Complexity in Quantum Field Theories

Pawel Caputa

UNIVERSITY OF WARSAW

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Outline:

- History/Motivation
- Krylov basis complexity of states and operators
- Applications
- Conclusions and Open Questions

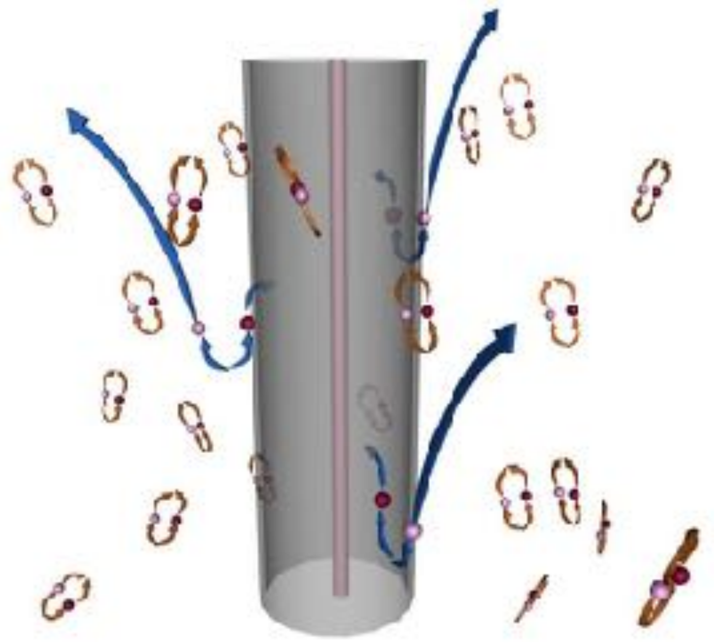
Based on works with:

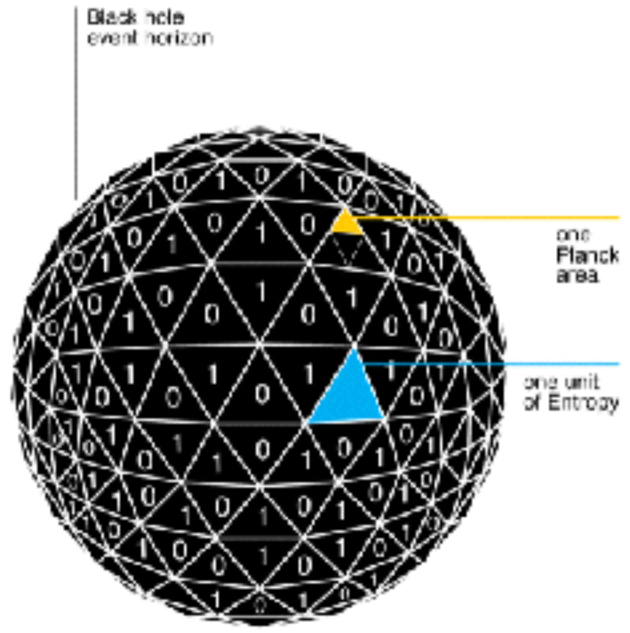
V. Balasubramanian, J. Magan, Q. Wu, D. Patramanis, S. Liu

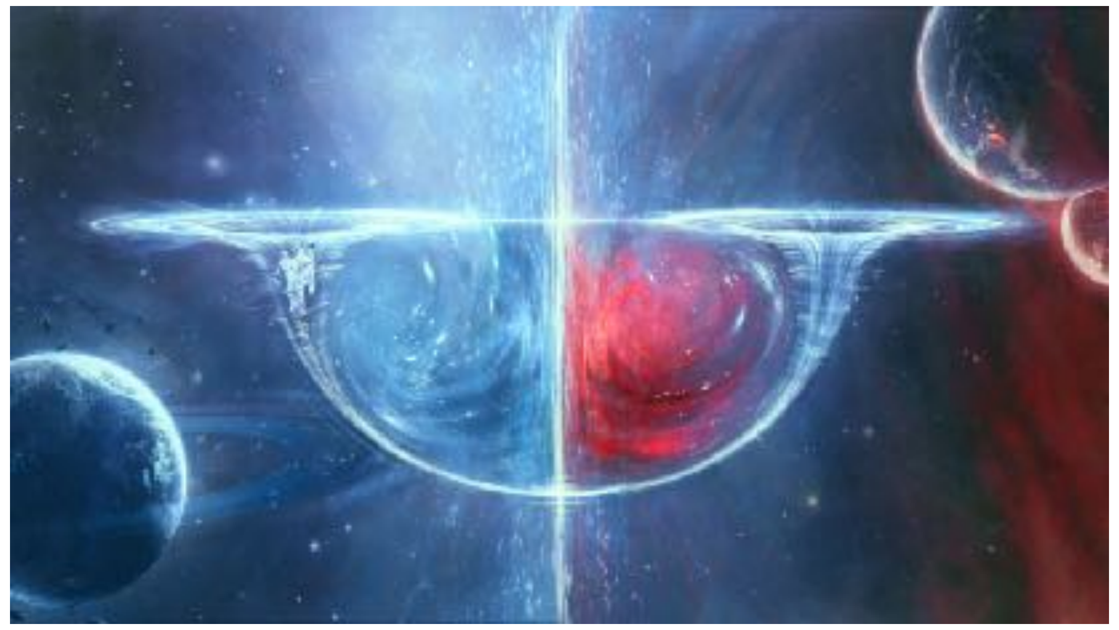




Black Holes: (Theoretical) Laboratory for QM and QGR







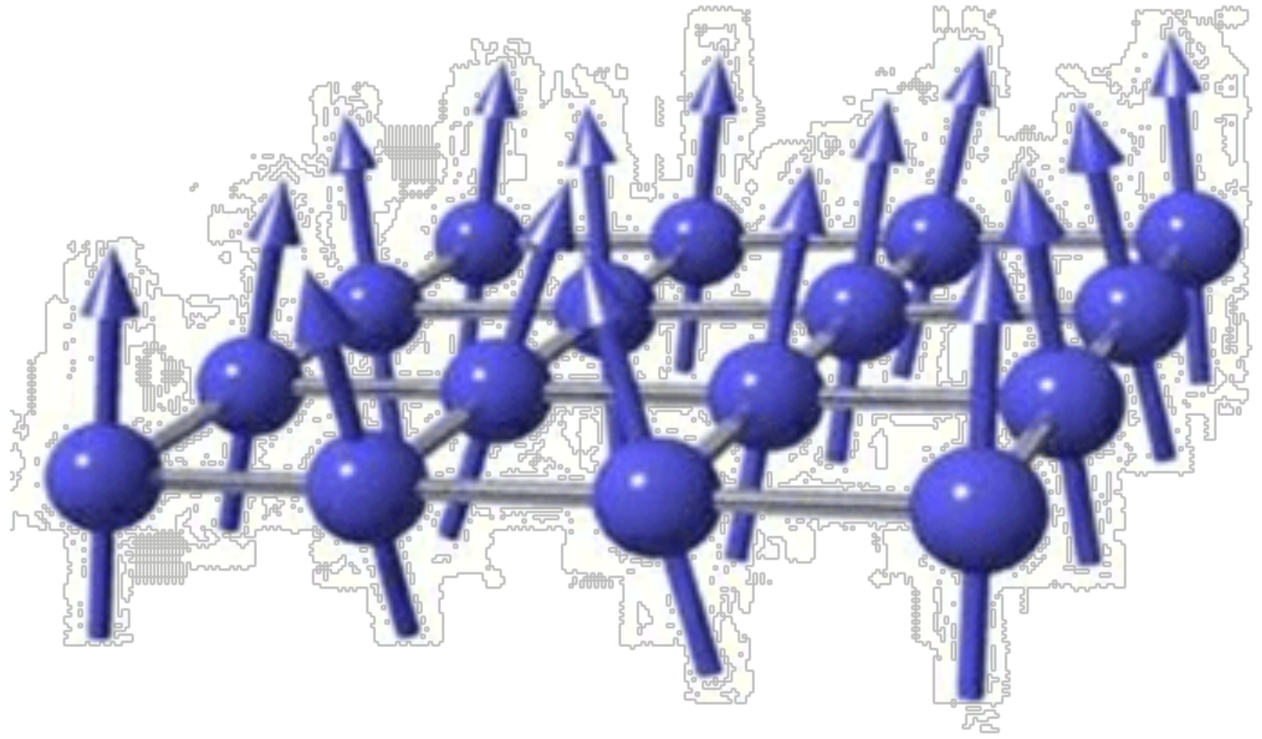
They radiate (Information paradox), holographic (entropy), the fastest scramblers of information, Page curve? Extreme in thermalisation,….

We need a concrete model of Quantum Gravity! Or maybe a new paradigm…?

AdS = CFT

[Maldacena ’97]

?



