Relativistic Hydrodynamic Fluctuations

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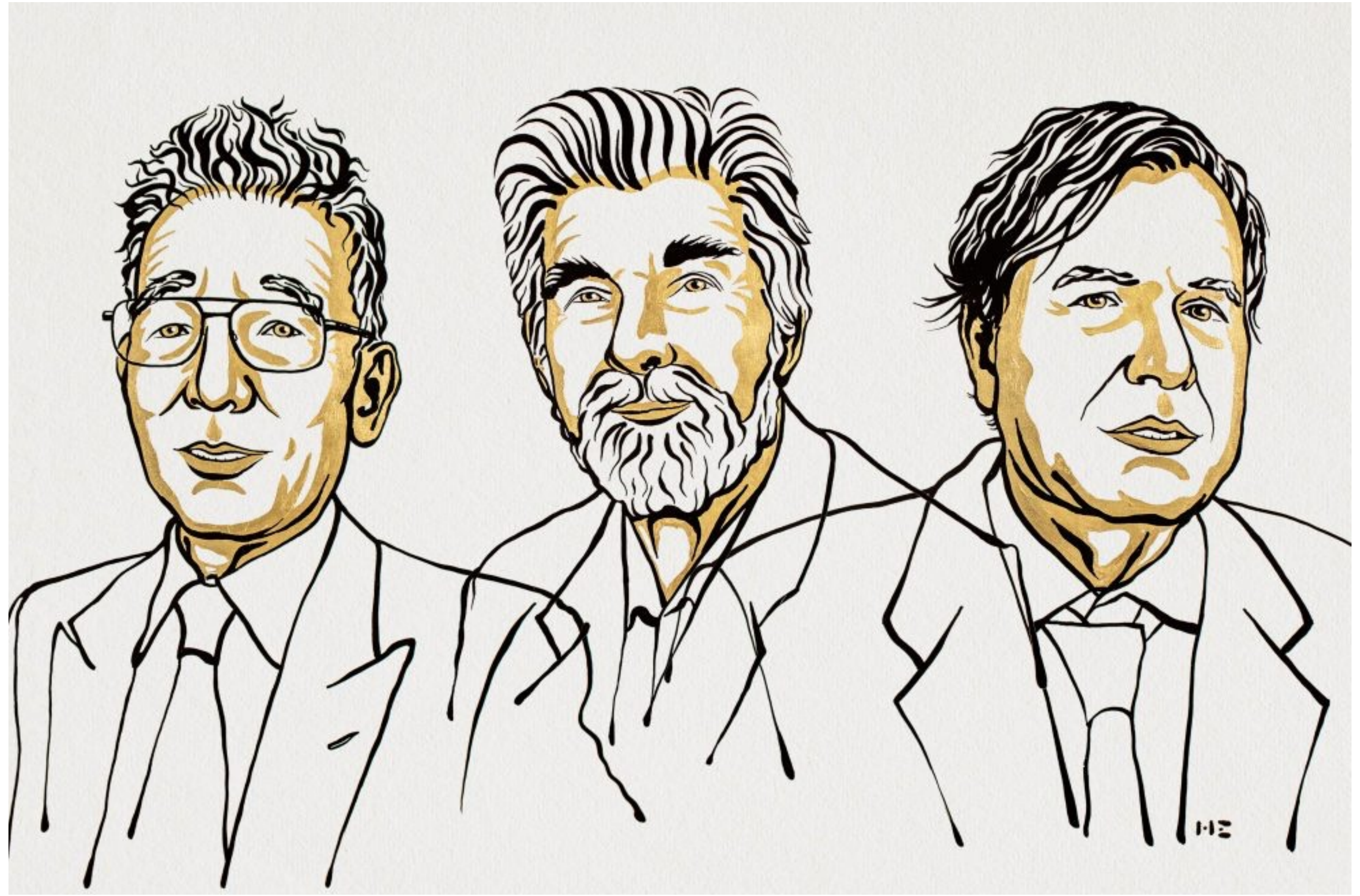
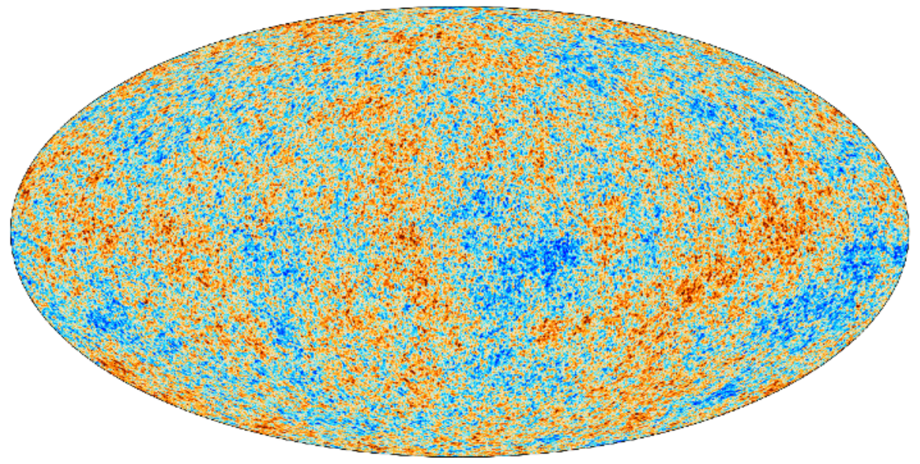
NCBJ

Fluctuations on all length scales

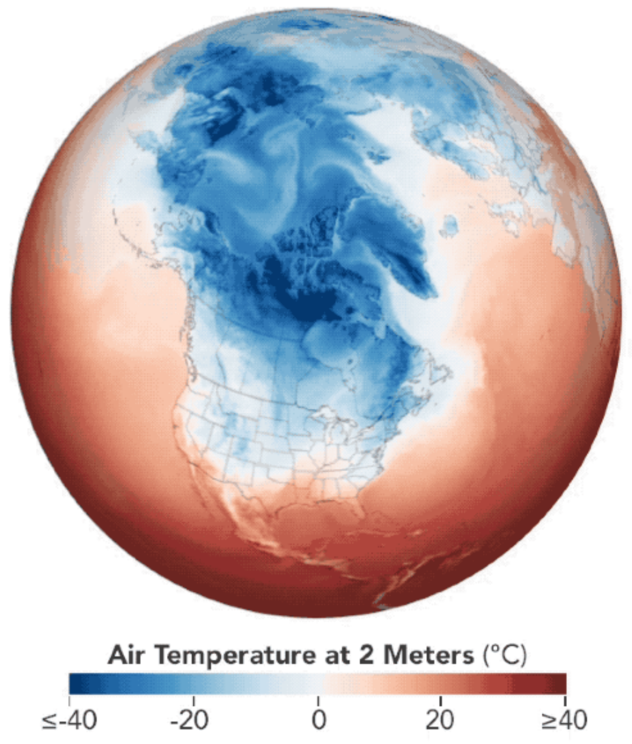
- Fluctuations are ubiquitous phenomena emerging on all length scales.



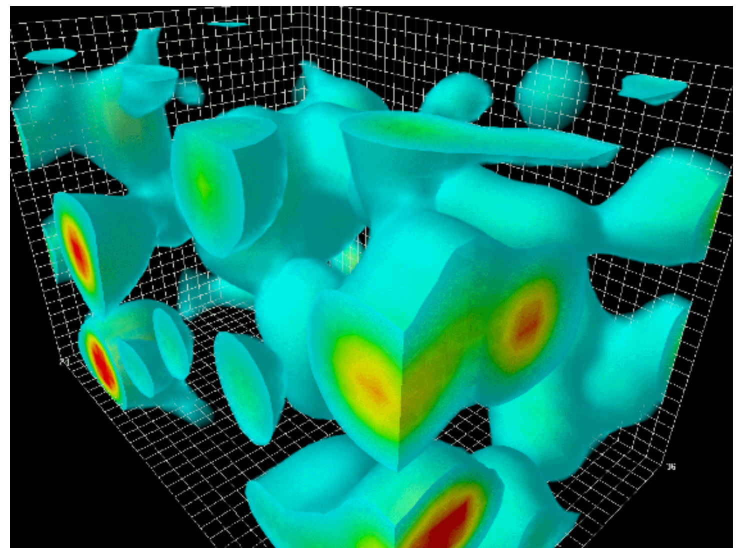
Nobel Prize in Physics 2021
S. Manabe, K. Hasselmann, G. Parisi



