# Exploring Phase Space Deformation through a Correspondence between Modified Gravity and GUP, and Earthquakes as Testing Grounds

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# Advertisement

I am spearheading an application, along with a number of colleagues, for COST Action 2024, with the goal of bringing together researchers in the areas of (quantum) gravity, particle physics, seismology, and solid-state physics.

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# Motivation and plan of the talk

Motivation:

- To understand effects of gravity on thermodynamic systems
- To constrain theories of modified and quantum gravity

Plan of the talk:

- Description of thermodynamic systems in the presence of gravity
- Seismic data as a probe of gravitational models
- Constraining QG and MG with seismic data
- Improving the method by
  - incorporating rotation and the newest data (better description of the Earth's interior)
  - taking into account gravity in the thermodynamic description and
  - more accurate depiction of inter-atomic interactions and the nature of excited states

# Gravity vs matter: motivation based on a number of indications

- Effective quantities: opacity[1], ...
- Modifications introduced by modified gravity to pressure[2]
- Chemical reactions rates depend on gravity[3]
- Specific heat and crystallization depend on modified gravity[4]
- Chemical potential depends on gravity[5]
- Elementary particle interactions modified by modified gravity (dependence of the metric on the local energy-momentum distributions[6]
- EoS depends on relativistic effects introduced by GR[7]
- Thermonuclear processes...?[8]
- Fermi and Bose equations of state depend on (modified/quantum) gravity[9]

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[1] J. Sakstein, PRD 92 (2015) 124045; ...
[2] H-Ch. Kim, PRD 89 (2014) 064001
[3] P. Lecca, J. Phys.: Conf. Ser. 2090 (2021) 012034
[4] S. Kalita, L. Sarmah, AW, PRD 107 (2023) 4, 044072
[5] I.K. Kulikov, P.I. Pronin, Int. J. Theor. Phys. 34, (1995) 9
[6] A.D.I Latorre, G.J. Olmo, M. Ronco, PRB 780, 294 (2018)
[7] G.M. Hossain, S. Mandal, JCAP 02 (2021) 026; PRD 104 (2021) 123005
[8] J. Sakstein, PRD 92 (2015) 124045; AW, PRD 103 (2021) 4, 044037; M. Guerrero, AW, in preparation
[9] AW, PRD 107 (2023) 4, 044025; A. Pachol, AW, Class.Quant.Grav. 40 (2023) 19, 195021; AW, PRD 109 (2024) 2, 024011; AW arXiv:2401.01159

# Quantum gravity[10]

Observation 1:

Modifies Heisenberg uncertainty principle (GUP)

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}\Big(1 + \text{modification}\Big)$$

or/and dispersion relation

$$E^2 + p^2\Big(1 + \text{modification}\Big) = m^2$$

[10] LQG, Doubly Special Relativity, String Theory, Noncommunative geometry,...

# Quantum gravity and thermodynamics

Observation 2:

The weighted phase space volume is modified ($D$ - dim of the phase space).

$$\frac{d^D\mathbf{x}d^D\mathbf{p}}{1 + \text{modification}}$$

Consequence: modified partition function ($z = e^{\mu/k_B T}$)

$$\ln \mathcal{Z} = \frac{V}{(2\pi\hbar)^3}\frac{g}{\pm 1}\int \ln\left(1 \pm z e^{-E/k_B T}\right)\frac{d^3p}{1 + \text{modification}}$$

Conclusion: Quantum Gravity modifies equations of state since

$$P = k_B T \frac{\partial}{\partial V}\ln \mathcal{Z},$$

$$n = k_B T \frac{\partial}{\partial \mu}\ln \mathcal{Z}\ |_{T,V}\ ,$$

$$U = k_B T^2 \frac{\partial}{\partial T}\ln \mathcal{Z}\ |_{z,V}$$

Observation 3: MG as an effective theory derived from QG

# MG vs QG[11]

**Palatini gravity ($f(R)$, EiBI)**

$$\nabla^2\phi = \frac{\kappa}{2}\Big(\rho + \bar{\alpha}\nabla^2\rho\Big)$$

$\downarrow$ with $\frac{d\Phi}{dr} = -\rho^{-1}\frac{dP_p}{dr}$

$$P_{T\to 0} = K\rho^{\frac{5}{3}} + \sigma K_2\rho^2$$

$\downarrow$ with $f(E) = \left(1 + z^{-1}e^{E/k_BT}\right)^{-1}$

$$P = \frac{g_s}{(2\pi^2\hbar^3)^3}\int \frac{c^2 p}{E} f(E)\left(\frac{p^3}{3} + \sigma\frac{p^4}{4}\right)4\pi dp$$