Strongly coupled Many-Body system at finite temperature

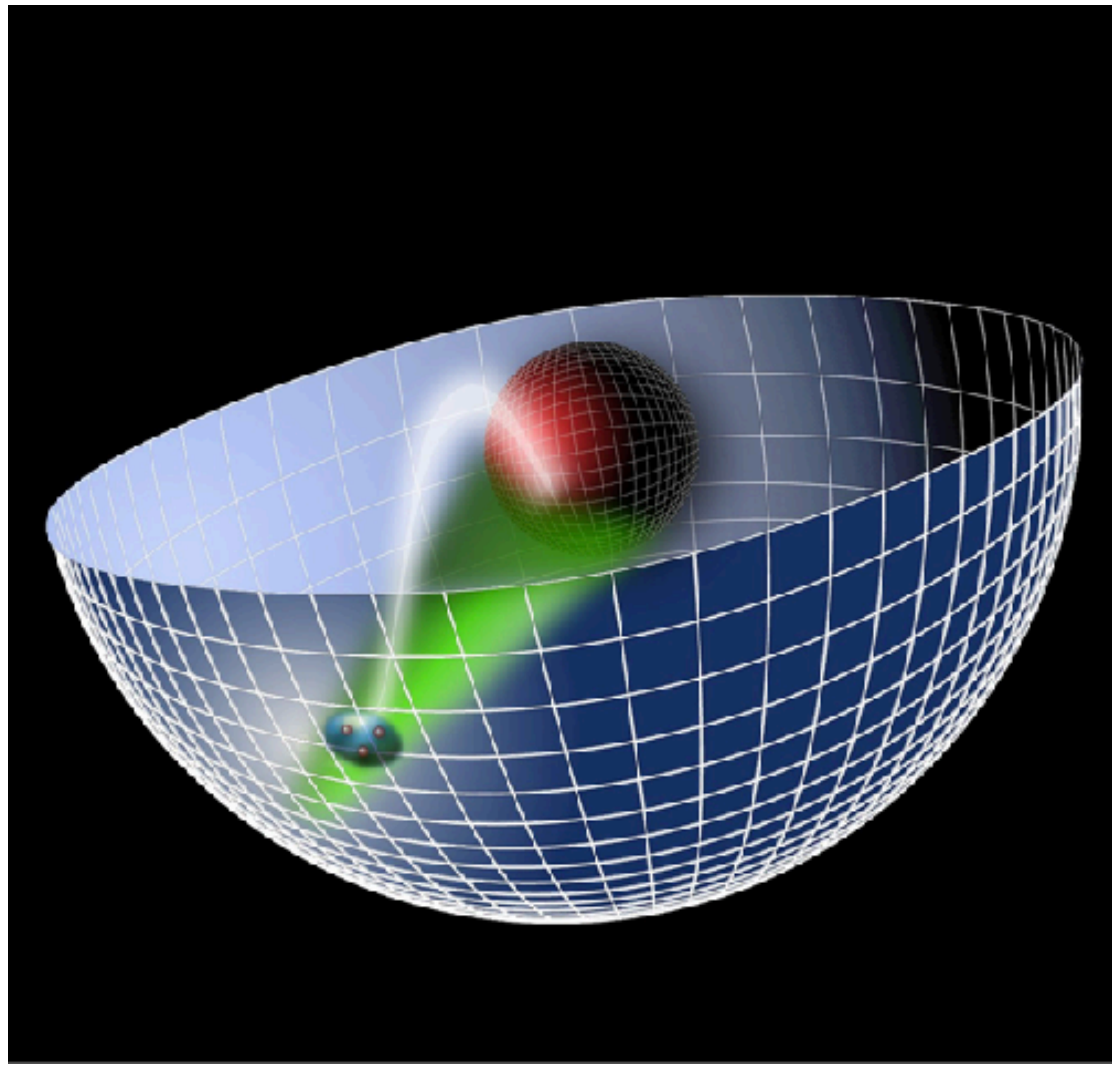
Logic (abductive reasoning): “Black Hole test”:

If it has entropy of a BH, scrambles like a BH and is complex like a BH, then it probably is a BH…

N=4 SYM, SYK, Random Matrix models…

AdS = CFT

Quantum Gravity



Quantum Field Theory

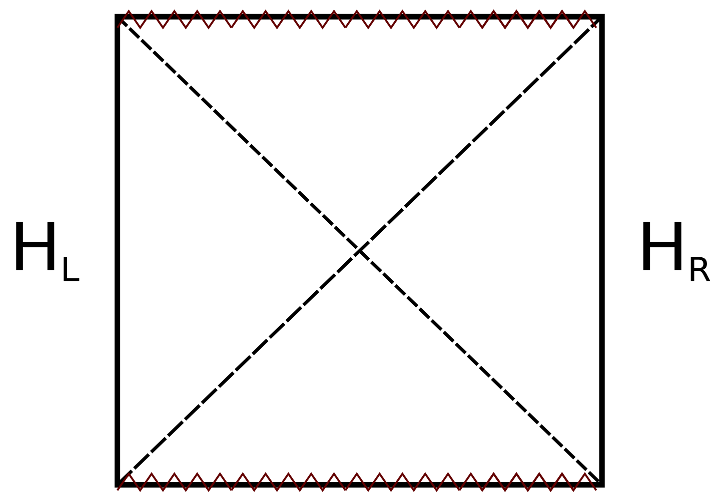
"Hilbert spaces are isomorphic"

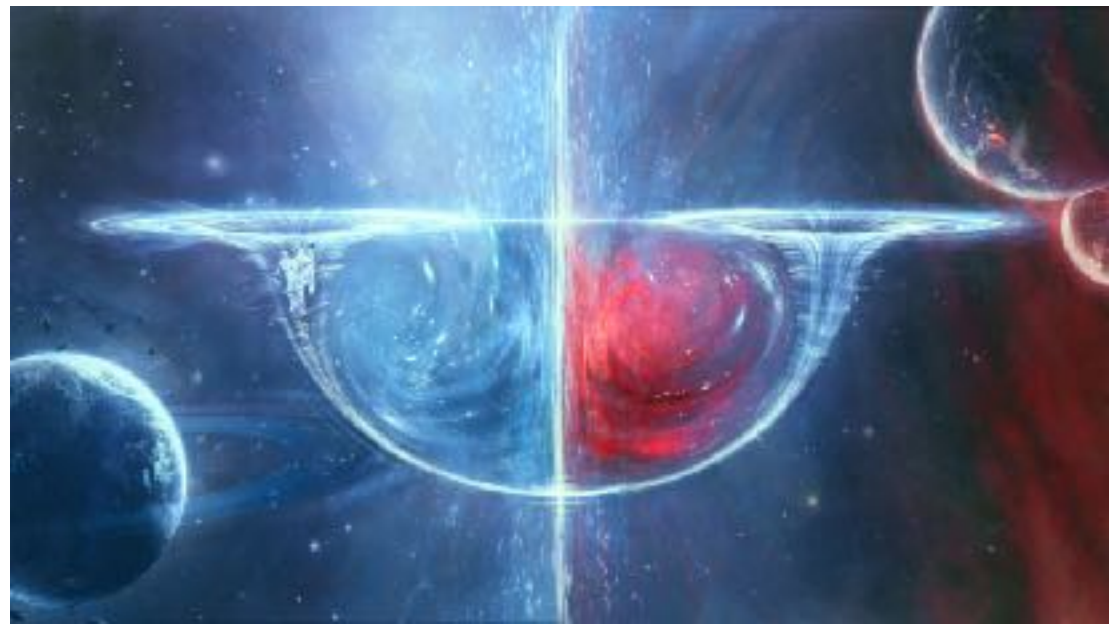
QGR Basis ⟷ ? QFT Basis

Black Hole interiors grow with time!

[Maldacena ’01]

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2} E_n} |n, n\rangle$$

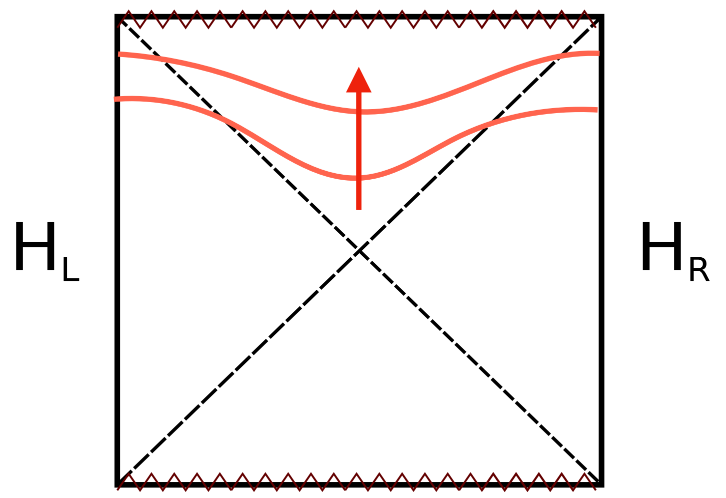


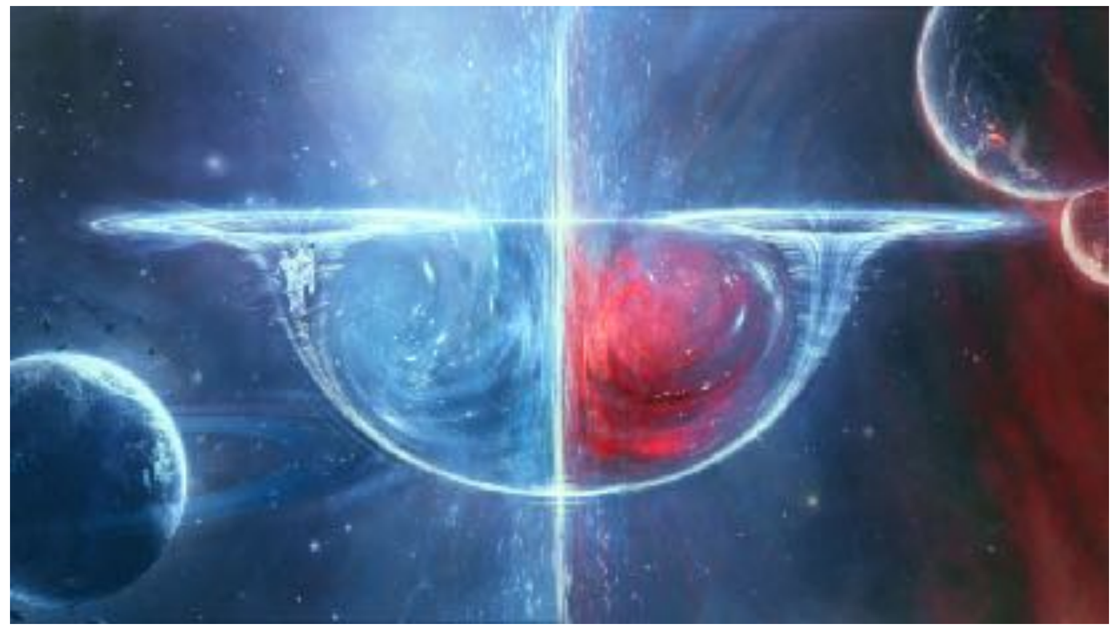


Black Hole interiors grow with time!