CMB



Atmosphere



Quantum fluctuations

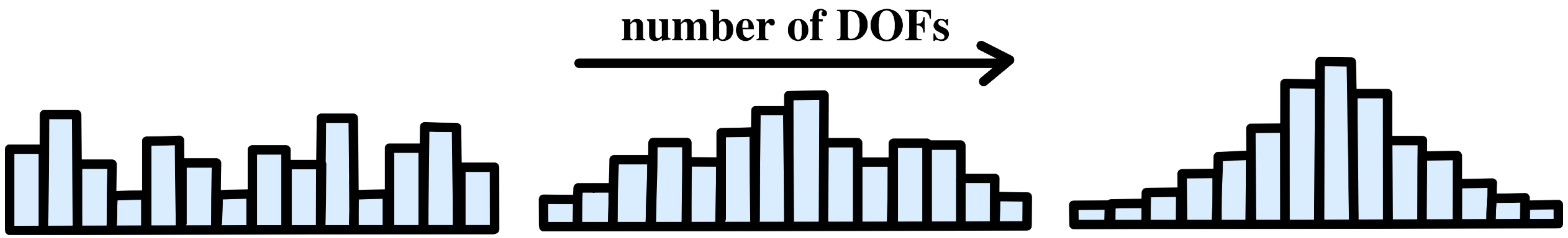
Fluctuations in equilibrium

The subject of thermodynamics is complicated.

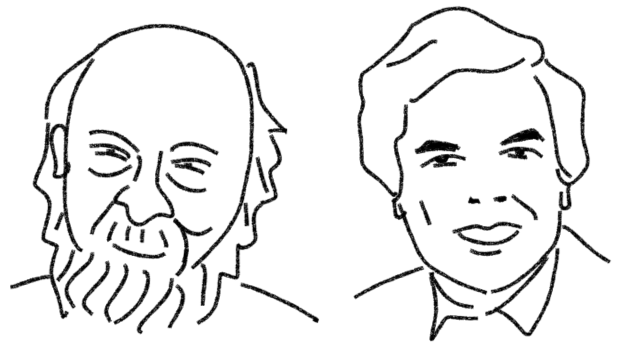
Feynman

Thermodynamic fluctuations

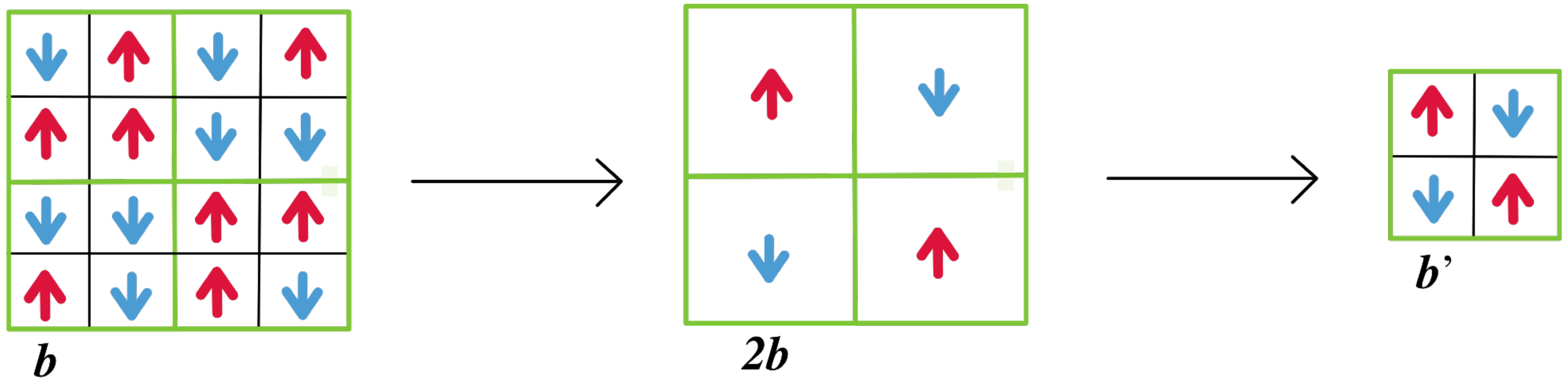
- Thermodynamic fluctuations: systems possess *large* number of DOFs; *small* deviation from *Gaussian* distribution due to the central limit theorem.



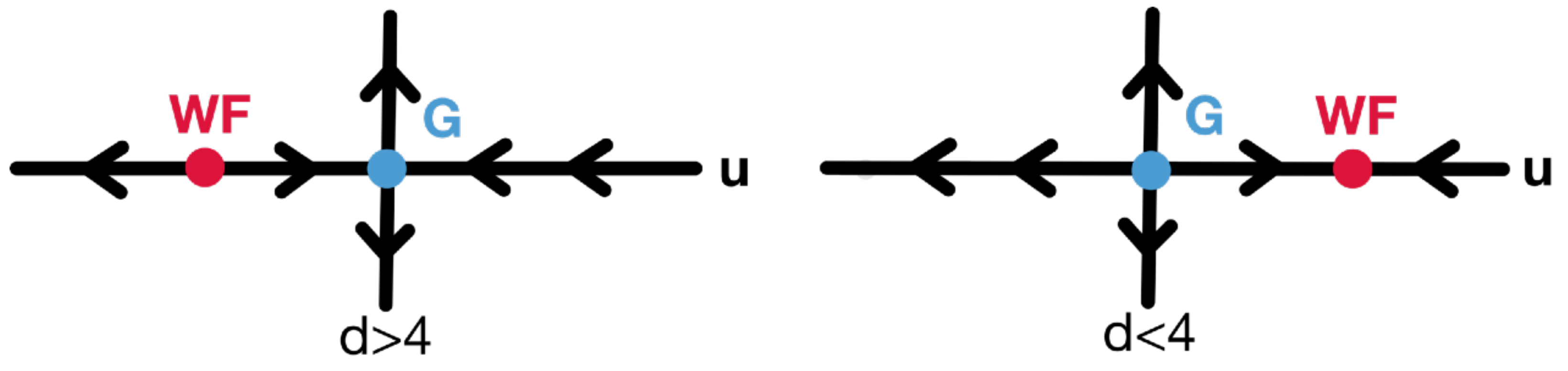
- Fluctuations on UV scales *renormalize* observed quantities on IR scales; *irrelevant* parameters flow to fixed point.