$\downarrow$

$$P = k_BT\frac{g_s}{a(2\pi^2\hbar^3)^3}\int \ln(1 + aze^{-\frac{E}{k_BT}})\frac{4\pi p^2 dp}{(1-\sigma p)^b}$$

The partition function in the grand-canonical ensemble:

$$\ln Z = \frac{V}{(2\pi\hbar)^3}\frac{g}{a}\int \ln\left[1 + aze^{-E/k_BT}\right]\frac{d^3p}{(1-\sigma p)^b}$$

$\rightarrow$ linear GUP with $b = 1$

**Snyder model (NCG, qGUP)**

$$\nabla^2\phi = 4\pi G\rho - \tilde{\epsilon}\nabla^2\rho^{\frac{4}{3}}$$

$\uparrow$ with $\frac{d\Phi}{dr} = -\rho^{-1}\frac{dP_p}{dr}$

$$P_{T\to 0} = K_1\rho^{\frac{5}{3}}\left[1 - \varepsilon\rho^{\frac{2}{3}}\right]$$

$\uparrow$ with $f(E) = 1$ if $E \leq E_F$; 0 otherwise

$$P = \frac{1}{\pi^2\hbar^3}\int \frac{1}{3}p^3\,{}_2F_1\left(\frac{3}{2}, 1, \frac{5}{2}, -p^2\Omega\right)f(E)\frac{c^2 p}{E}dp,$$

$\uparrow$

The partition function in the grand-canonical ensemble:

$$\ln Z = \frac{V}{(2\pi\hbar)^3}\frac{g}{a}\int \ln\left[1 + aze^{-E/k_BT}\right]\frac{4\pi p^2 dp}{(1+\Omega p^2)}$$

$\rightarrow$ quadratic GUP ($\Omega = \beta\left(4\chi - \frac{3}{2}\right)$)

$$[p_i, \hat{x}_j] = -i\hbar\delta_{ij}\left(1 + \beta\left(\chi - \frac{1}{2}\right)p_k p_k\right) - 2i\hbar\chi\beta p_i p_j + O(\beta^2)$$

[11] A.Pachoł, AW, EPJC 83 (2023) 12, 1097; AW, PRD 109 (2024) 2, 024011; AF Ali, AW, arXiv:2401.05941

# Terrestrial (exo-)planets in modified gravity[12]

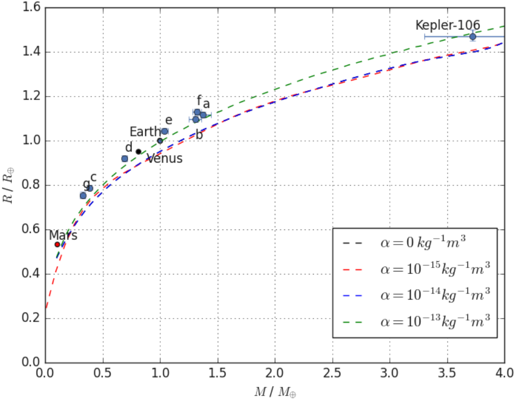



Figure: Mass-radius relation for small planets composed of two layers: iron core and perovskite silicate mantle. The results were obtained for different values of the Starobinsky parameter $\alpha = 2c^2\kappa^2\beta$ in Palatini gravity. The solar-system planets were included, as well as some TRAPPIST-1 exoplanets, denoted by letters.

[12] A. Kozak, AW, IJGMMP 19 (2022) Supp01, 2250157; E. Agol et al., Planet. Sci. 2(2021) 1.

