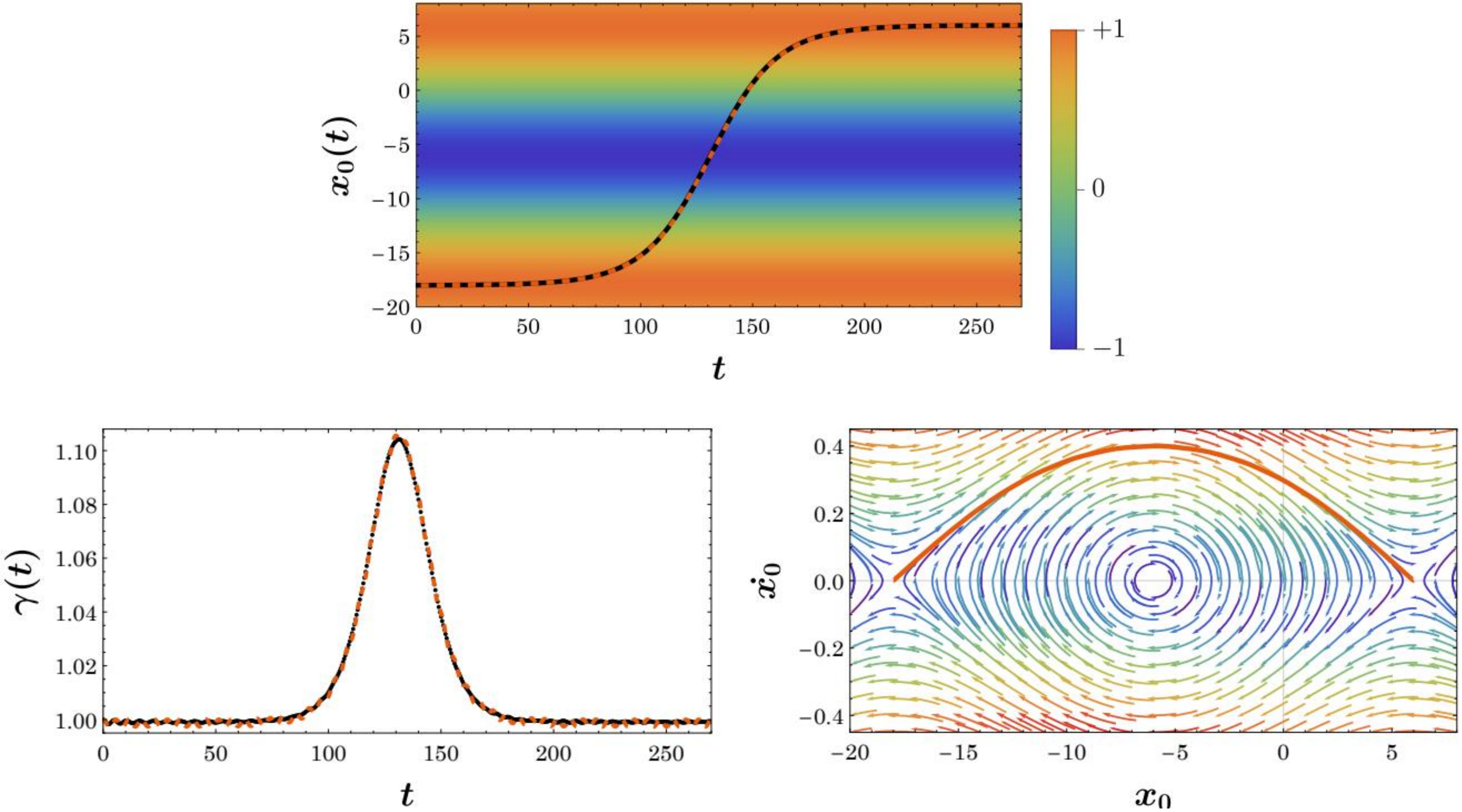
➢ Hamiltonian Case



Comparison of the solution from the original field model (black line) and the approximate model (orange line).