Kadanoff Wilson



Kadanoff's block spin coarse-graining procedure in 2D Ising model

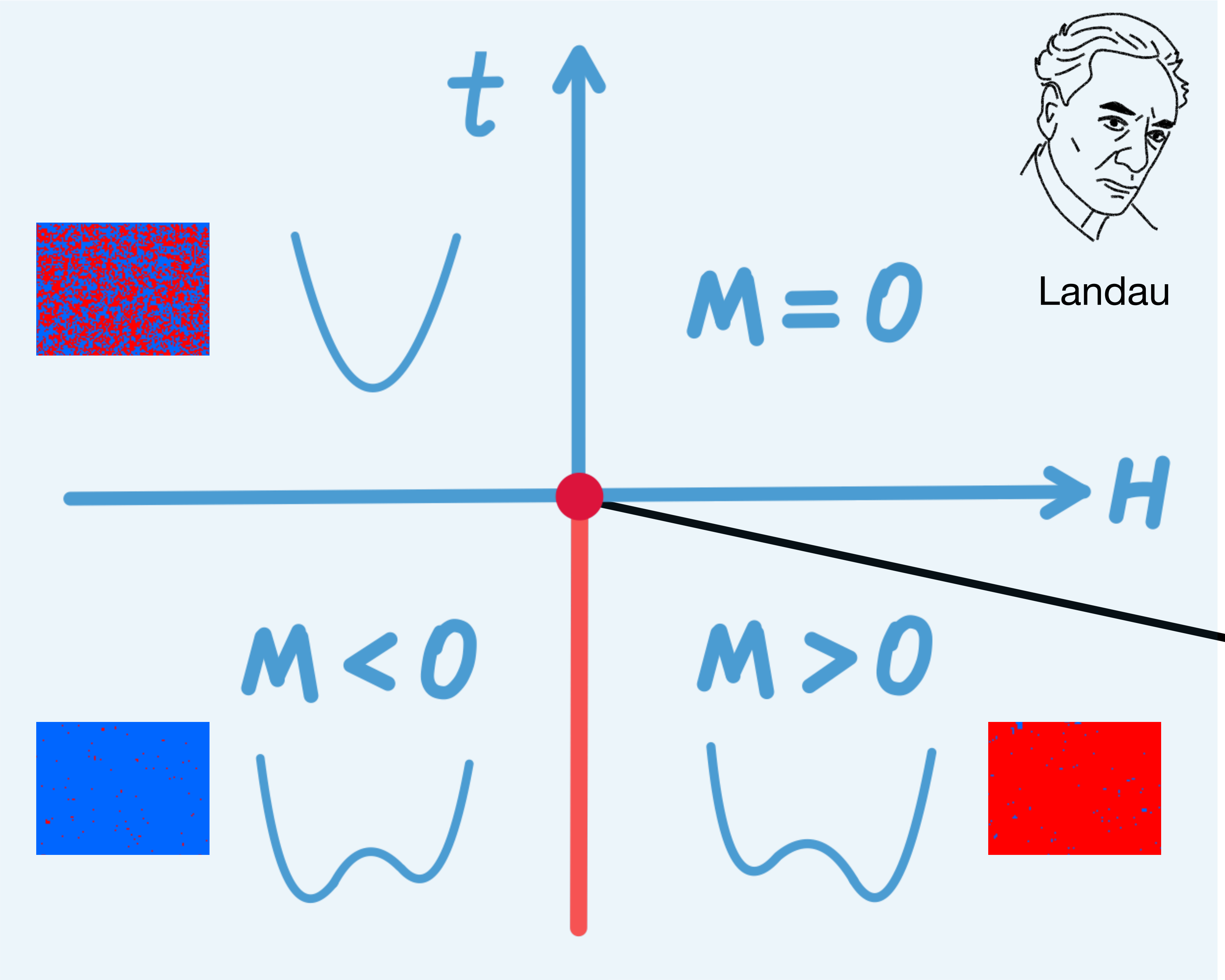


$$\beta(u) \equiv \frac{du}{db} = -(4-d)u + \frac{3}{2}u^2 + \dots$$

$$\beta(u_*) = 0 \implies u_G = 0,\ u_{WF} \sim 4 - d$$

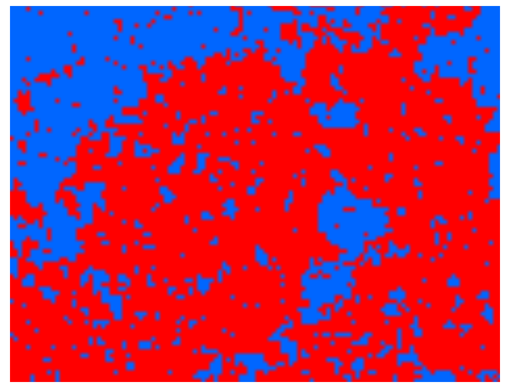
Critical fluctuations

- Critical point: end point of phase transition curve in *relevant* parameter space, where the correlation length ξ diverges and universal behavior manifests.
  Smoluchowski, 1908; Einstein, 1910



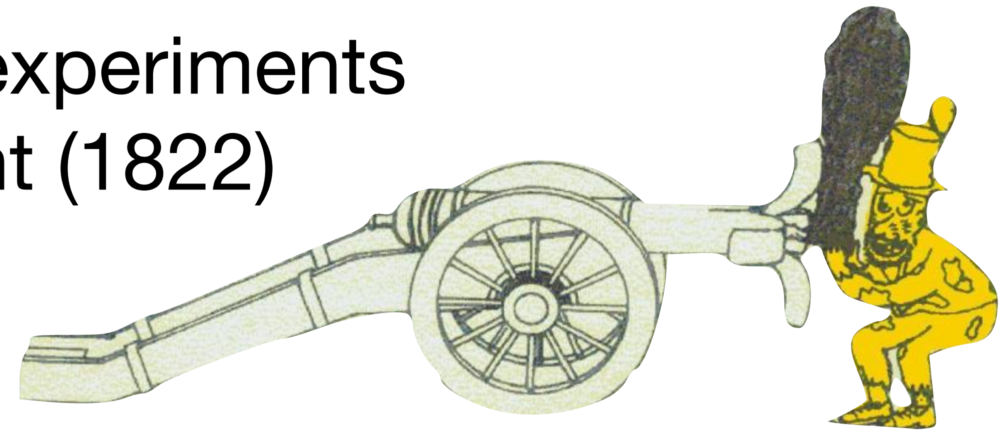
Ising phase diagram

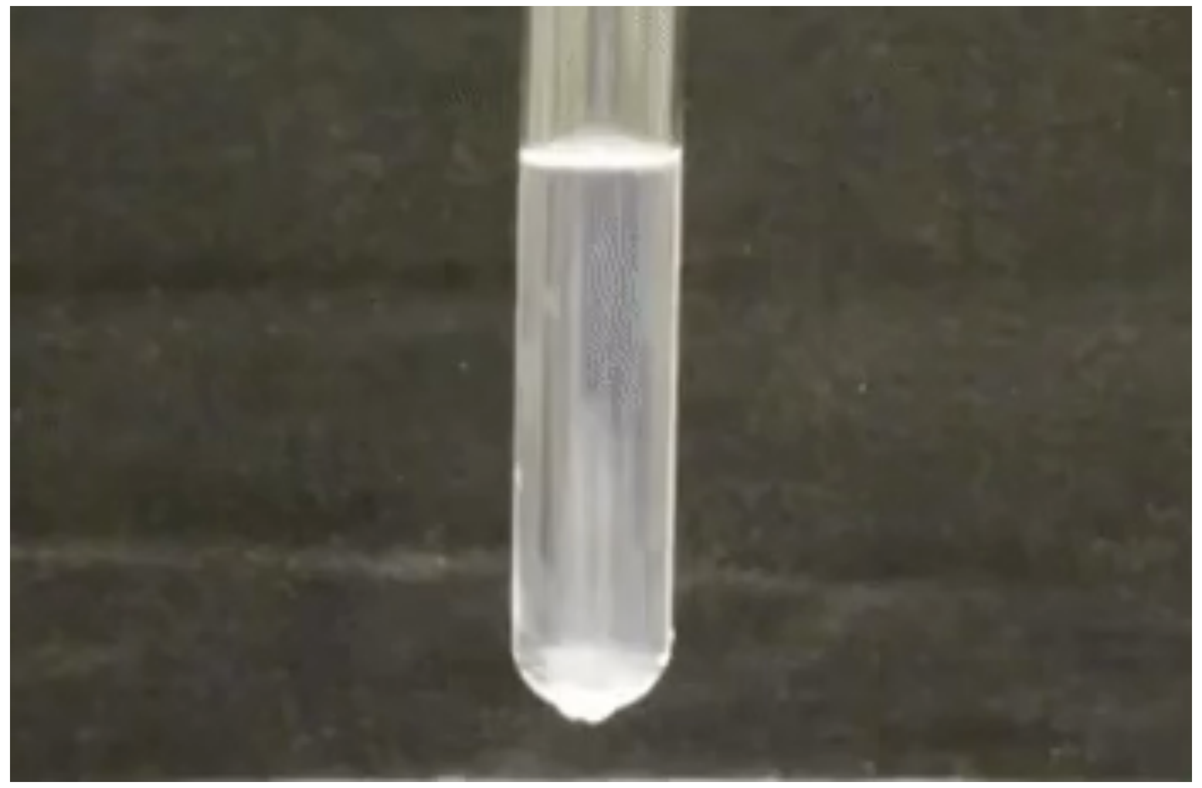
$$\langle \phi(x_1)\phi(x_2)\rangle \sim e^{-\frac{|x_1 - x_2|}{\xi}}$$



$\xi \to \infty$

Charles Cagniard's gun barrel experiments for the discovery of critical point (1822)





Critical opalescence: $\xi \leftrightarrow \lambda_{\text{light}}$

Near the critical point, systems possess *smaller* number of effective DOFs and *non-Gaussian* fluctuations become more important (due to CLT).