# Non-relativistic equations of modified and quantum gravity

Modified Poisson equation

$$\nabla^2\Phi \approx \frac{1}{2}(\rho + \text{modification})$$

For spherical-symmetric spacetime the gravitational potential the hydrostatic equilibrium equation

$$\frac{d\Phi}{dr} = -\rho^{-1}\frac{dP}{dr},$$
$$M = \int 4\pi' \tilde{r}^2 \rho(\tilde{r}) d\tilde{r},$$

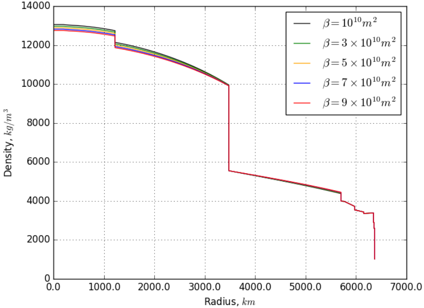
+ matter description (EoS or **seismic data**, temperature dependence,...)
+ eventual equations for additional fields

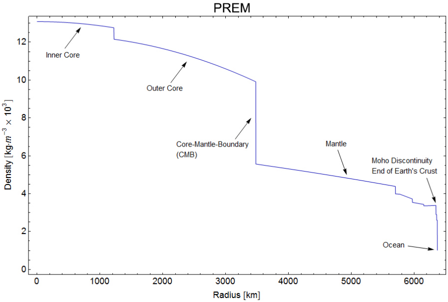
A new method of testing theories of gravity proposed[13]

[13] A. Kozak, AW, Phys.Rev.D 104 (2021) 8, 084097

# Terrestrial planets - seismology vs gravity I [15]

The Earth's density profile (inner and outer core, mantle + outer layers given by the Birch law)




On the RHS: Palatini gravity ($\Delta\rho = 600$, $\rho_m = 5550$); on the left: Preliminary Reference Earth Model (PREM) A. M. Dziewonski, D. L. Anderson, Phys. Earth Plan. Int. 25 (1981) 297.

Exoplanets properties: central values and CMB are affected by modified gravity[14]

[14] A. Kozak, AW, Universe 8 (2021) 1, 3

[15] A. Kozak, AW, PRD 104 (2021) 8, 084097; IJGMMP 19 (2022) Supp01, 2250157; Phys. Rev. D 108 (2023) 4, 044055

# Terrestrial planets - seismology vs gravity II [16]

- No exchange of heat between different layers (adiabatic compression)
- The planet is a spherical-symmetric ball in hydrostatic equilibrium
- The planet consists of radially symmetric shells with the given density jump between the inner and outer core $\Delta\rho = 600$, central density $\rho_c = 13050$ and density at the mantle's base $\rho_m = 5563$ (in kg/m$^3$) - PREM
- Mass $M = 4\pi \int_0^R r^2 \rho(r) dr$ and moment of inertia $I = \frac{8}{3}\pi \int_0^R r^4 \rho(r) dr$ where $R$ is Earth's radius, play a role of the constraints (given by observations with a high accuracy)
- The outer layers' density profile described by Birch law $\rho = a + bv_p$

$v_p$ is the longitudinal elastic wave. It contributes, together with the transverse elastic wave $v_s$, to the seismic parameter $\Phi_s$ and the elastic properties of an isotropic material

$$\Phi_s = v_p^2 - \frac{4}{3}v_s^2 = \frac{K}{\rho}, \quad K = \frac{dP}{d\ln\rho}$$

The hydrostatic equilibrium equation in MG:

$$\frac{d\rho}{dr} = -\rho\left(\frac{GM(r)}{r^2} + \text{modification}\right)\Phi_s^{-1},$$

[16] A. Kozak, AW, Phys.Rev.D 108 (2023) 4, 044055

# Theories of gravity constrained so far

Modified Poisson equation

$$\nabla^2\phi(\mathbf{x}) = 4\pi G\Big(\rho(\mathbf{x}) + \nabla^2\alpha\big(\mathbf{x}, \rho(\mathbf{x})\big)\Big),$$