[Hartman,Maldacena '13]

$$|\Psi_\beta(t)\rangle = e^{-i(H_L+H_R)t} \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n} |n,n\rangle$$





What is the "CFT dual" of this growth? "Complexity" of the TFD state? [Susskind,'14]

Is there a universal (useful/computable in QFT) notion of "Complexity"?

COMPLEXITY

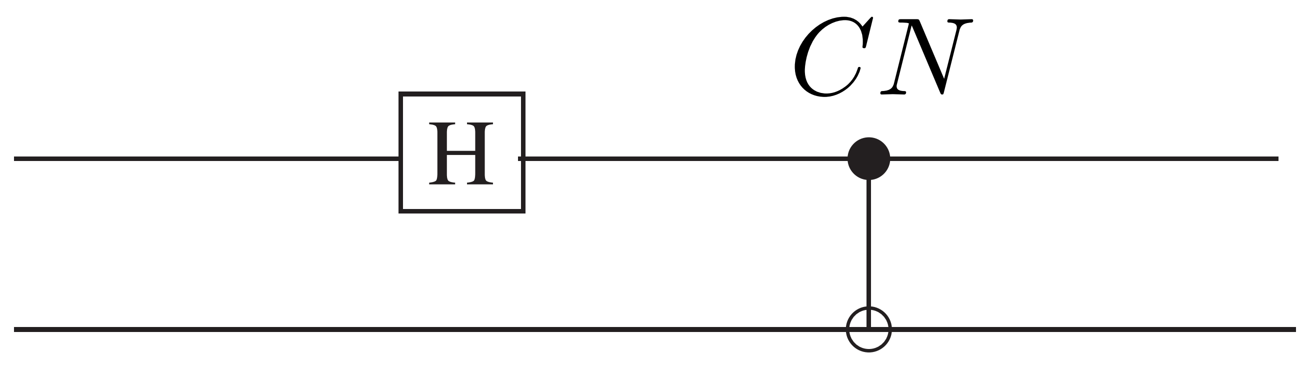
Hardness of tasks given limited resources

COMPLEXITY

Hardness of tasks given limited resources

Circuit Complexity: **Minimal number of gates**

$|\downarrow\downarrow\rangle$



$$\frac{|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle}{\sqrt{2}}$$

Complexity = 2

C. Shannon, Bell System Technical Journal. (1949)

Hardness of tasks given limited resources

Kolmogorov complexity:

Length of the shortest computer program

ABABABABABABABABABABABABABABAB

“write AB 15 times”

Complexity = 17

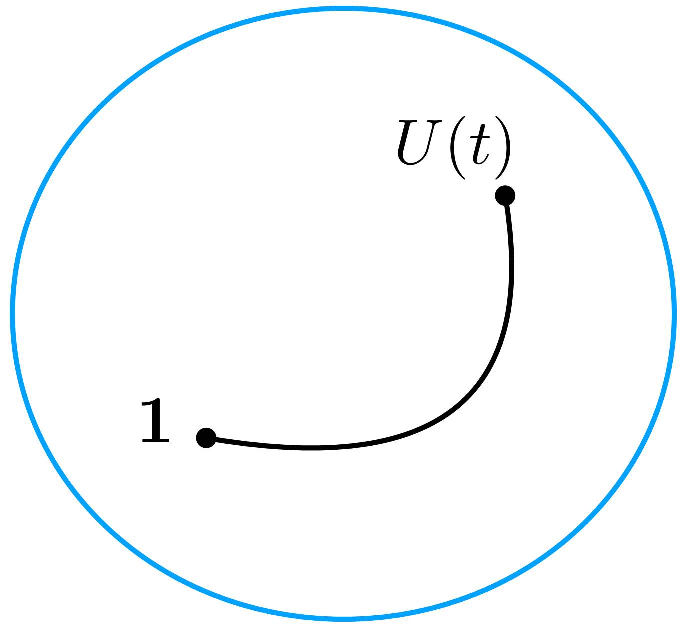
A. Kolmogorov,
Theoretical Computer Science
(1963)

J. Rissanen, (1986)

Hardness of tasks given limited resources

Geometric Complexity: **Geodesic Length**

$$|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle$$



M. Nielsen et al.
Science (2006)

Applications to QFT

Nielsen complexity in free QFTs, Conformal Field Theories [Jefferson,Myers'17][PC,Magan'18]

Lessons: First and Second Laws of Quantum Complexity [Brown,Susskind'17]

Ambiguities… (choice of gates, cost functions)

Physical definition of Complexity?

Physics Problems

Unitary evolution of states or operators:

$$i\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle \qquad \partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)]$$

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle \qquad \mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}$$

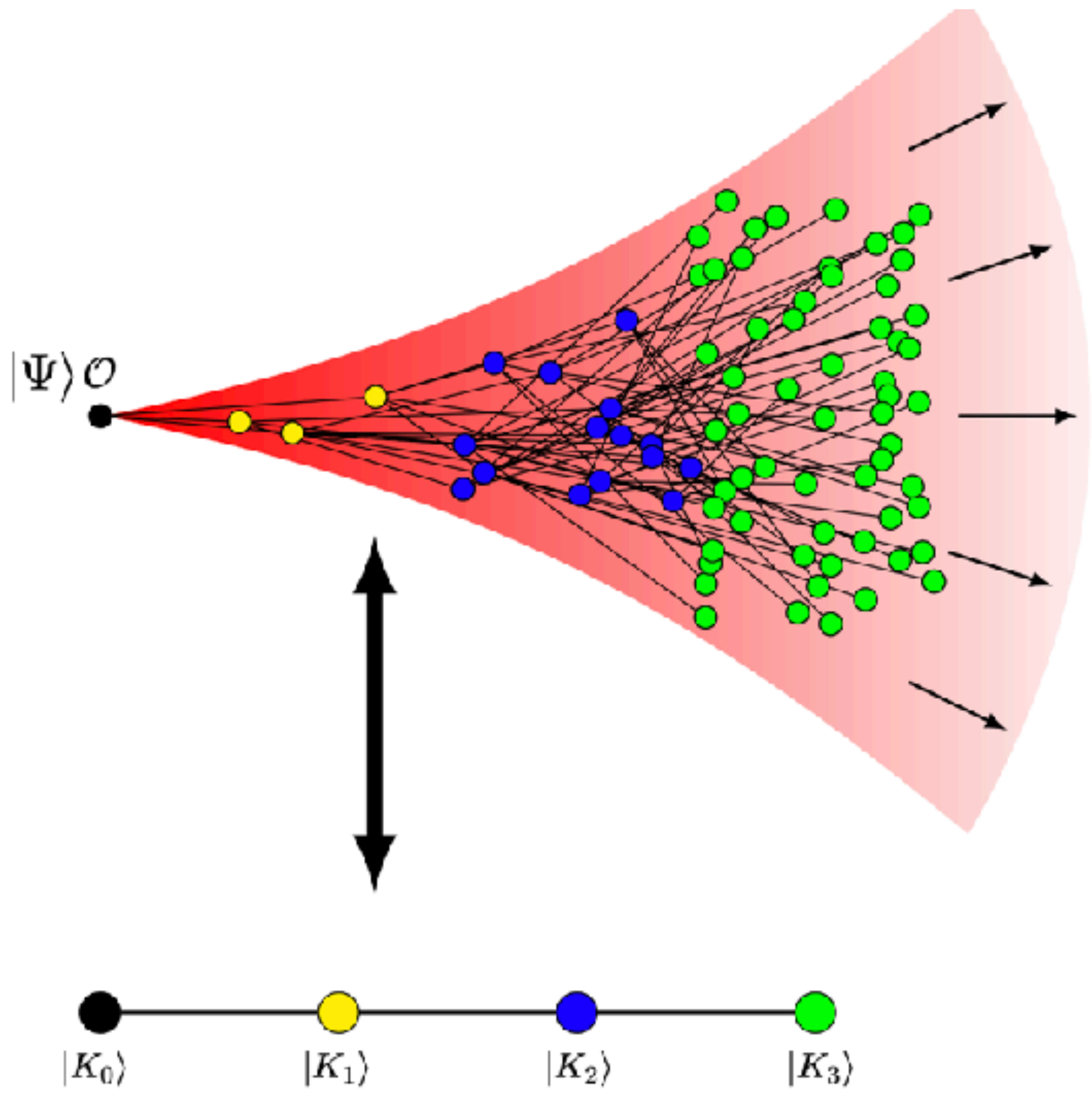
Generically, a “simple” reference quantum state $|\Psi(0)\rangle$ “**spreads**” and becomes “complex” (in Hilbert space)

Generically, a “simple” operator $\mathcal{O}(0)$ “grows” and becomes “**complex**” (in operator space)

Q: How to quantify this “**Complexity**”?