EOS with fluctuations

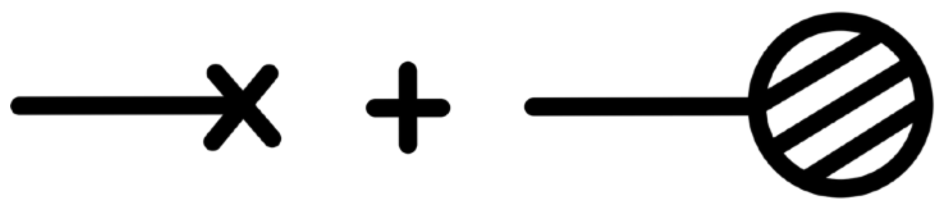
- Defining $\phi = \psi - \langle\psi\rangle$, the partition function reads

$$Z(J) = e^{\mathscr{W}[J]} = \int \mathscr{D}\psi\, e^{-\int_x (S_{\rm eff}(\psi) + J\psi)} = e^{-V(S_{\rm eff}(\langle\psi\rangle)+J\langle\psi\rangle)} \int \mathscr{D}\phi e^{-\int_x (S_{\rm eff}(\phi)+J\phi)}$$

E.g., for Ising model where $\langle\psi\rangle = M, J = H,$ Brezin, Wallace, Zia et al, 1970s; Wilson and Kogut, 1974

$$S_{\rm eff}(\phi) = \underbrace{\frac{1}{4}u\phi^4 + uM\phi^3}_{\text{perturbation}} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(r + 3uM^2)\phi^2 + \left(rM + uM^3\right)\phi$$

$$\langle\phi\rangle = 0 \quad \Longrightarrow \quad rM + uM^3 + H + g(M, r, u) = 0$$



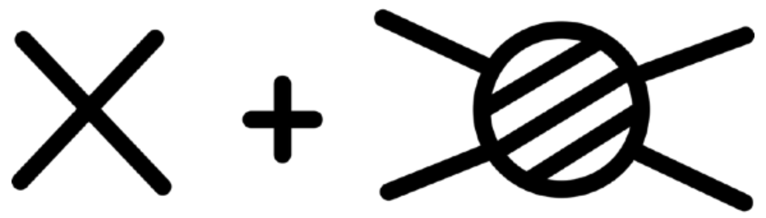
one-point diagrams

Widom's scaling relation $\quad H = M^\delta f(r/M^{1/\beta}; u = u_*)$

$$\beta(u) \equiv \frac{du}{db} = -\varepsilon u + \frac{3}{2}u^2 + \dots$$

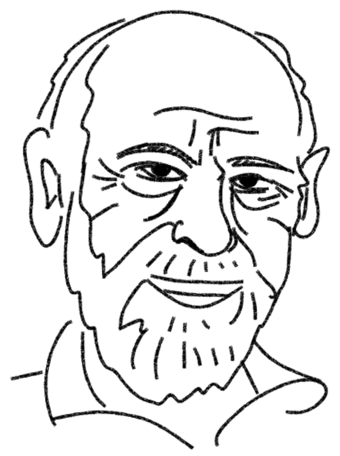
where $\quad \delta = 3 + \varepsilon + \mathscr{O}(\varepsilon^2), \quad \beta = \frac{1}{2} - \frac{1}{6}\varepsilon + \mathscr{O}(\varepsilon^2), \quad \varepsilon = 4 - d$

four-point diagrams



Fluctuations out of equilibrium

Thermal equilibrium is extremely boring.



Susskind

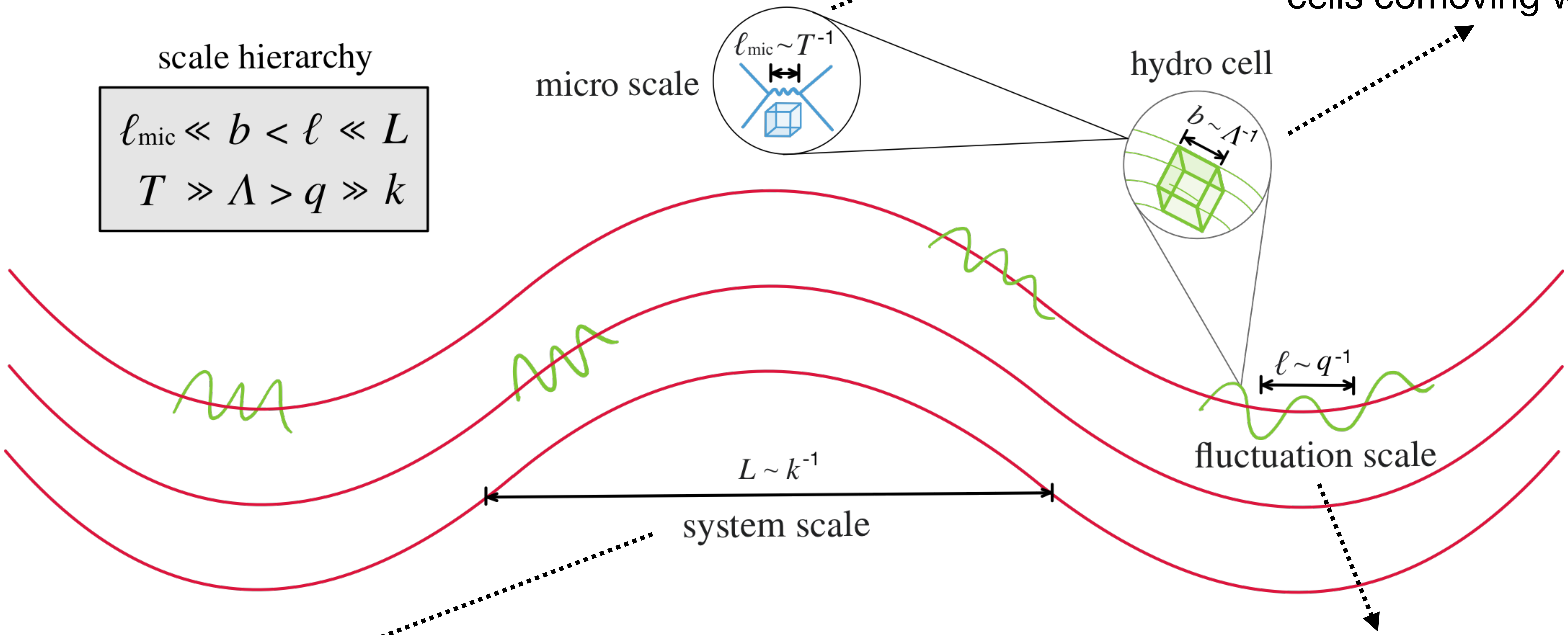
Hydrodynamic fluctuations

noise

large number of *locally* thermalized cells comoving with fluid

scale hierarchy

$$\ell_{\text{mic}} \ll b < \ell \ll L$$
$$T \gg \Lambda > q \gg k$$



fluctuations *not* equilibrated at *large* scales

evolution described by a set of conservation equations

$$\partial_t \psi = \nabla \cdot (\text{flux}\,[\psi]) \quad \text{where} \quad \psi = \begin{pmatrix} n \\ \epsilon \\ \pi_i \end{pmatrix} \begin{matrix} \longrightarrow \text{charge} \\ \longrightarrow \text{energy} \\ \longrightarrow \text{momentum} \end{matrix}$$

Fluctuation dynamics in Brownian motion

- Einstein's formula for diffusion coefficient in $\partial_t \rho = D \nabla^2 \rho$: Einstein, 1905

$$D = \lim_{t\to\infty} \frac{1}{2t} \langle \Delta x^2(t) \rangle = \int_0^\infty d\tau \langle v(\tau) v(0) \rangle$$

Kubo formula

- Long-time behavior:

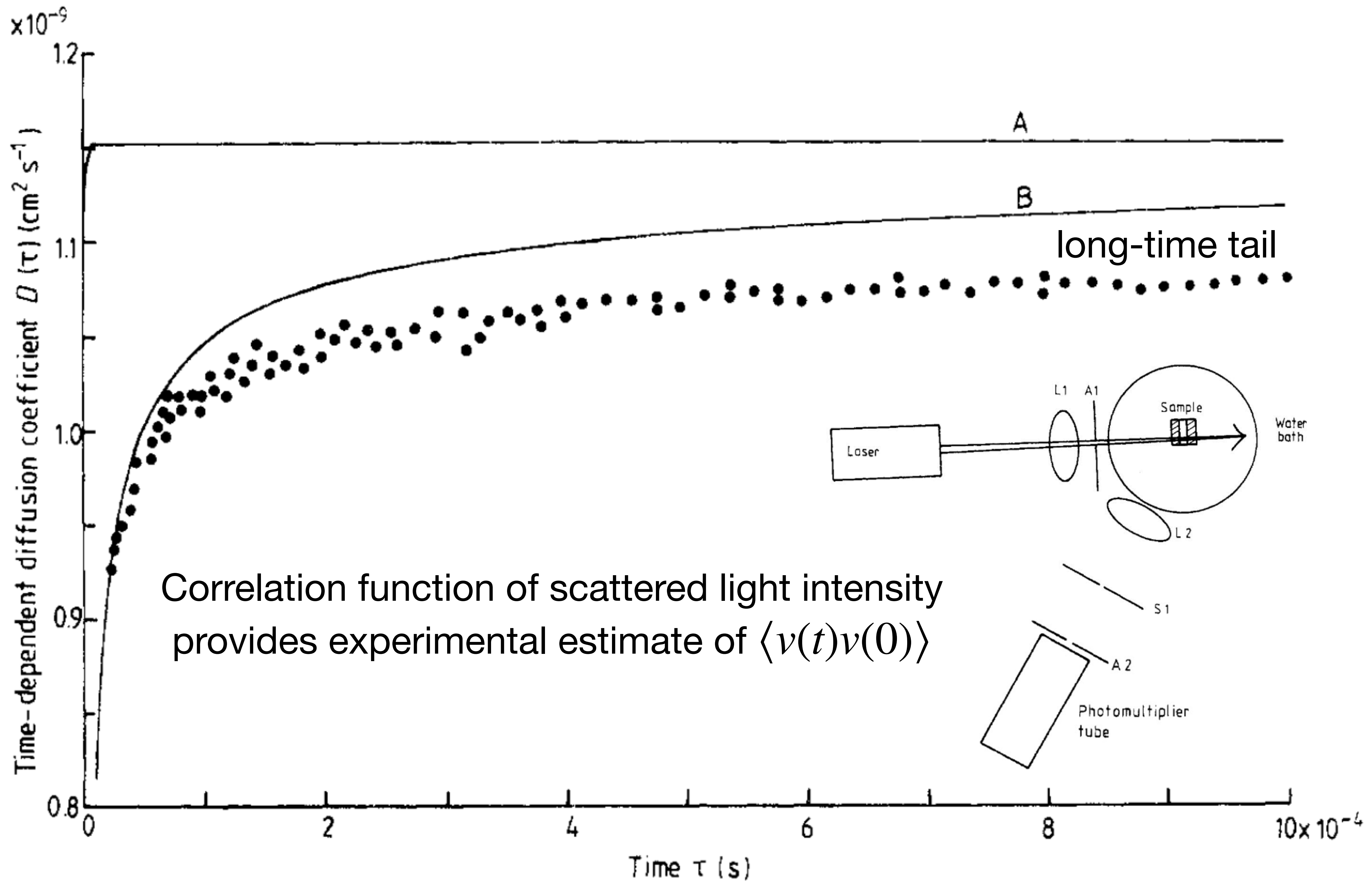
$$\langle v(t)v(0)\rangle \sim e^{-\mu t} \quad \rightarrow \quad D \sim \mu^{-1}$$

With only dissipation

$$\langle v(t)v(0)\rangle \sim t^{-3/2} \quad \rightarrow \quad D \sim t^{-1/2}$$