- Palatini $f(R)$ and Eddington-inspired Born-Infeld gravity (Ricci-based)[17]: $\alpha(r,\rho) = \epsilon/2\rho(r)$, and $\epsilon = 4\beta$

$$-2\times 10^9 \lesssim \beta \lesssim 10^9 \mathrm{m}^2 \text{ for Palatini }, -8\times 10^9 \lesssim \epsilon \lesssim 4\times 10^9 \mathrm{m}^2 \text{ for EiBI}$$

- DHOST theories $\alpha(r,\rho) = \frac{\Upsilon}{4} r^2 \rho(r)$

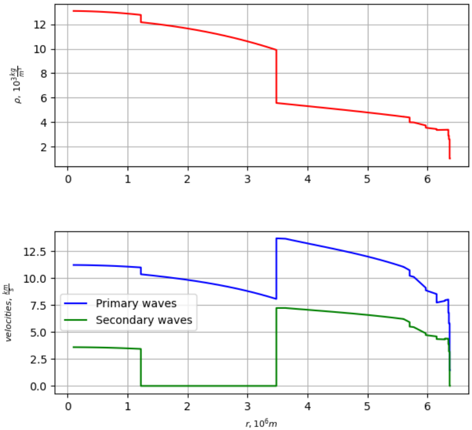
$$-10^{-3} \lesssim \Upsilon \lesssim 10^{-3}$$

- Quantum gravity: Snyder and qGUP ($\beta_0 := \beta M_P^2 c^2$): $\beta_0 < 4.67\times 10^{44}$
- Quantum gravity: linear GUP: $-6\times 10^{22} \lesssim \sigma \lesssim 3\times 10^{22}$ s/kg m

[17] New cosmological data provides bounds $|\beta| < 10^{49}$ m$^2$, D.Aguiar+,AK,AW, JCAP 01 (2024) 011

# The density profile given by the PREM (Newtonian)

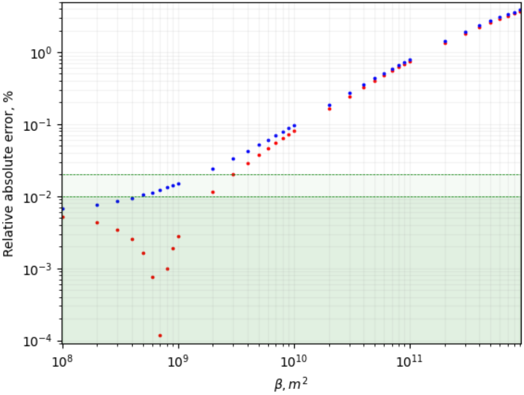
- The density profile given by the preliminary reference Earth model in which Newtonian gravity is assumed.
- The velocities' plots are obtained from data without using any theory of gravity.
- The primary waves are the same as the longitudinal waves, while the secondary waves are transverse in nature.
- The units are in km/s for velocities, while the densities are in kg/$m^3$.



# Terrestrial planets - seismology vs gravity III [18]

Constraining theory (moment of inertia $I = 8.01736 \pm 0.00097 \times 10^{37}$kg m$^2$ and mass $M = 5.9722 \pm 0.0006 \times 10^{24}$kg)

- Relative absolute error for the mass and the moment of inertia of Earth. Red dots represent errors for the moment of inertia, while blue ones correspond to the mass.
- The dark green stripe represents a $1\sigma$ region for both quantities, while the light green denotes a $2\sigma$ region.
- The green region denotes the uncertainties for both mass and moment of inertia because, for either of them, the ratio of $\sigma$ to the mean value is similar ($\approx 0.01\%$).
- The values of $(\rho_m, \rho_c, \Delta\rho)$ chosen for numerical calculations are $(5563, 13050, 600)$kg/m$^3$, respectively.