Basic Idea

Map the unitary evolution into a “1d chain” and quantify **Complexity** as a distance from the origin



Basic Idea

Map the unitary evolution into a "1d chain" and quantify **Complexity** as a distance from the origin

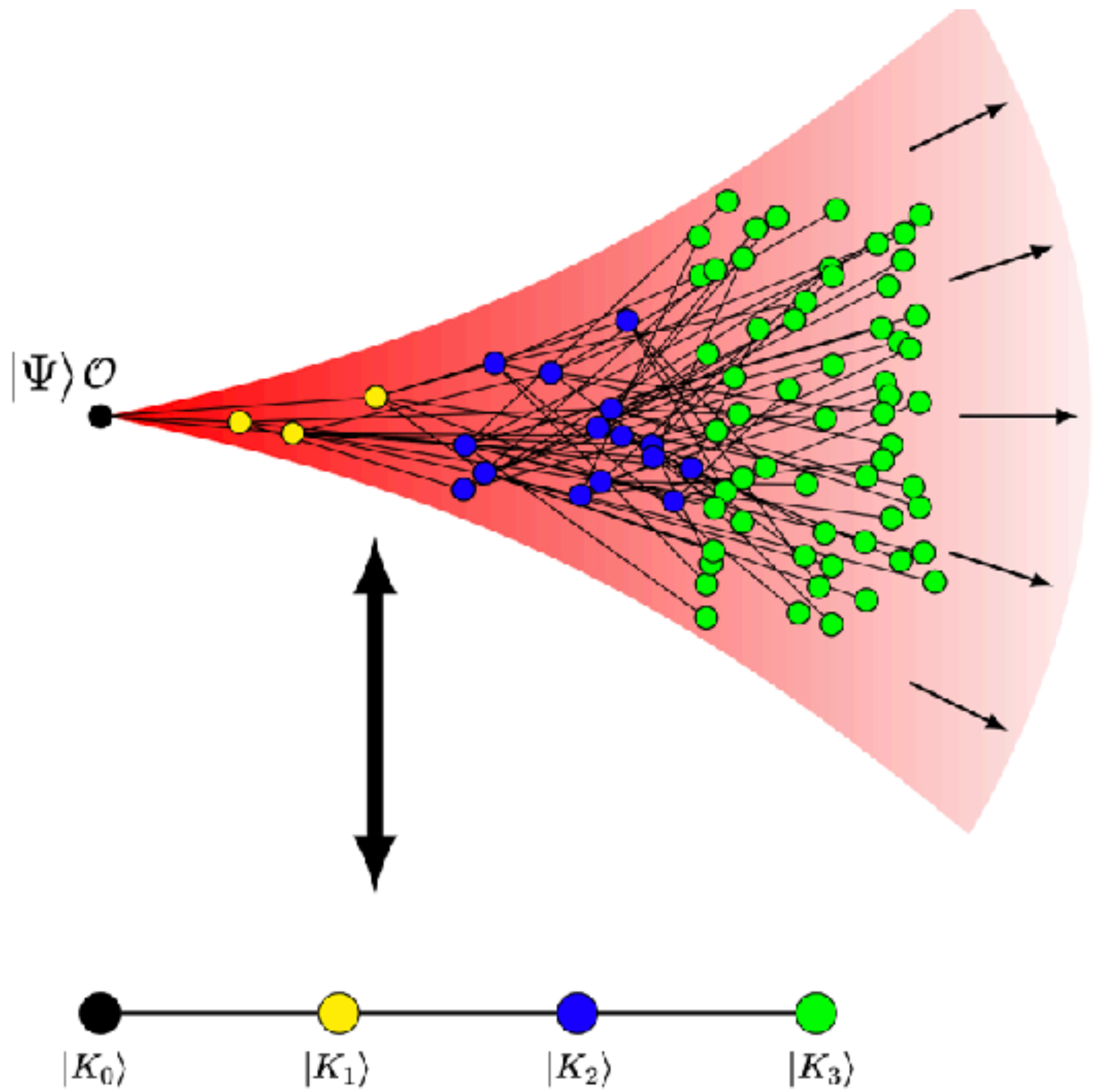
Technically:

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_n \phi_n(t)|K_n\rangle$$

Coefficients of the expansion = probability distribution

$$\sum_n |\phi_n(t)|^2 \equiv \sum_n p_n = 1$$

Use it to characterise **spread and growth**



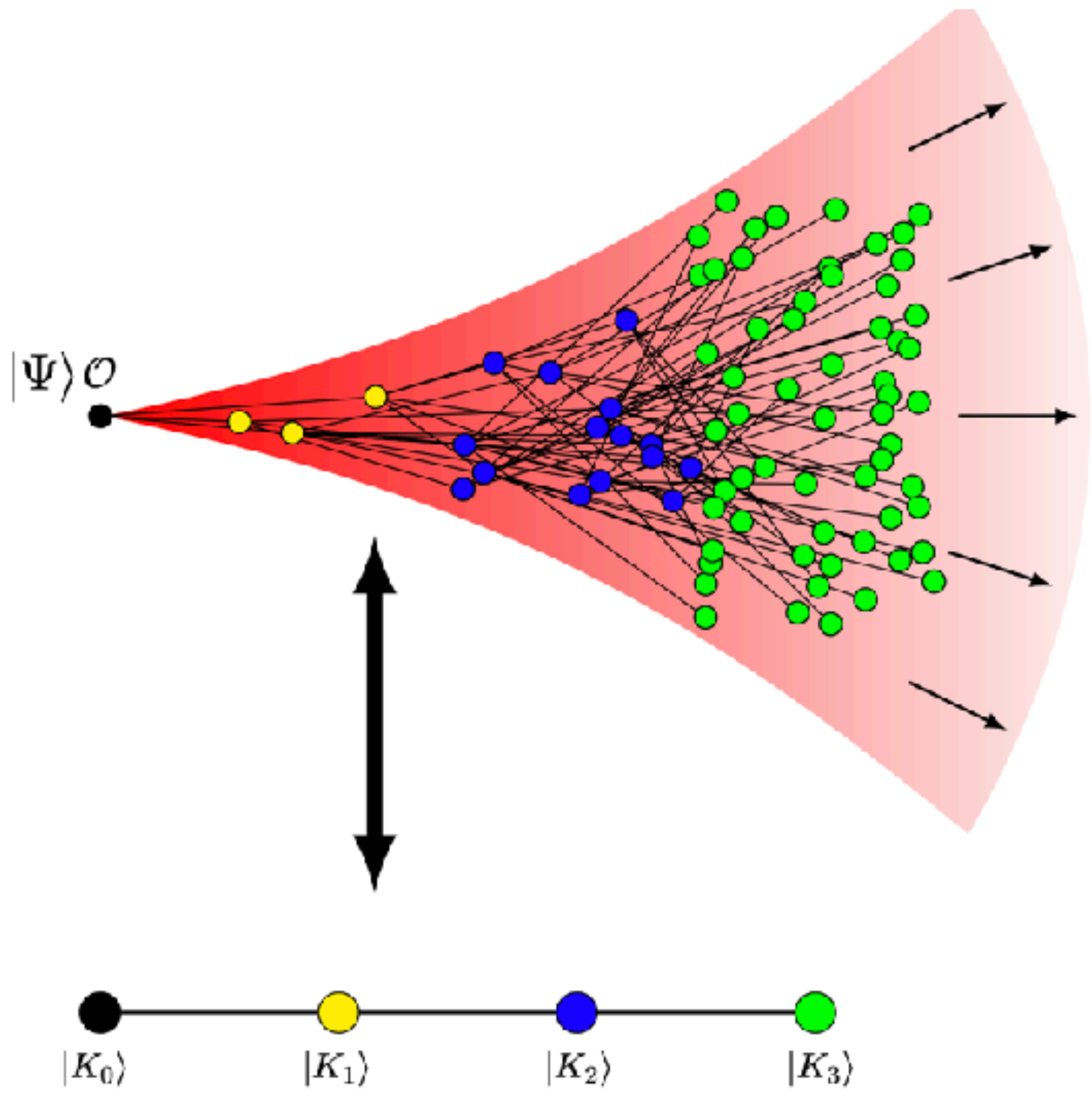
[Roberts, Stanford, Susskind '16][Qi,Streicher '18]
[Parker, Cao, Avdoshkin, Scaffidi, Altman '19]
[Balasubramanian,PC,Magan,Wu'22]

Basic Idea

Map the unitary evolution into a “1d chain” and quantify **Complexity** as a distance from the origin