➢ Hamiltonian Case



Comparison of the solution from the original field model (black line) and the approximate model (orange line).

➢ Dynamics in the presence of dissipation

With dissipation and external forcing we used method based on a *nonconservative Lagrangian* density $\mathcal{L}_N = \mathcal{L}(\phi_1) - \mathcal{L}(\phi_2) + \mathcal{R}$

In this notation, the equation of motion is of the form $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = \left[\frac{\partial \mathcal{R}}{\partial \phi_-} - \partial_\mu \left(\frac{\partial \mathcal{R}}{\partial(\partial_\mu \phi_-)}\right)\right]_{PL}$

Here, the left-hand side of this equation takes the form

$$\partial_t^2 \phi - \partial_x(\mathcal{F}(x)\partial_x \phi) + \sin\phi = \left[\frac{\partial \mathcal{R}}{\partial \phi_-} - \partial_\mu \left(\frac{\partial \mathcal{R}}{\partial(\partial_\mu \phi_-)}\right)\right]_{PL}$$

We taking the following form of the non-conservative term $\mathcal{R} = -\Gamma\phi_- - \alpha\phi_-\partial_t\phi_+$

The equations of motion at the effective ODE level are then obtained as follows

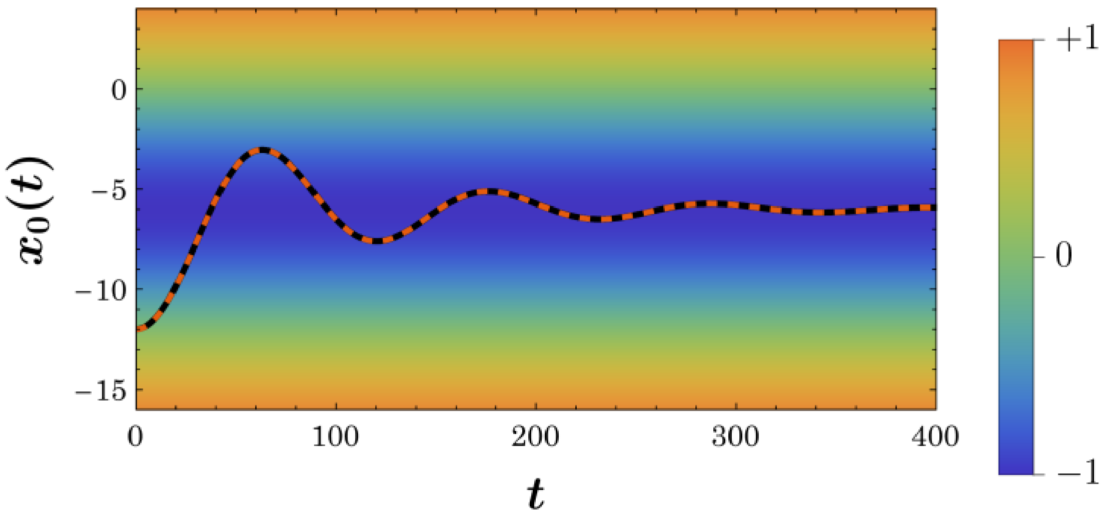
$$\frac{d}{dt}\left(\frac{\partial L_{eff}}{\partial \dot{x}_0}\right) - \frac{\partial L_{eff}}{\partial x_0} = \left[\frac{\partial R_{eff}}{\partial x_-} - \frac{d}{dt}\left(\frac{\partial R_{eff}}{\partial \dot{x}_-}\right)\right]_{PL} \qquad \frac{d}{dt}\left(\frac{\partial L_{eff}}{\partial \dot{\gamma}}\right) - \frac{\partial L_{eff}}{\partial \gamma} = \left[\frac{\partial R_{eff}}{\partial \gamma_-} - \frac{d}{dt}\left(\frac{\partial R_{eff}}{\partial \dot{\gamma}_-}\right)\right]_{PL}$$