[18] A. Kozak, AW, Phys.Rev.D 108 (2023) 4, 044055

# The uncertainties for the models' parameters I

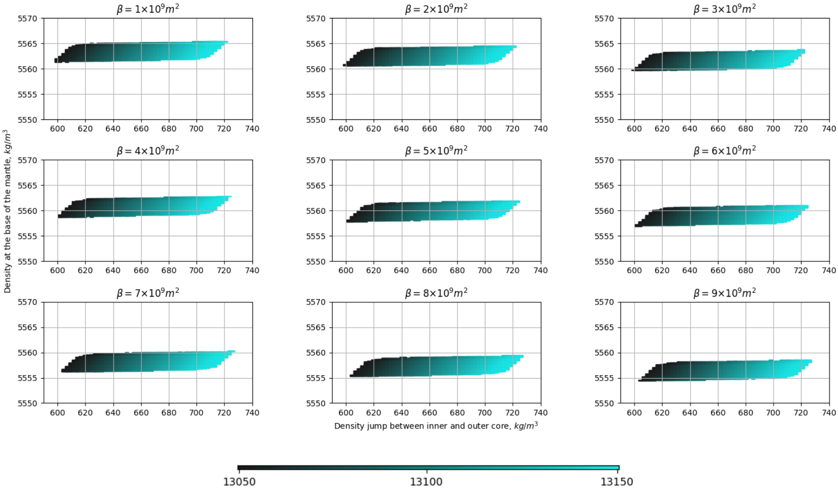



Figure: $1\sigma$ confidence regions of the theory parameters $(\rho_c, \rho_m, \Delta\rho)$ for different values of the $\beta$ parameter, being of order $10^9$ m$^2$. The darker color corresponds to lower values of the central density, while the brighter one - to higher. The range of the central density is shown in the color bar below the figures. The units are kg/m$^3$.

# The uncertainties for the models' parameters II

- There always exists a region for a given value of the theory parameter for which all three density parameters result in a good agreement with experimental measurements
- $\Delta\rho$ and $\rho_c$ admit much wider ranges of their values, not taking out of the $1\sigma$ region.
- $\rho_m$ can differ by no more than $2-3$ kg m$^{-3}$ from the value assumed in our calculations in order to remain within the $1\sigma$ region
- To incorporate bigger uncertainty of $\rho_m$, increase in the range of $\rho_m$ and $\Delta\rho$, and/or the range of $\beta$ would be necessary
- Large uncertainty in the determination of $\rho_m$ is related to a bigger range of $\beta$ parameter's allowed values
- Example: for $\beta = 10^9$m$^2$, deviations from the PREM $\rho_m$ $(\beta = 0)$ leading to the same values of $M$ and $I$, is 0.02% while, in the worst case, for the uncertainty of the PREM model 50 kg m$^{-3}$, is 0.9% ($\Delta\rho$ and $\rho_c$ unchanged). It increases the bound to $10^{11}$m$^2$.

# Astrophysical bounds on Generalized Uncertainty Principle[19]

Our bound when more realistic physics taken into account

$\beta_0 \leq 1.36 \times 10^{48}$ from low-mass stars (A. Pachol, AW, Eur.Phys.J.C 83 (2023) 12, 1097)

$\beta_0 < 4.67 \times 10^{44}$ from Earthquakes (A. Kozak, A. Pachol, AW, arXiv:2310.00913)

| experiment | ref. | upper bound on $\beta$ |
|---|---|---|
| equivalence principle (pendula) | [240] | ~~$10^{20}$~~ $10^{73}$ |
| gravitational bar detectors | [387, 388] | ~~$10^{33}$~~ $10^{93}$ |
| equivalence principle (atoms) | [389] | $10^{45}$ |
| perihelion precession (solar system) | [123, 155] | $10^{69}$ |
| perihelion precession (pulsars) | [123] | $10^{71}$ |
| gravitational redshift | [155] | $10^{76}$ |
| black hole quasi normal modes | [251] | $10^{77}$ |
| light deflection | [123, 155] | $10^{78}$ |
| time delay of light | [155] | $10^{81}$ |
| black hole shadow | [247] | $10^{90}$ |
| black hole shadow | [251, 259] | $10^{90}$ |

19 See review by Bosso+ 2023, arXiv:2305.16193

# Improving the method and future constraints

- Spherical-symmetric 1-dim Earth with adiabatic compression:
  - to introduce the complexities of Earth's true geometry (it rotates)
  - to estimate the equatorial moment of inertia relative to the polar moment by applying travel time ellipticity corrections to PREM[20]
  - to recognize the imperfections of layers and accounting for variable density jumps
  - to take into account a temperature variation with depth.
- Core description:
  - PREM does not describe well the boundaries of the outer and inner core
  - to use a more precise model like AK135-F[21] - it incorporates the complexities of core waves
  - to use equations of state for modeling core density and bulk moduli[22] (improving the uncertainties in density jumps at the inner and outer core boundaries).
- Birch law - a probable reevaluation when dealing with seismic data from Mars (the coefficients obtained experimentally).