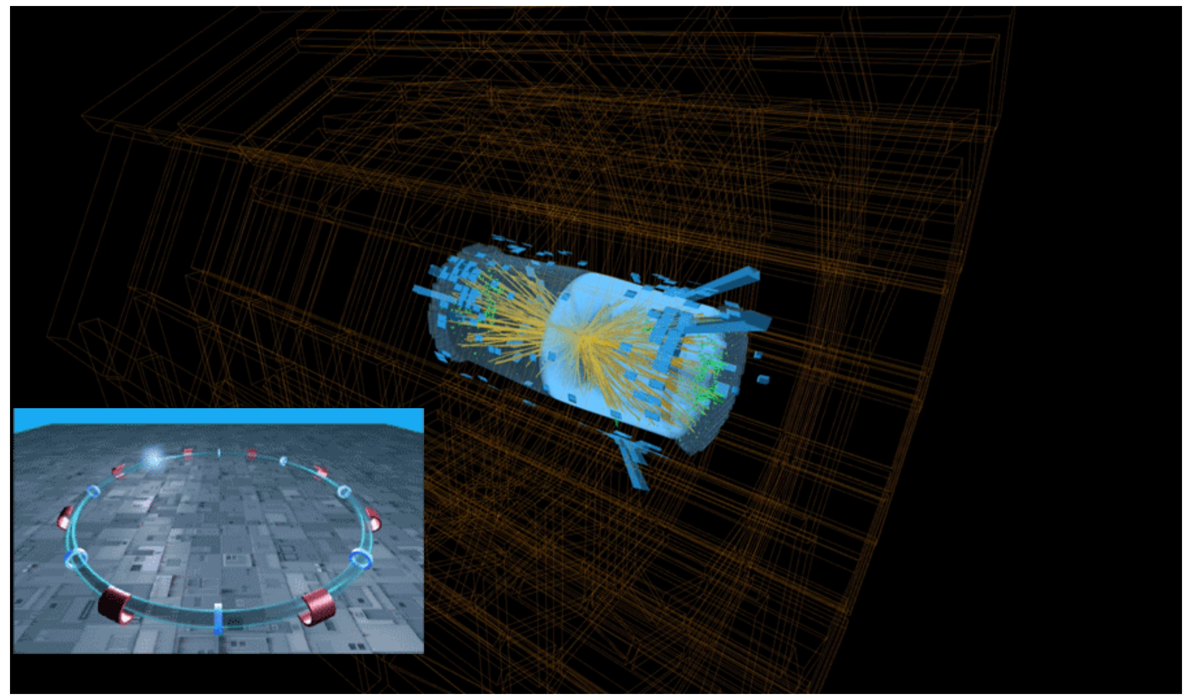
With also fluctuations

Paul et al, 1981, J. Phys. A: Math. Gen. 14 3301



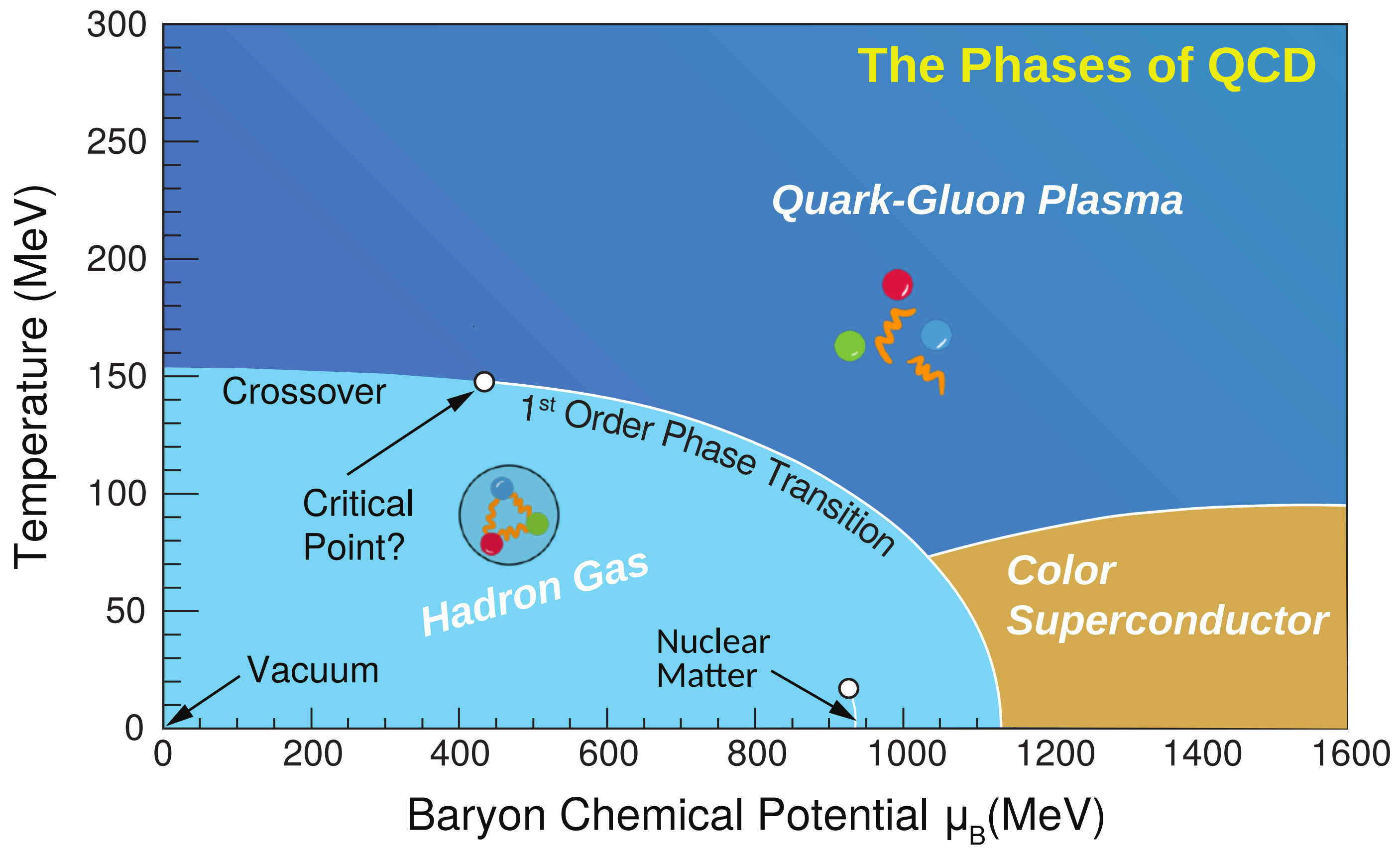
Fluctuation dynamics in heavy-ion collisions

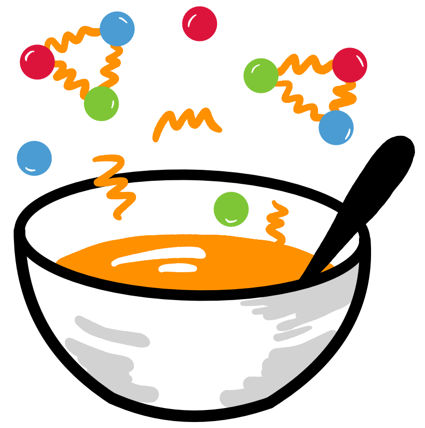
- Fluctuating hydrodynamics is a *non-equilibrium* approach to unraveling the *equilibrium* properties of QCD matters in different phases.



Collision events at LHC (CERN)

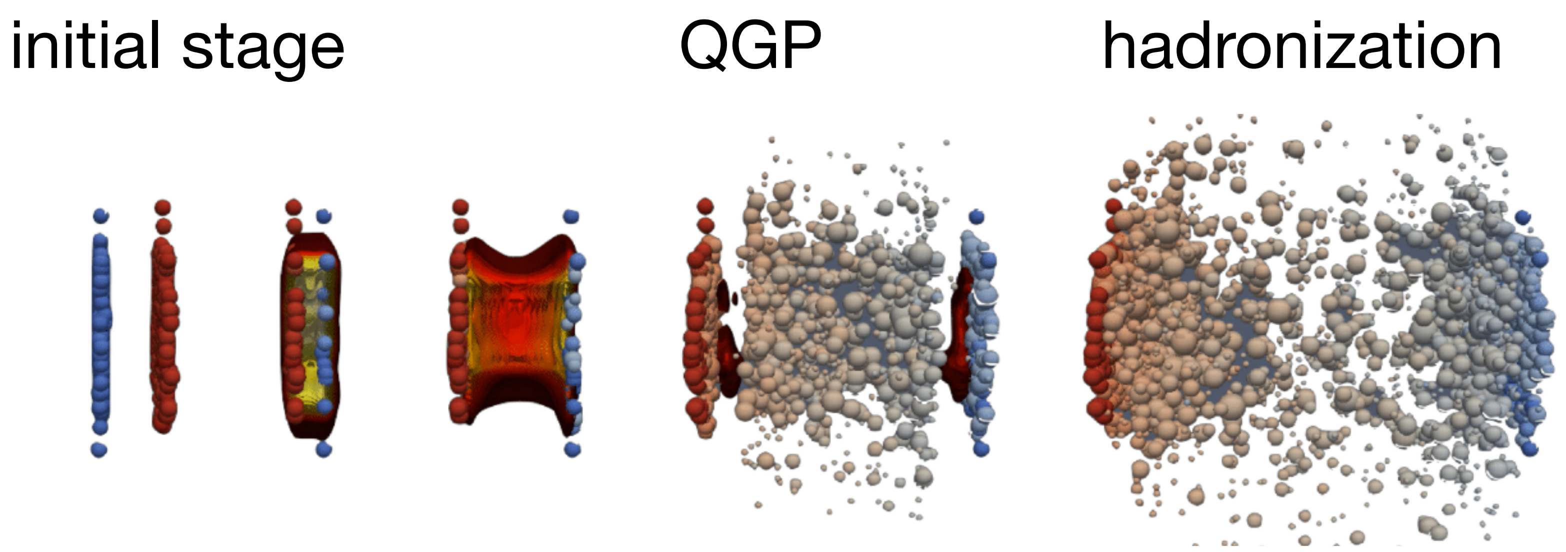
Out of equilibrium; observables fluctuate event-by-event





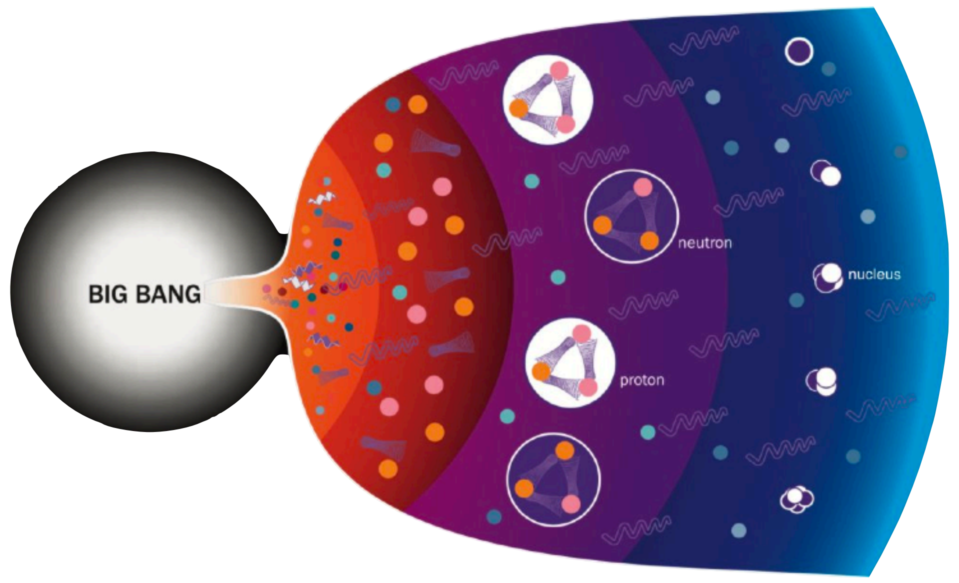
In equilibrium; observables fluctuate ensemble-by-ensemble

- Small bang vs Big bang: extreme initial state; particle synthesis; system expands, cools followed by freezeout and thermalization.



High statistics; measured in *momentum* coordinate

History of a heavy-ion collision



History of Universe

Cosmic variance; measured in *space* coordinate

Theory

EFTs (top-down like)

Starting from effective action with first principles

e.g., Martin-Siggia-Rose (MSR), Schwinger-Keldysh (SK), Hohenberg-Halperin (HH), n-particle irreducible (nPI), etc.

Glorioso et al, 1805.09331
Jain et al, 2009.01356
Sogabe et al, 2111.14667
Chao et al, 2302.00720
…

EOMs (bottom-up like)

Starting from phenomenological equations with required properties

e.g., Langevin equations in *stochastic* description, Fokker-Planck (FP) equations in *deterministic* description.