Krylov/Spread complexity

$$\mathcal{C}(t) = \langle n \rangle = \sum_n n\, p_n(t)$$



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Basic Idea

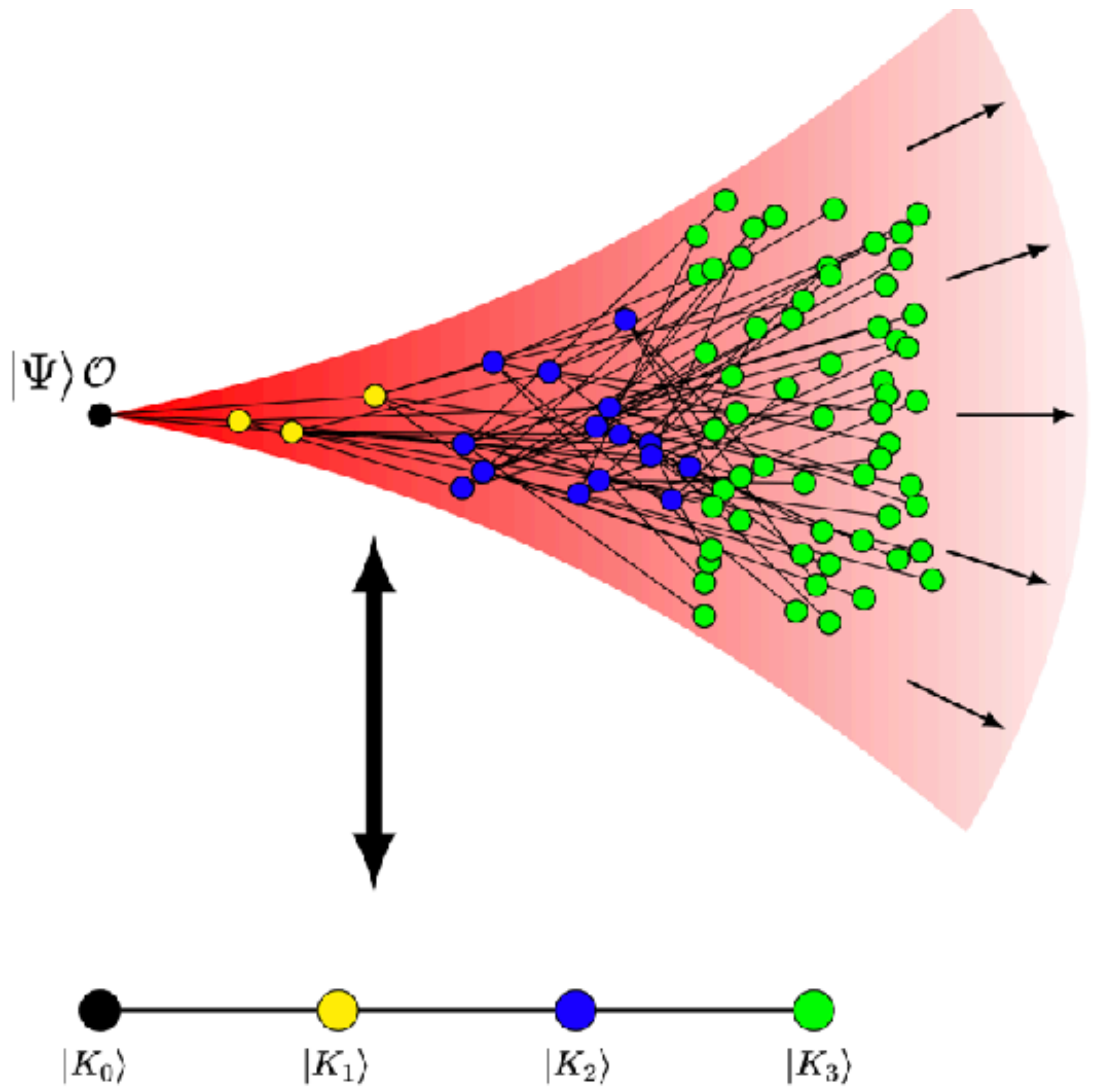
Map the unitary evolution into a "1d chain" and quantify **Complexity** as a distance from the origin

Krylov/Spread complexity

$$\mathcal{C}(t) = \langle n \rangle = \sum_n n\, p_n(t)$$

Krylov entropy (Shannon)

$$S(t) = -\sum_n p_n(t) \ln p_n(t)$$



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Aleksey Nikolaevich Krylov (1863-1945)

Russian naval engineer and applied mathematician
Became famous for pioneering “Theory of oscillating motions of the ship”

Alekandr Lyapunov was his cousin

In 1931 he wrote a paper on Krylov subspace: A NxN matrix and b N-vec

$$\mathcal{K}_r(A, b) = \operatorname{span}\{b, Ab, A^2 b, \ldots, A^{r-1} b\}$$

Goal: efficient diagonalization of matrices and computation of characteristic polynomial coefficients.

“… he was concerned with efficient computations and
counted **computational work/complexity** as the number of separate numerical multiplications”.

[Recursion Method: Viswanath,Muller '63]

Krylov Basis

Unitary evolution/Q-circuit

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}|\Psi_n\rangle$$

Goal: Given states (Krylov subspace)

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

construct an orthonormal basis $|K_n\rangle$ recursively (Lanczos algorithm, Gram-Schmidt):

$$|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle, \qquad |K_n\rangle = b_n^{-1}|A_n\rangle$$

with "**Lanczos coefficients**"

$$a_n = \langle K_n|H|K_n\rangle, \qquad b_n = \langle A_n|A_n\rangle^{1/2}$$

Such that $b_0 = 0$ and $|K_0\rangle = |\Psi_0\rangle$

Krylov Basis

[Recursion Method: Viswanath,Muller '63]

In the Krylov basis, the Hamiltonian becomes tri-diagonal

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

$$\langle K_m|\, H\, |K_n\rangle = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

"Hessenberg form"

When expanding our state in the Krylov basis

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle = \sum_n \phi_n(t)\,|K_n\rangle$$

$$\sum_n |\phi_n(t)|^2 \equiv \sum_n p_n = 1$$

by construction, we have a Schrödinger equation for the coefficients (amplitudes)

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

$$\phi_n(0) = \delta_{n,0}$$

QI: Complexity = “Spread in Hilbert space”

[Balasubramanian, PC, Magan, Wu ’22]

Starting from the state: $|\psi(t)\rangle \; = \; e^{-iHt}|\psi(0)\rangle$

Take a basis: $\mathcal{B} = \{|B_n\rangle : n = 0, 1, 2, \cdots\}$ and a “cost function” (a family, $c_n = n$)

$$C_{\mathcal{B}}(t) = \sum_n c_n |\langle\psi(t)|B_n\rangle|^2 \equiv \sum_n c_n \, p_{\mathcal{B}}(n,t)$$

Define Complexity as the minimum over basis choices

$$C(t) = \min_{\mathcal{B}} C_{\mathcal{B}}(t)$$

minimum (finite t) for the Krylov basis!

Summary

States

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle = \sum_n \phi_n(t)\,|K_n\rangle$$

$$i\partial_t \phi_n(t) = a_n \phi_n(t) + b_n \phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

$$S(t) \equiv \langle \Psi(t)|\Psi(0)\rangle = \langle \Psi_0|e^{iHt}|\Psi_0\rangle = \phi_0^*(t)$$

Operators

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) \equiv \sum_n i^n \varphi_n(t)|\mathcal{O}_n) \qquad \mathcal{L} = [H, \;]$$

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

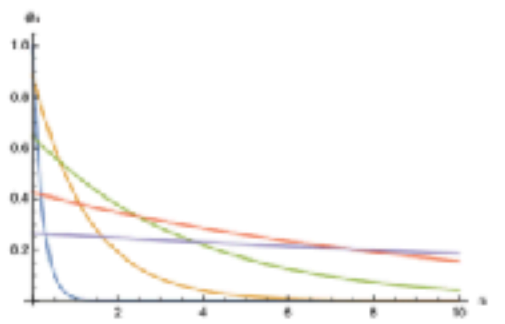
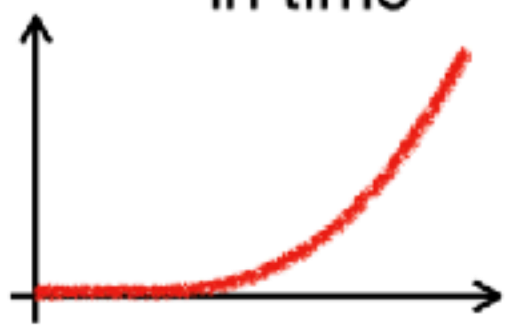
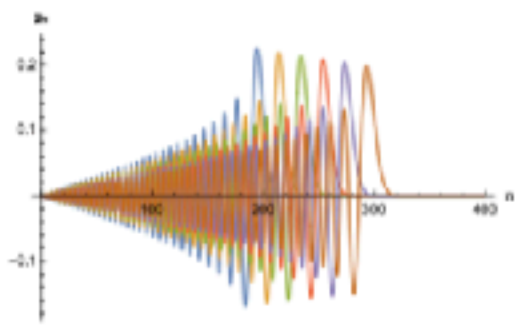
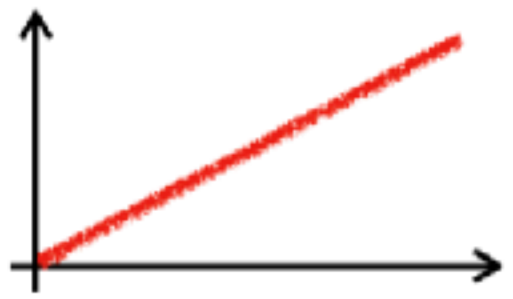
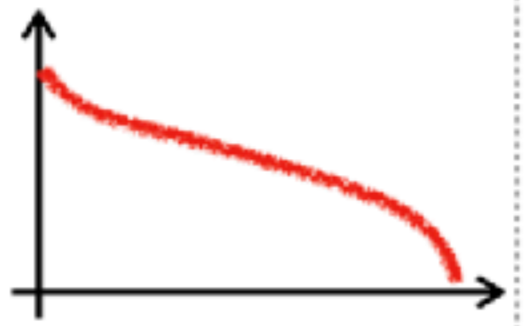
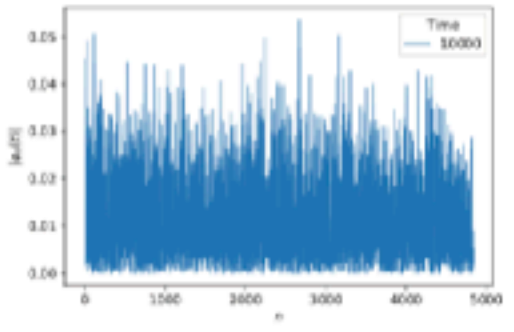
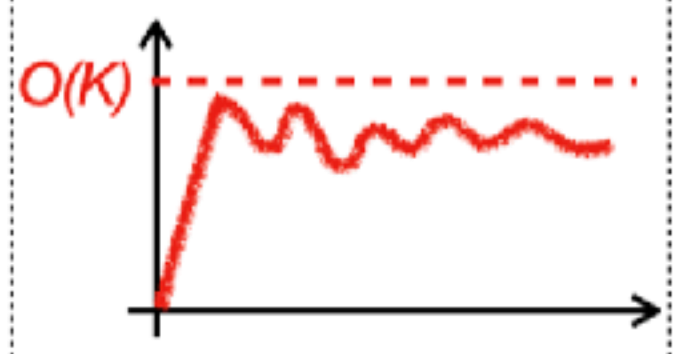
$$S(t) = (\mathcal{O}_0|\mathcal{O}(t)) = \big(\mathcal{O}_0|e^{i\mathcal{L}t}|\mathcal{O}_0\big) = \varphi_0(t)$$