[20] B. L. N. Kennett, O. Gudmundsson, Geophysical Journal International 127.1 (1996): 40-48.

[21] B. L. N. Kennett, E. R. Engdahl, R. Buland, Geophysical Journal International 122.1 (1995): 108-124.

[22] J. C. E. Irving, S. Cottaar, V Lekic, Science advances 4.6 (2018): eaar2538.

# Improving the thermodynamic description and future plans

- To consider gravity effects in the elasti moduli and lattice description of the Earth's materials - corrections to the thermal energy (in progress)
- To take into account gravity effects in equations of state, melting and transport properties (in progress)
- To consider modified dispersion relation in the above calculations

I am spearheading an application, along with a number of colleagues, for COST Action 2024, with the goal of bringing together researchers in the areas of (quantum) gravity, seismology, and solid-state physics.

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# Summary and conclusions

- Tests of gravity with the use of stars and substellar objects (BD, (exo)-planets, seismology)
- We must be consistent in describing physical systems in different scales
- We should consider more realistic models on both sides: gravity and matter - rotating bodies, magnetic fields, ..., opacities (atmosphere), mirophysics description - to obtained better bounds and understand the gravity effects
- More research on matter properties in the MG and QG frameworks is necessary

# Thanks!

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# Palatini gravity in a nutshell

$$S = S_{\rm g} + S_{\rm m} = \frac{1}{2\kappa}\int \sqrt{-g} f(\hat{R}) d^4x + S_{\rm m}(g_{\mu\nu}, \psi_m),$$

where $\hat{R} = \hat{R}^{\mu\nu}(\hat{\Gamma})g_{\mu\nu}$. Modified field equations wrt $g_{\mu\nu}$ and $\hat{\Gamma}$ are

$$f'(\hat{R})\hat{R}_{\mu\nu} - \frac{1}{2}f(\hat{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \text{GR:} \quad R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\hat{\nabla}_\beta(\sqrt{-g}f'(\hat{R})g^{\mu\nu}) = 0 \quad \rightarrow \quad h_{\mu\nu} = f'(\hat{R})g_{\mu\nu}.$$

The trace of the first MFE wrt $g_{\mu\nu}$ gives the structural equation

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = \kappa\mathcal{T},$$

where $\mathcal{T}$ is a trace of e-m tensor $T_{\mu\nu}$ wrt $g_{\mu\nu}$, provides $\hat{R} = \hat{R}(\mathcal{T})$.

- Non-linear system of a second order PDE.
- $f(\hat{R}) = \hat{R} - 2\Lambda$ is fully equivalent to the Einstein $R - 2\Lambda$.
- Any $f(\hat{R})$ vacuum solution $\rightarrow$ Einstein vacuum solution with the cosmological constant.
- Modifies non- and relativistic stellar structure equations[23].

[23] K. Kainulainen et al, PRD. 76 (2007) 043503; **AW**, EPJC 78 (2018) 421; **AW** EPJC 79 (2019) 51; A. Sergyeyev, **AW**, EPJC 80

# Snyder model (Snyder 1947, Chang et al. 2002, Battisti&Meljanac 2009)

One can consider many realizations of deformed phase spaces which correspond to Snyder non-commutative space time $[\hat{x}_\mu, \hat{x}_\nu] = i\hbar\beta M_{\mu\nu}$. $\beta$ is related with the minimal (Planck) length.
The deformation of the quantum-mechanical phase space (Heisenberg algebra), in the most general realization (parametrized by $\chi$) of Snyder model, up to the linear order in the non-commutativity parameter $\beta$, can be written as:

$$[p_i, \hat{x}_k] = -i\hbar\delta_{ik}\left(1+\beta\left(\chi-\frac{1}{2}\right)p_j p_j\right) - 2i\hbar\chi\beta p_i p_k + O(\beta^2).$$