Akamatsu et al, 1606.07742
Nahrgang et al, 1804.05728
Singh et al, 1807.05451
Chattopadhyay et al, 2304.07279
…

Two bottom-up approaches

Stochastic

Langevin equation

Newton's equation + noise

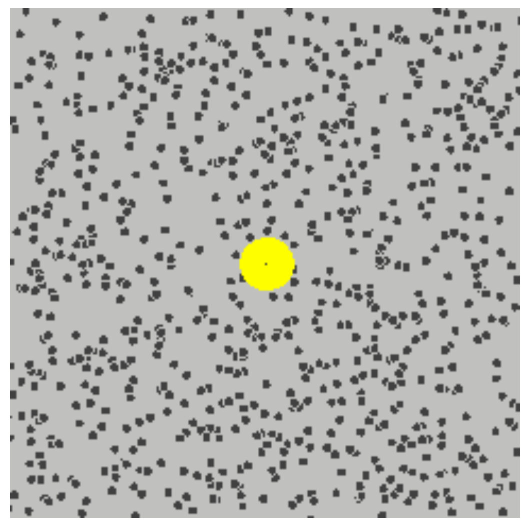
$$\partial_t \psi_i = F_i(\psi) + \eta_i$$

↓ drift

$$\langle \eta_i(x_1)\, \eta_j(x_2) \rangle = 2Q_{ij}(\psi)\, \delta^{(4)}(x_1 - x_2)$$

↙ multiplicative noise ↘ infinite noise

Langevin Landau Lifshitz



Pros: *one* equation, albeit *millions* of samples

Cons: divergence due to infinite noise; ambiguity due to multiplicative noise

Deterministic

Fokker-Planck equation

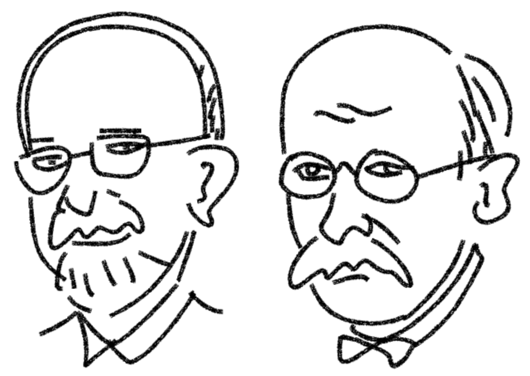
probability evolution equation (Ito's)

$$\partial_t P = (-F_i P + (M_{ij} P)_{,j})_{,i}$$

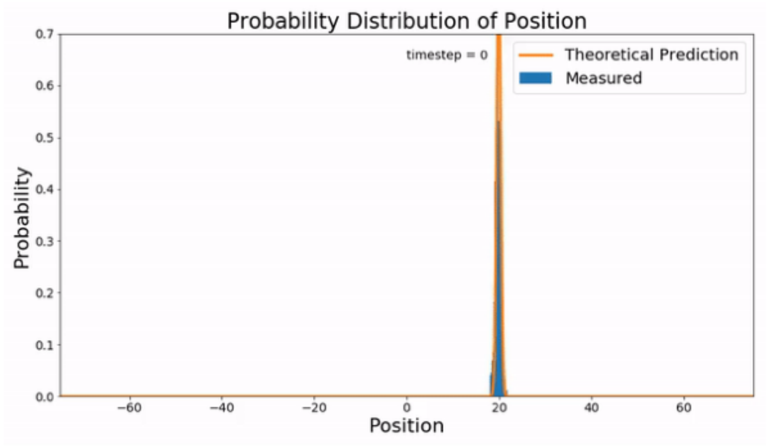
$$F_i = M_{ij} S_{,j} + M_{ij,j} \qquad M_{ij} = Q_{ij} + \Omega_{ij} \qquad P_{\mathrm{eq}} = e^S$$

Q_{ij} : Onsager matrix (symmetric)

Ω_{ij} : Poisson matrix (anti-symmetric)



Fokker Planck



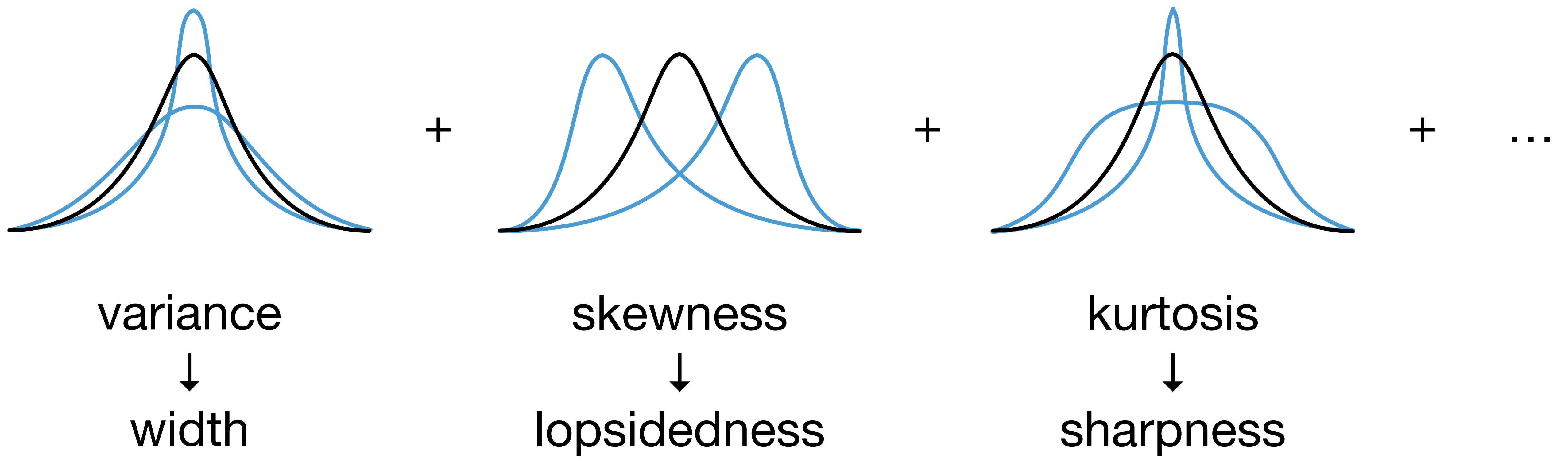
Pros: infinite noise regularized analytically; multiplicative noise well defined

Cons: *millions* of equations, albeit *one* sample

Dynamics of n-point correlators

- The cumulant generating function $e^{\mathscr{W}[J;t]} \equiv Z[J;t] = \int \mathscr{D}\psi\, P[\psi;t]\, e^{J_i \psi_i}$ expands as

$$\mathscr{W} = \sum_{i=1}^{\infty} \frac{1}{n!} G_{i_1 \ldots i_n} J_{i_1} \ldots J_{i_n} \qquad \text{where} \qquad G_{i_1 \ldots i_n} \equiv \langle \phi_{i_1} \ldots \phi_{i_n} \rangle = \left. \frac{\delta^{(n)} \mathscr{W}}{\delta J_{i_1} \ldots \delta J_{i_n}} \right|_{J=0}$$



$$\phi \equiv \psi - \langle \psi \rangle$$

n-pt correlators are related to *cumulants* by space integration

- Evolution equations for generating function $\mathscr{W}$: XA et al, 2009.10742, 2209.15005

$$\partial_t \mathscr{W} = e^{-\mathscr{W}} (J_i F_i + J_i J_j M_{ij})\, e^{\mathscr{W}}$$

where $F_i = F_i(\delta/\delta J_i), M_{ij} = M_{ij}(\delta/\delta J_i)$.