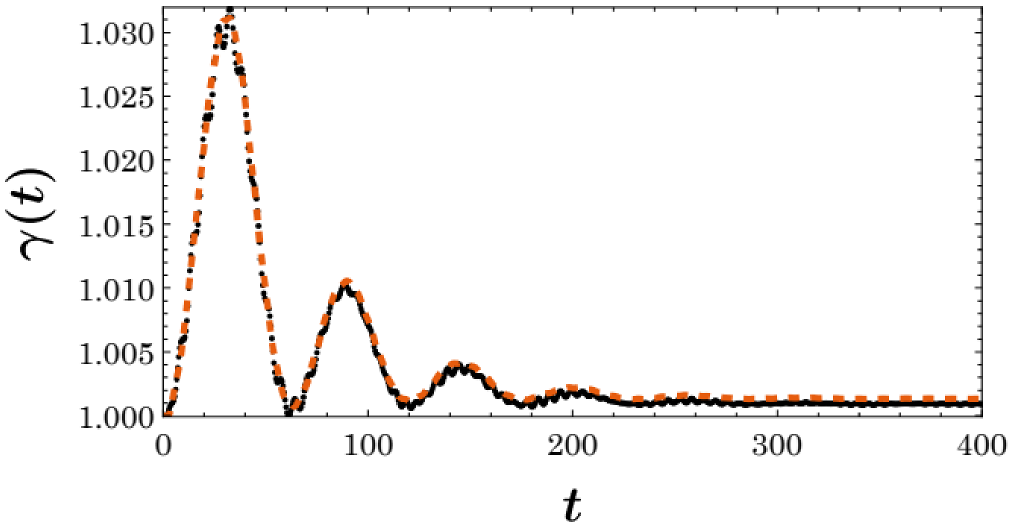
$$M\ddot{x}_0 - \kappa\ddot{\gamma} + \frac{1}{2}(\partial_{x_0}M)\dot{x}_0^2 - \frac{1}{2}(\partial_{x_0}m)\dot{\gamma}^2 - (\partial_\gamma\kappa)\dot{\gamma}^2 + (\partial_\gamma M)\dot{\gamma}\dot{x}_0 + \partial_{x_0}V =$$

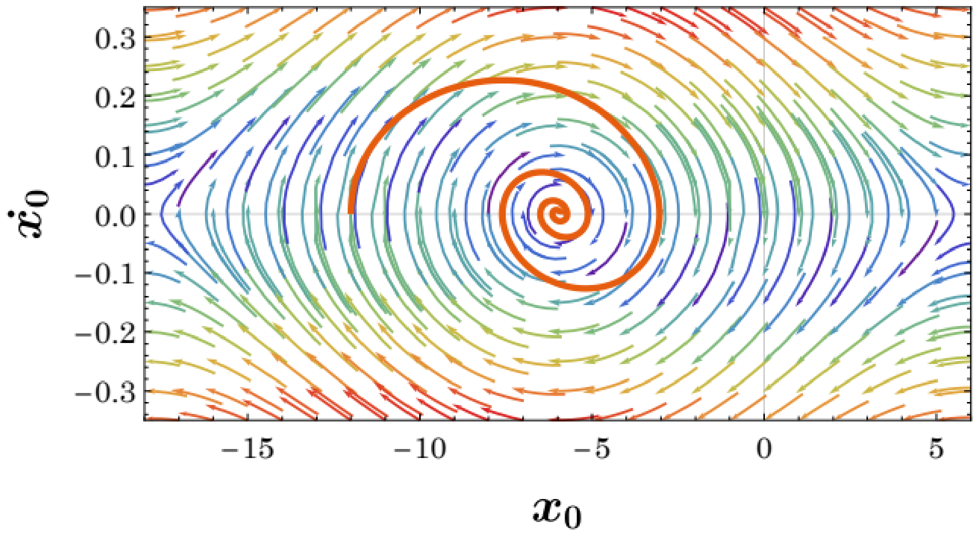
$$2\pi\Gamma - \alpha\frac{\pi^2}{6\gamma}\sqrt{\mathcal{F}(x_0)}\left(\frac{\partial_{x_0}\mathcal{F}(x_0)}{\mathcal{F}(x_0)}\right)^2\dot{x}_0 - 8\alpha\frac{\gamma}{\sqrt{\mathcal{F}(x_0)}}\dot{x}_0 + \alpha\frac{\pi^2}{3\gamma^2}\frac{\partial_{x_0}\mathcal{F}(x_0)}{\sqrt{\mathcal{F}(x_0)}}\dot{\gamma}$$