$$= \frac{1}{Z}\sum_{n,m} |\langle n|\mathcal{O}|m\rangle|^2 e^{-\left(\frac{\beta}{2}-it\right)E_n} e^{-\left(\frac{\beta}{2}+it\right)E_m}$$

Applications

Extensive studies of the operator growth

[Parker, Cao, Avdoshkin, Scaffidi, Altman ’19]
[Barbon, Rabinovici, Shir, Sinha ’19]
[Rabinovici, Sanchez-Garrido, Shir, Sonner ’21’22]

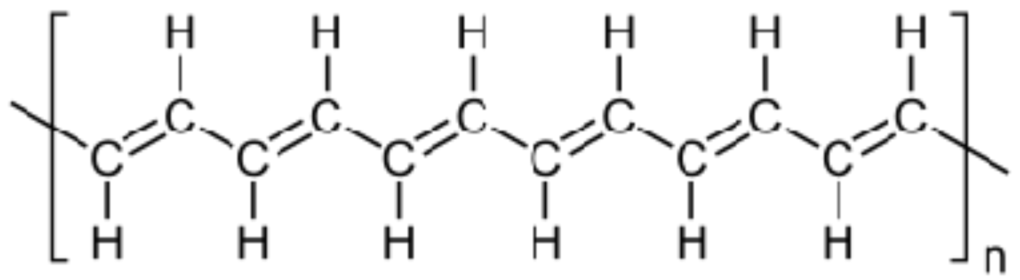
n	**Lanczos coefficients**	**wavefunction**	**K-complexity**	**time scales**
$1 \ll n < S$	Linear growth in *n* $b_n \sim \alpha n$		Exponential growth in time 	$0 \lesssim t \lesssim \log S$
$n \gg S$	Plateau, constant in *n* $b_n \sim \Lambda S$		Linear growth in time 	$t \gtrsim \log S$
$n \sim e^{2S}$	Descent 		Saturation 	$t \sim e^{2S}$

S=#dof

[PC, Liu '22]

Detecting topological phases?

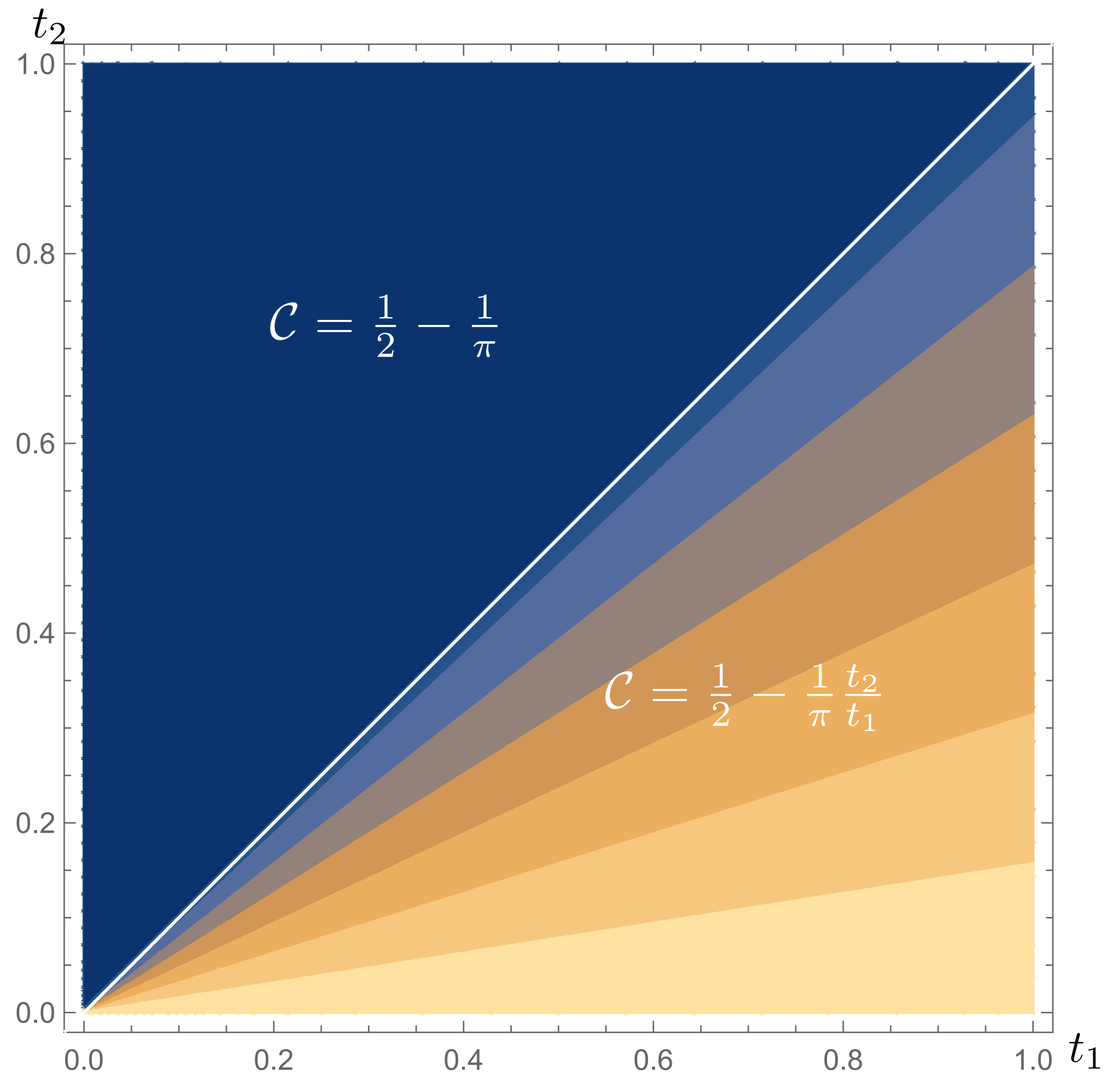
Su–Schrieffer-Hegger model (polyacetylene)



$$H = t_1 \sum_i \left(c^\dagger_{Ai} c_{Bi} + \mathrm{h.c.}\right) - t_2 \sum_i \left(c^\dagger_{Bi} c_{A,i+1} + \mathrm{h.c.}\right)$$

Depending on t's the ground state of the model:

$$|\Omega\rangle = \prod_{k>0} \mathcal{N}_k e^{-i\tan\left(\frac{\phi_k}{2}\right)(J_+^{(k)} + J_+^{(-k)})} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_k$$



non-topological phase (t1>t2) or topological insulator (t1<t2).