The original Snyder case is recovered for $\chi = 1/2$. The most popular GUP is given by $\chi = 1/2$ and $\chi = 0$.
The Heisenberg algebra generators $\hat{x}_i$ and $p_j$ can be represented on momentum space wave functions $\phi(p)$ as:

$$\hat{x}_i\phi(p) = i\hbar\left(\left(1+\beta\left(\chi-\frac{1}{2}\right)p_k p_k\right)\frac{\partial}{\partial p_i} + 2\chi\beta p_i p_j \frac{\partial}{\partial p_j} + \gamma p_i\right)\phi(p),$$

$$p_i\phi(p) = p_i\phi(p)$$

$\gamma$ is an arbitrary constant, which does not enter the commutation relations, but affects the definition of the scalar product in momentum space (physical choice for $\gamma = 0$).
To define symmetric operators the new inner product in momentum space must take the form:

$$\langle\psi,\phi\rangle = \int \frac{d^D p}{(1+\beta(3\chi-\frac{1}{2})p^2)^\alpha}\psi^*(p)\phi(p) =: \int \frac{d^D p}{(1+\omega p^2)^\alpha}\psi^*(p)\phi(p)$$

where $\alpha = \frac{\beta(2\xi+D\xi-\frac{1}{2})-\gamma}{\beta(3\chi-\frac{1}{2})}$ and $\omega = \beta(3\chi-\frac{1}{2})$. Note that for $\xi = 1/6$ there is no deformation in the measure.
One can introduce $\alpha\omega = \beta(5\xi-\frac{1}{2})-\gamma$ for D=3 which includes deformation parameter $\beta$ and choice of realization $\chi$.

# Fermi-Dirac equation of state[24]

Let us consider a system of $N$ fermions with the energy states $E_i$. The partition function in the grand-canonical ensemble

$$\ln Z = \sum_i \ln\left[1 + z e^{-E_i/k_B T}\right]$$

where $T$ is the temperature, $k_B$ Boltzmann constant, $z = e^{\mu/k_B T}$ while $\mu$ is the chemical potential.
Considering a large volume and 3D

$$\sum_i \to \frac{1}{(2\pi\hbar)^3}\int \frac{d^3x d^3p}{(1+\omega p^2)^\alpha},$$

the partition function (note that $f(E) = \left(1 + z e^{-E/k_B T}\right)^{-1}$ is the F-D distribution function)

$$\ln Z = \frac{V}{(2\pi\hbar)^3} g \int \ln\left[1 + z e^{-E/k_B T}\right] \frac{d^3p}{(1+\omega p^2)^\alpha}\ ,$$

where $g$ is a spin of a particle, $V := \int d^3x$ is the volume of the cell (of the configuration space), while $E = (p^2c^2 + m^2c^4)^{1/2}$.
The pressure is given by

$$P = \frac{1}{\pi^2\hbar^3}\int \frac{1}{3}p^3\, {}_2F_1\left(\frac{3}{2},\alpha,\frac{5}{2},-p^2\omega\right) f(E)\frac{c^2 p}{E}dp \overset{|\omega p^2|<<1}{=} \frac{1}{\pi^2\hbar^3}\int \frac{p^3}{3}\left(\sum_{k=0}^{\infty}\frac{(\alpha)_k\left(\frac{3}{2}\right)_k(-\omega p^2)^k}{\left(\frac{5}{2}\right)_k k!}\right) f(E)\frac{c^2 p}{E}dp$$

[24] A. Pachol, AW, Class.Quant.Grav. 40 (2023) 19, 195021

# Non-relativistic ($E \approx \frac{p^2}{2m_e}$) degenerate Fermi gas[25]

$$P = \frac{1}{3\pi^2\hbar^3}\int\left(\frac{(2m_eE)^{\frac{3}{2}}}{5} - \frac{3\alpha\omega}{35}(2m_eE)^{\frac{5}{2}}\right)f(E)dE.$$

In the limit $T \to 0$, the chemical potential $\mu \approx E_F$

$$f(E) = \begin{cases} 1 & \text{if } E \leq E_F \\ 0 & \text{otherwise.} \end{cases}$$

$$P_{T\to 0} = \frac{2}{5}vE_F^{\frac{5}{2}}\left(1 - \frac{3\alpha\omega}{7}(2m_e)E_F\right),$$

where we have defined $v = \frac{(2m_e)^{\frac{2}{3}}}{3\pi^2\hbar^3}$.