$$m\ddot{\gamma} - \kappa\ddot{x}_0 + \frac{1}{2}(\partial_\gamma m)\dot{\gamma}^2 - \frac{1}{2}(\partial_\gamma M)\dot{x}_0^2 - (\partial_{x_0}\kappa)\dot{x}_0^2 + (\partial_{x_0}m)\dot{x}_0\dot{\gamma} + \partial_\gamma V = \alpha\frac{\pi^2}{3\gamma^2}\frac{\partial_{x_0}\mathcal{F}(x_0)}{\sqrt{\mathcal{F}(x_0)}}\dot{x}_0 - \alpha\frac{2\pi^2}{3\gamma^3}\sqrt{\mathcal{F}(x_0)}\dot{\gamma}$$

➢ Dynamics in the presence of dissipation

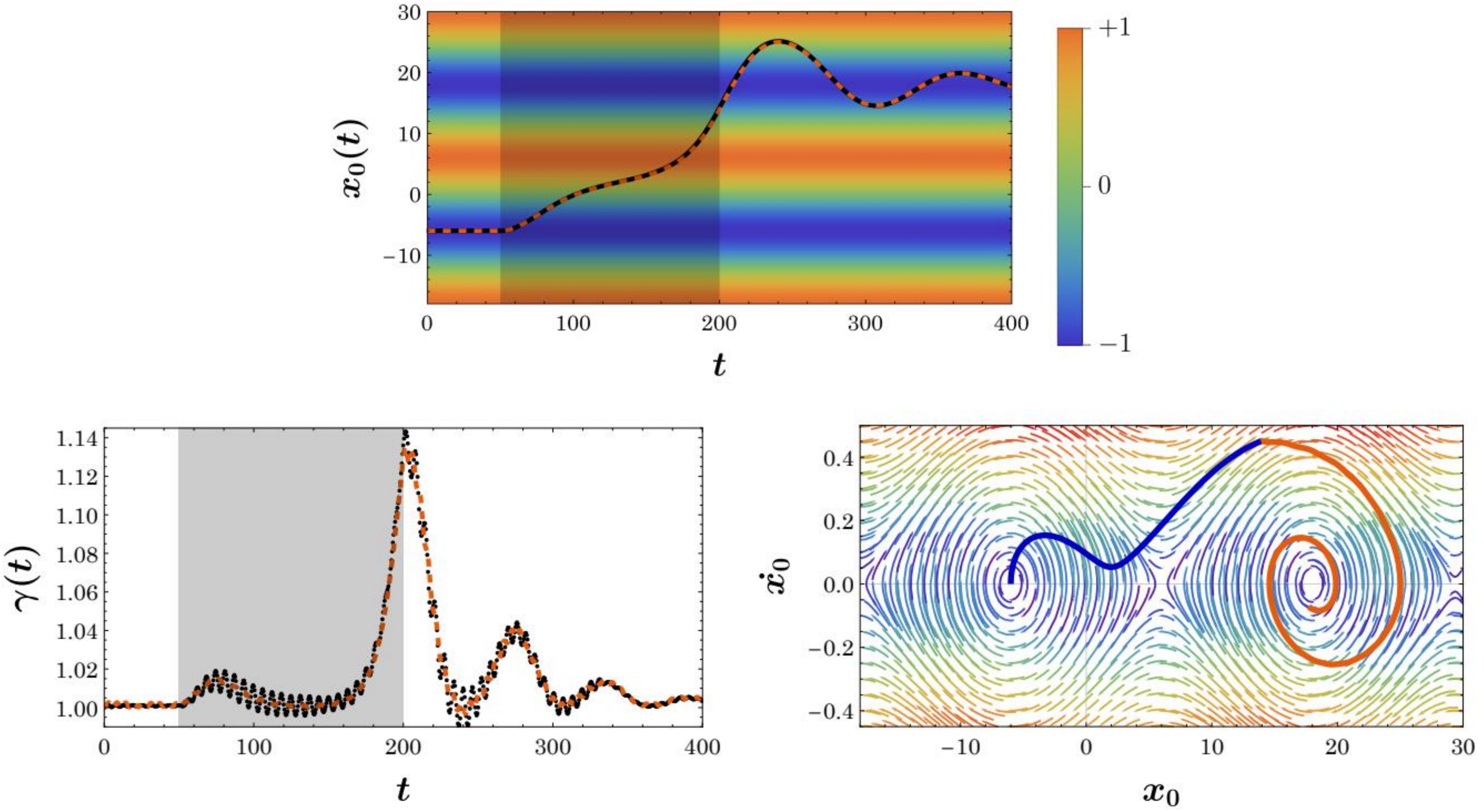






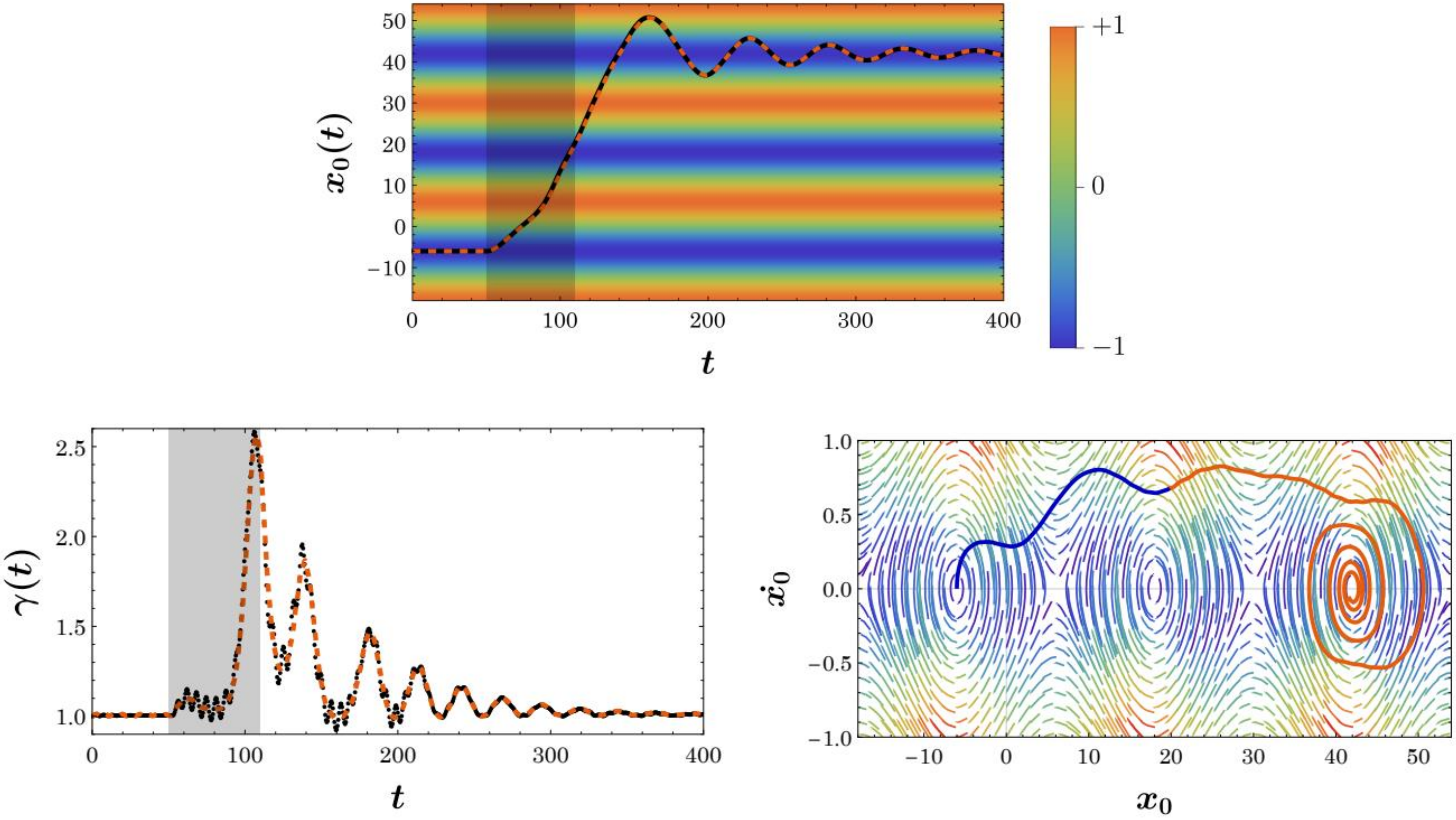
A decelerating kink eventually stopping at a potential minimum through the effect of dissipation.