Kitaev chain: [PC, Gupta, Haque, Liu, Murugan '22]

Complexity of the TFD evolution

[Balasubramanian, PC, Magan, Wu ’22]

Consider the TFD state as initial state

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n} |n,n\rangle \qquad Z(\beta) = \sum_n e^{-\beta E_n}$$

And evolution

$$|\psi_\beta(t)\rangle = e^{-iHt}|\psi_\beta\rangle$$

Lanczos coefficients encoded in the spectral form factor

$$S(t) = \langle \Psi_\beta(t)|\Psi_\beta\rangle = \frac{Z(\beta - it)}{Z(\beta)}$$

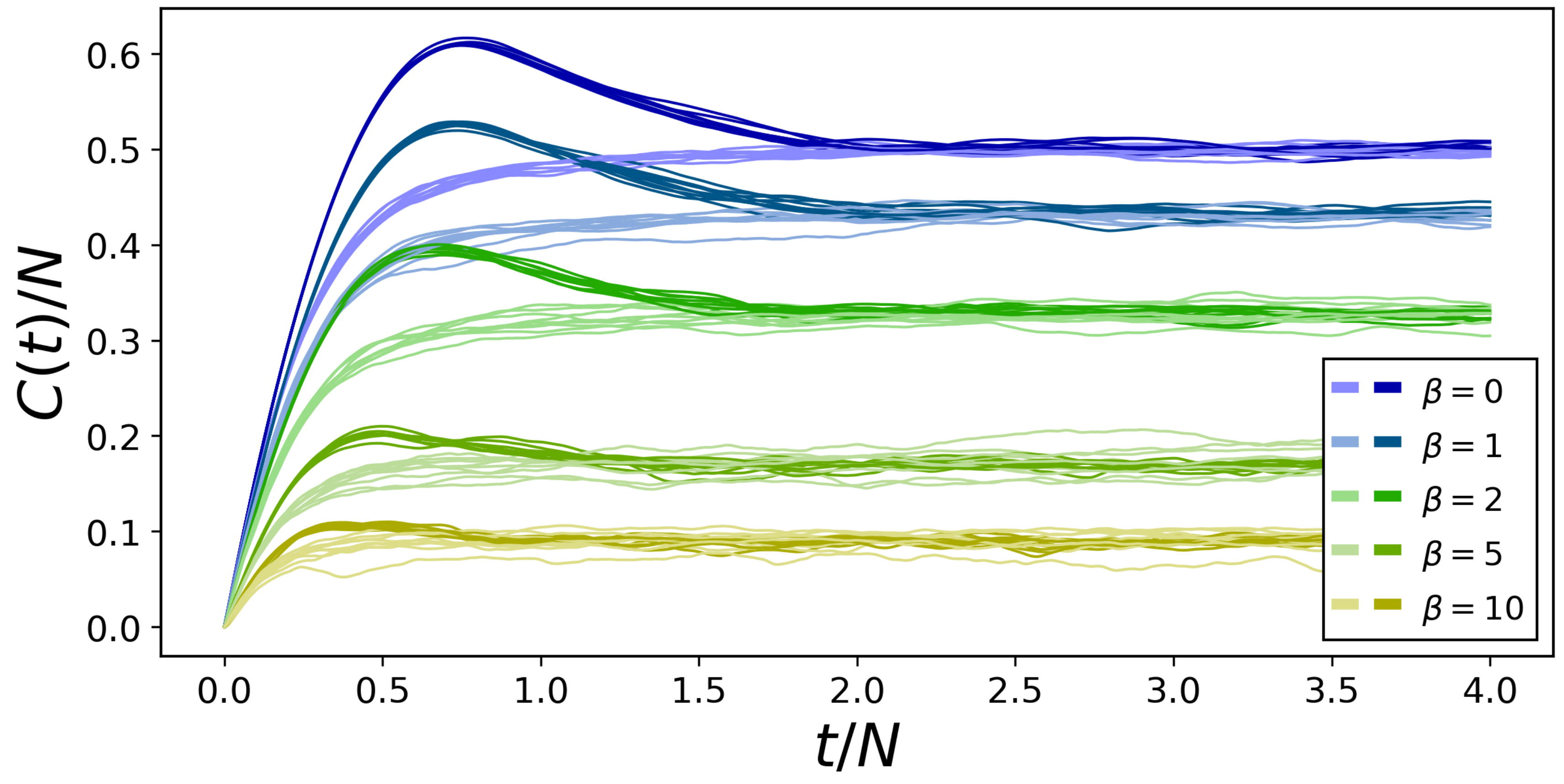
Direct sensitivity to the spectrum!

Evolution of the TFD for RMT

[Balasubramanian, PC, Magan, Wu ’22]

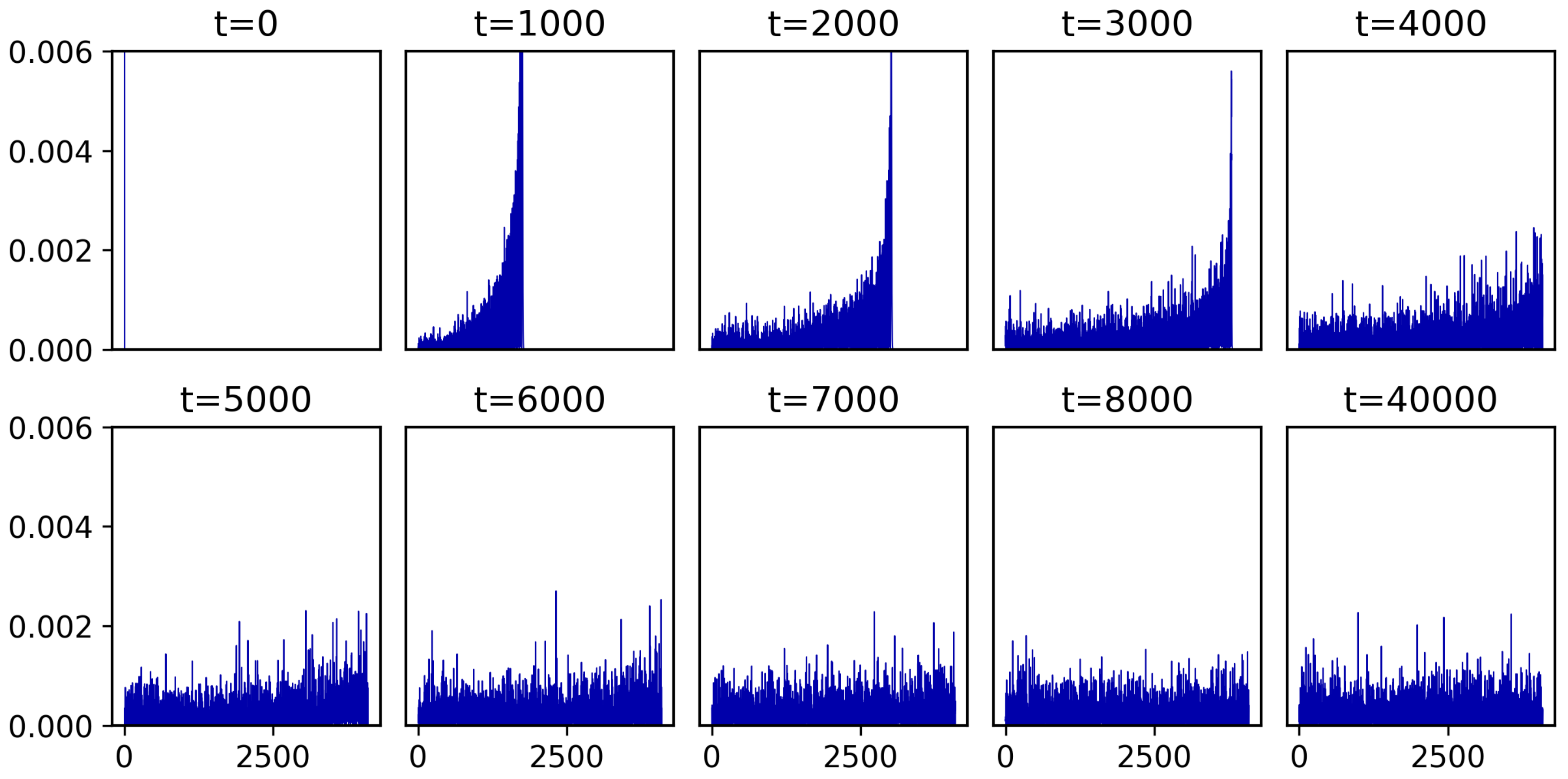
Spread Complexity for TFD evolved with GUE Hamiltonian

Ramp, Peak, Slope, Plateau



$$N = \{1024, 1280, 1536, 1792, 2048, 2560, 3072, 3584, 4096\}$$

p(t)