[25] A. Pachol, AW, Class.Quant.Grav. 40 (2023) 19, 195021

# Non-relativistic ($E \approx \frac{p^2}{2m_e}$) degenerate Fermi gas[27]

Let us use the definition of the measure of electron degeneracy ($u = (3\pi^2 \hbar^3 N_A)^{\frac{2}{3}} / 2m_e$)

$$\psi = \frac{k_B T}{E_F} = \frac{2m_e k_B T}{\left(3\pi^2 \hbar^3\right)^{2/3}} \left[\frac{\mu_e}{\rho N_A}\right]^{2/3} \equiv u^{-1} k_B T \left[\frac{\mu_e}{\rho}\right]^{2/3}$$

to rewrite the pressure as a mixture of two polytropes (compare to MG case[26])

$$P_{T\to 0} = \frac{2}{5} v u^{\frac{5}{2}} \left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}} \left[1 - \frac{3u}{7} \alpha\omega (2m_e) \left(\frac{\rho}{\mu_e}\right)^{\frac{2}{3}}\right] =: K_1 \rho^{\Gamma_1} - \alpha\omega K_2 \rho^{\Gamma_2}$$

where $K_1 = \frac{2}{5} v u^{\frac{5}{2}} \mu_e^{-\frac{5}{3}}$, $\Gamma_1 = 5/3$ and $K_2 = \frac{12}{35} v u^{\frac{7}{2}} m_e \mu_e^{-\frac{7}{3}}$, $\Gamma_2 = 7/3$.

Interpretation from the bulk modulus (incompressibility)

$$B = \frac{dP}{d\ln\rho} = (1 - \frac{6}{7}\alpha\omega m_e E_F)\rho^{\frac{5}{3}},$$

For incompressible solids $B \to \infty$ ($\alpha\omega < 0$) while for infinitely compressible one $B = 0$ ($\alpha\omega > 0$).

26 Kim, H. C. (2014), PRD 89(6), 064001; AW, PRD 107 (2023) 4, 044025

27 A. Pachol, AW, Class.Quant.Grav. 40 (2023) 19, 195021

# Non-relativistic objects[29]

Non-relativistic Poisson, hydrostatic equilibrium, and mass equations

$$\nabla^2\phi = 4\pi G\rho, \qquad \frac{d\phi}{dr} = -\rho^{-1}\frac{dP}{dr}, \qquad M = \int 4\pi' \tilde{r}^2\rho(\tilde{r})d\tilde{r}.$$

Applying our Fermi equation of state ($\varepsilon = \frac{6}{7}\left(\frac{3\pi^2\hbar^3}{N_A\mu_e}\right)^{\frac{2}{3}}\alpha\omega$)

$$P = K\rho^{\frac{5}{3}}\left[1-\varepsilon\rho^{\frac{2}{3}}\right]$$

we can write down modified Lane-Emden equation

$$\frac{d}{d\xi}\left\{\xi^2\frac{d\theta}{d\xi}\left[1-\epsilon\theta\right]\right\} = -\xi^2\theta^{\frac{3}{2}} \ ,$$

Such an equation also results from MG[28] with a non-modified polytrope ($\tilde{\epsilon} = \frac{7}{4}K\epsilon$)

$$\nabla^2\phi = 4\pi G\rho - \tilde{\epsilon}\nabla^2\rho^{\frac{4}{3}}$$

[28] For review, see G. Olmo, D. Rubiera-Garcia, AW, Phys. Rept. 876 (2020) 1-75

[29] A. Pachol, AW, arXiv:2307.03520