➢ Dynamics in the presence of dissipation



A decelerating kink eventually stopping at a potential minimum through the effect of dissipation.

➢ Dynamics in the presence of dissipation



Shifting the kink from a minimum to a minimum that is not directly adjacent to it.

J. Gatlik, T. Dobrowolski, and Panayotis G. Kevrekidis, arXiv:2409.05436 [nlin.PS] (2024)

➢ Kink-barrier interaction in the presence of bias current and dissipation

Linear approximation of the system

$$M_s \delta\ddot{x}_0 - \kappa_s \delta\ddot{\gamma} + \Omega_{x_0}^2 \delta x_0 = -\alpha M_s \delta\dot{x}_0 + \alpha \kappa_s \delta\dot{\gamma}$$
$$m_s \delta\ddot{\gamma} - \kappa_s \delta\ddot{x}_0 + \Omega_{\gamma}^2 \delta\gamma = -\alpha m_s \delta\dot{\gamma} + \alpha \kappa_s \delta\dot{x}_0$$

The exponents describing the time evolution of the trajectory in the linear approximation correspond to the solution of the following quartic equation

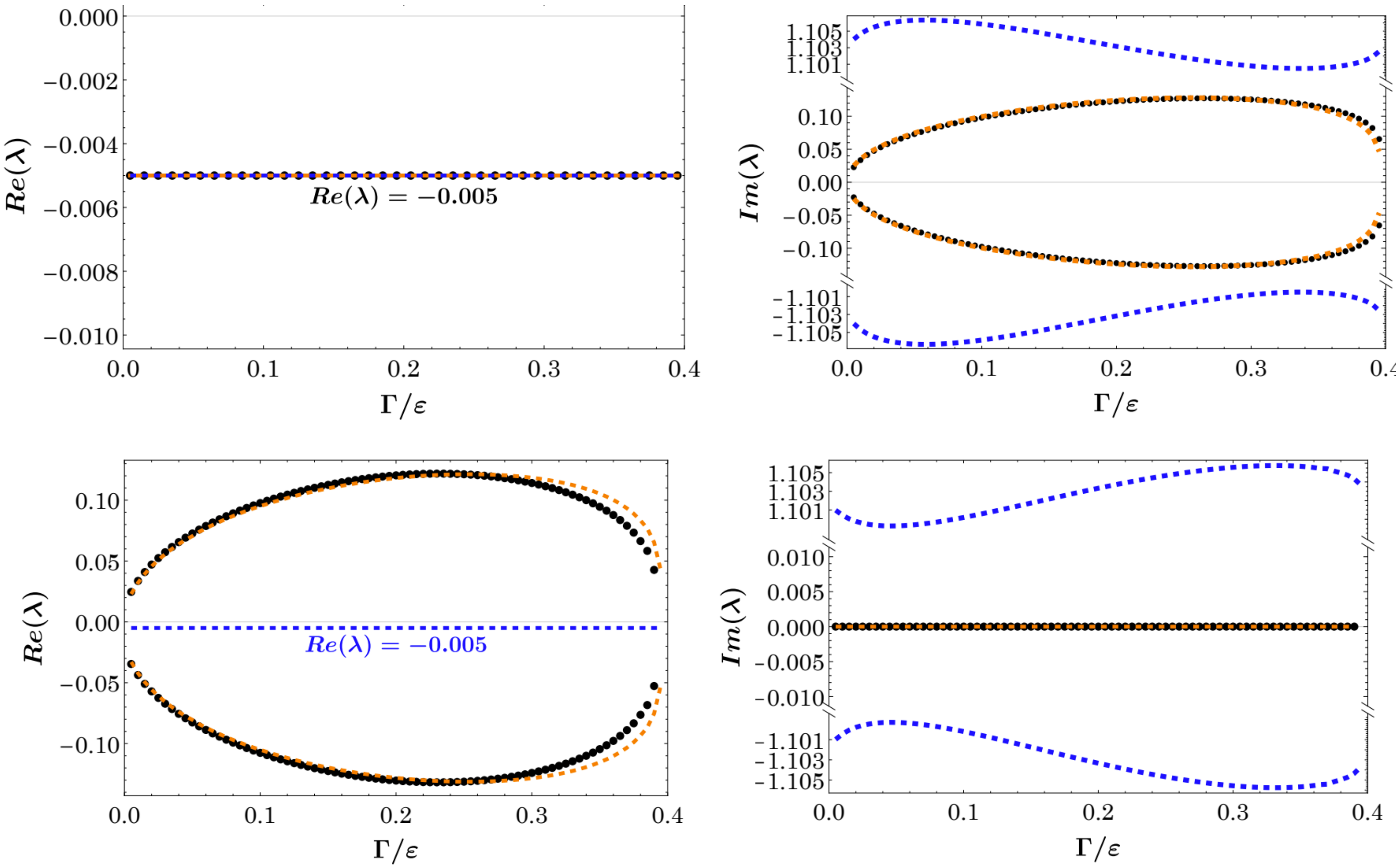