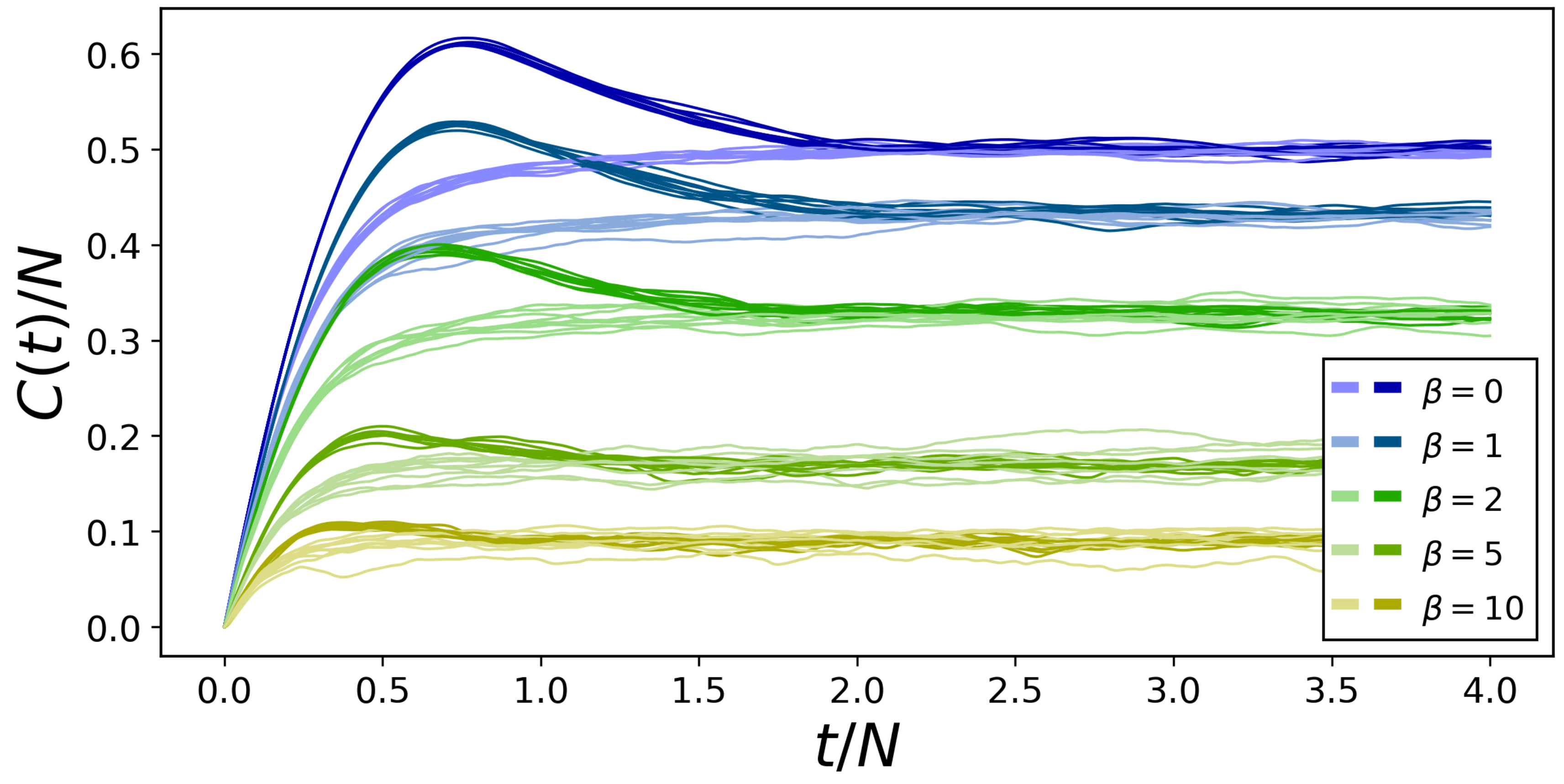


Quantum Chaos and Spread Complexity

[Balasubramanian, PC, Magan, Wu ’22]

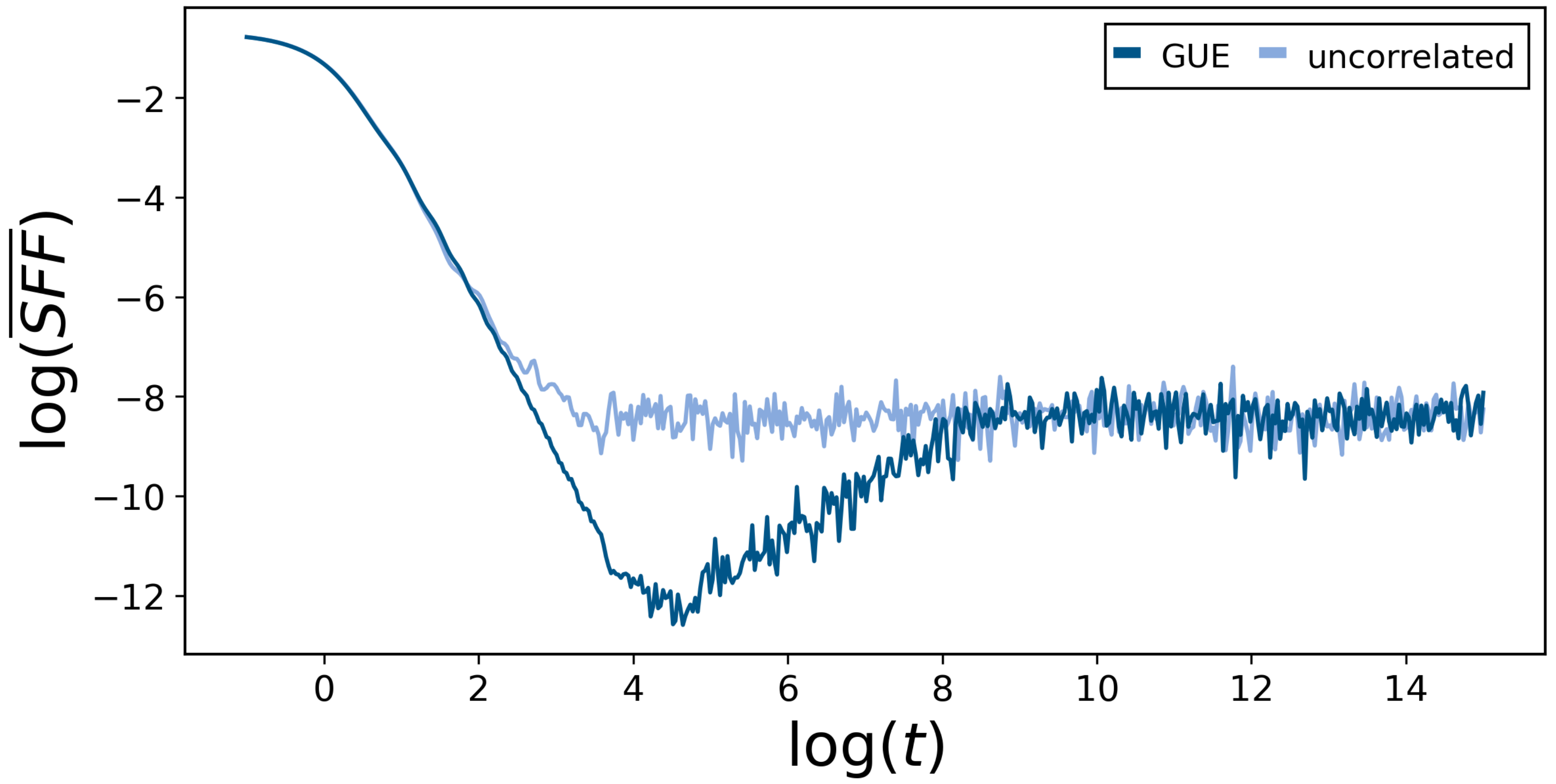
Spread Complexity for TFD evolved with GUE Hamiltonian

Ramp, Peak, Slope, Plateau



$N = \{1024, 1280, 1536, 1792, 2048, 2560, 3072, 3584, 4096\}$

Spectral Form Factor



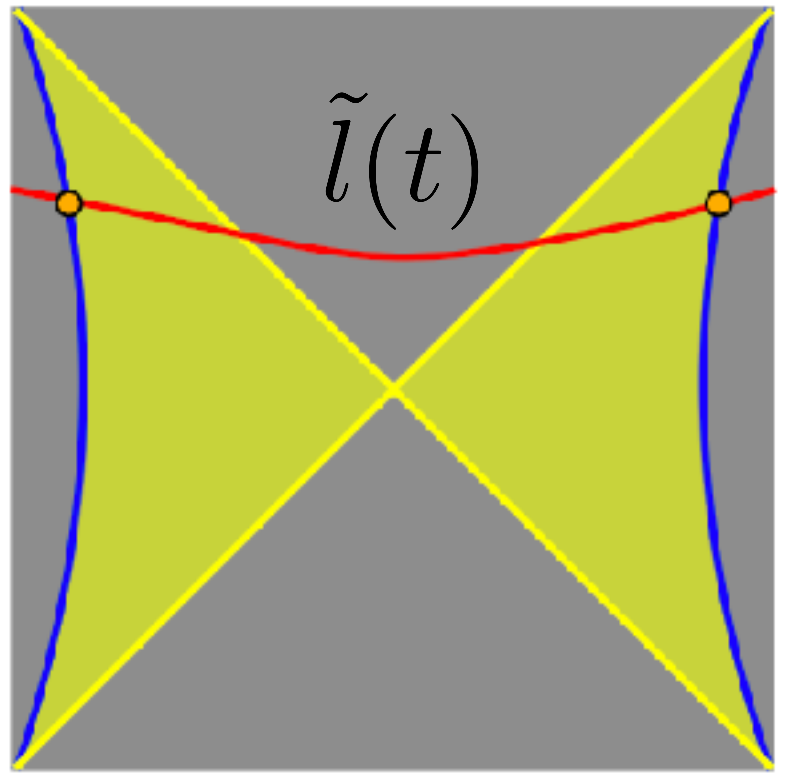
$N = 4096$ and $\beta = 1$, averaged over 10 samples of the GUE

Spread Complexity and Geodesic Length in JT gravity

Isomorphism of Hilbert spaces in SYK and JT gravity: [Berkooz,Narayan,Simon’18,…][Lin ’23]

Chord Basis = Krylov Basis

Spread Complexity in SYK matches the JT length [Rabinovici et al. ’23]



$$|\phi(t)\rangle = e^{-itT}|0\rangle \qquad T = \gamma\left(a + a^{\dagger}\right)$$

$$\lambda\widetilde{C_K}(t) = \frac{\tilde{l}(t)}{l_{AdS}}$$

Conclusions

- New "physical" complexity measures for operators/states in QFTs
- New tools for Quantum Many-Body, Quantum Chaos/Integrability, Quantum Gravity
- Reproduce the growth of Black Hole interiors in toy models (SYK, JT)
- Universal laws for Spread/Krylov complexity?
- Relation with QI or QC approaches? Circuit, Kolmogorov, Nielsen…?
- Quantum Black Hole Interiors? Infalling Observers? Singularities?

Thank You!