$$(m_s M_s - \kappa_s^2)\lambda^4 + 2\alpha(M_s m_s - \kappa_s^2)\lambda^3 + \Big(M_s\Omega_\gamma^2 + m_s\Omega_{x_0}^2 + \alpha^2(M_s m_s - \kappa^2)\Big)\lambda^2 + \alpha(M_s\Omega_\gamma^2 + m_s\Omega_{x_0}^2)\lambda + \Omega_\gamma^2\Omega_{x_0}^2 = 0.$$

$$m_s\lambda^2 + \alpha m_s \lambda + \Omega_\gamma^2 = 0$$

$$\longrightarrow \quad \lambda_\pm^{(\gamma)} = -\frac{\alpha}{2} \pm i\frac{\sqrt{4m_s\Omega_\gamma^2 - \alpha^2 m_s^2}}{2m_s}$$

$$M_s\lambda^2 + \alpha M_s \lambda + \Omega_{x_0}^2 = 0$$

$$\longrightarrow \quad \lambda_\pm^{(x_0)} = -\frac{\alpha}{2} \pm i\frac{\sqrt{4M_s\Omega_{x_0}^2 - \alpha^2 M_s^2}}{2M_s}$$



Analysis of the stability of the kink located in the minimum and maximum of the potential. The upper panels show the behavior of a kink located around the minimum potential, while the lower panels illustrate the instability of a kink located at the local maximum of the potential.

J. Gatlik, T. Dobrowolski, and Panayotis G. Kevrekidis, arXiv:2409.05436 [nlin.PS] (2024)

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Thank you for your attention!