Evolution equations and truncation

- Evolution equations for n-pt correlators $G_n = G_{i_1 \dots i_n}$: XA et al, 2009.10742, 2212.14029

$$\partial_t G_n = \mathscr{F}\left[\langle\psi\rangle, G_2, G_3, \dots, G_n, G_{n+1}, \dots G_\infty\right]$$

need ∞ equations to close the system!

$$\text{E.g., } \partial_t G_{ij} = \underbrace{F_{i,k}G_{kj} + F_{j,k}G_{ki} + 2M_{ij}}_{\text{leading, only trees}} + \underbrace{\frac{1}{2}F_{i,k\ell}G_{k\ell j} + \frac{1}{2}F_{j,k\ell}G_{k\ell i} + M_{ij,k\ell}G_{k\ell} + \dots}_{\text{higher order, including loops}}$$

- Introducing the loop expansion parameters $\varepsilon \sim 1/$number of DOFs, the evolution equations can be systematically truncated and iteratively solved:

XA et al, 2009.10742

$$\partial_t G_n = \mathscr{F}\left[\langle\psi\rangle, G_2, G_3, \dots, G_n\right] + \mathcal{O}(\varepsilon^n) \qquad \text{where} \qquad G_n \sim \varepsilon^{n-1}, \qquad F_i \sim 1, \qquad M_{ij} \sim \varepsilon\,.$$

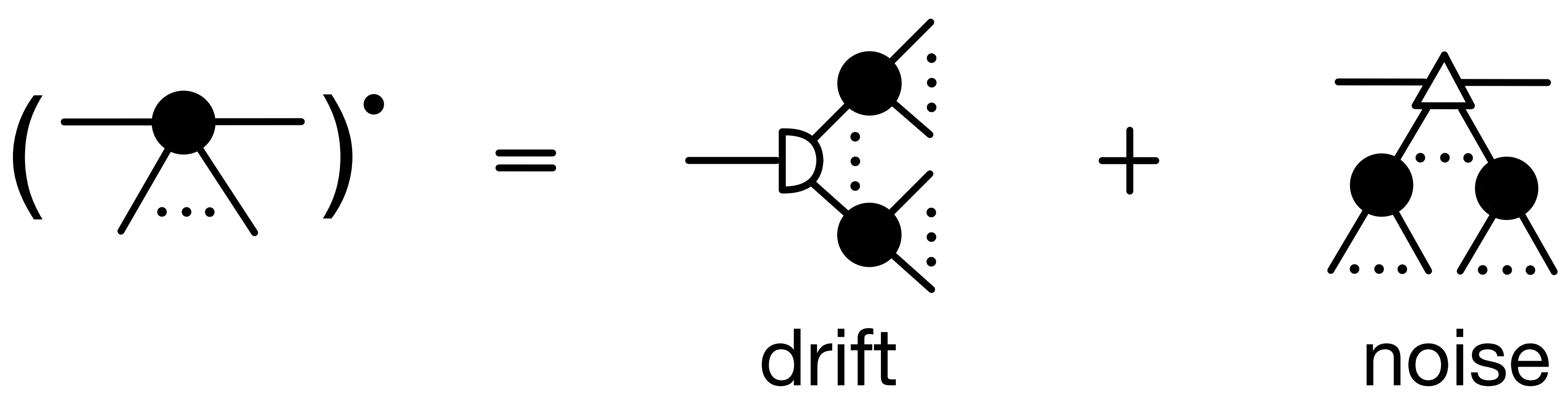
Hydrodynamics: $\varepsilon \sim (\xi/\ell)^3 \sim$ correlated volume / fluctuation volume

Holography: $\varepsilon \sim 1/N_c \sim$ 1 / number of colors

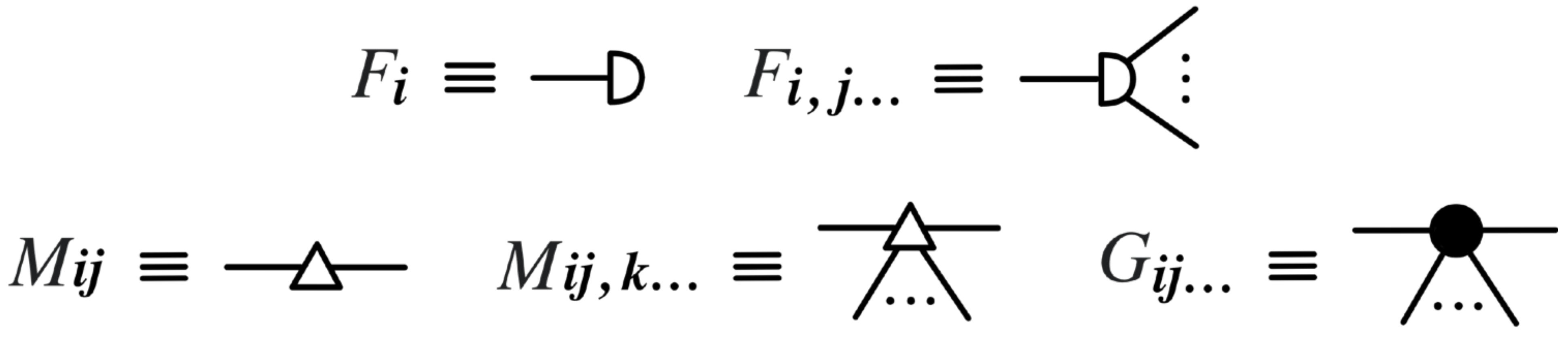
$\phi \sim \sqrt{\epsilon}$ CLT!

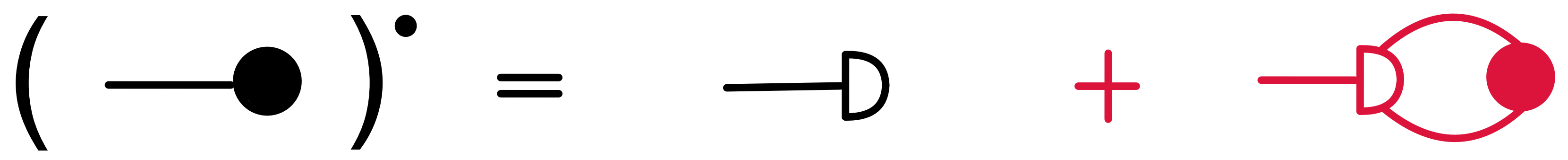
Diagram representation

- Truncated equations for n-pt correlators (diagrams): XA et al, 2009.10742, 2212.14029



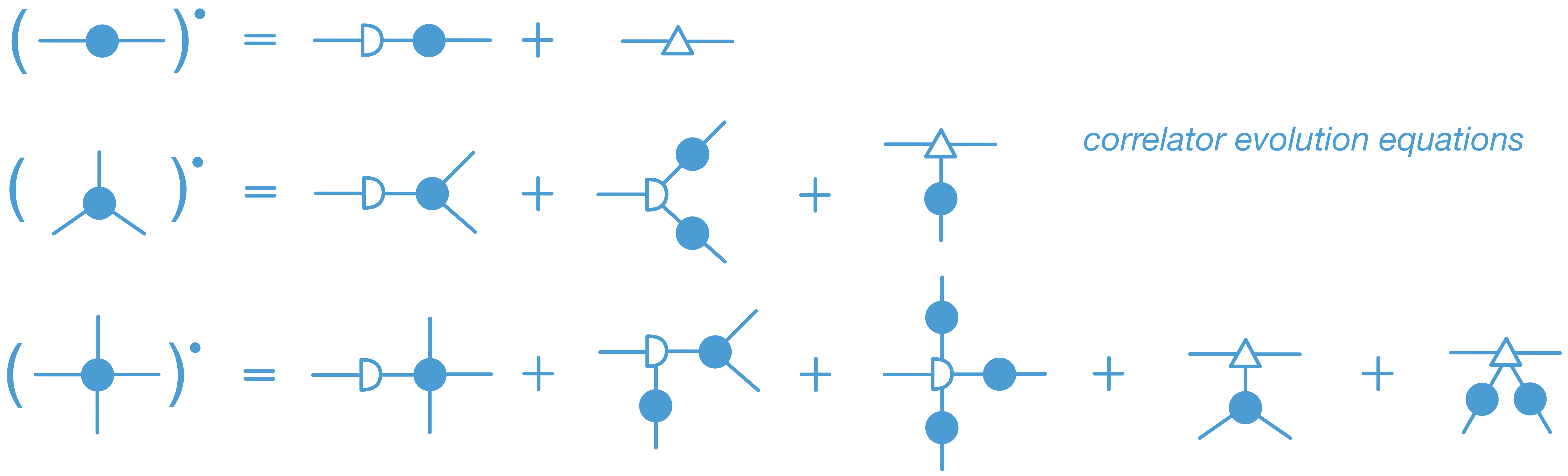
all *combinatorial* configurations of trees





conventional hydro equations

one loop (renormalization & long-time tails)

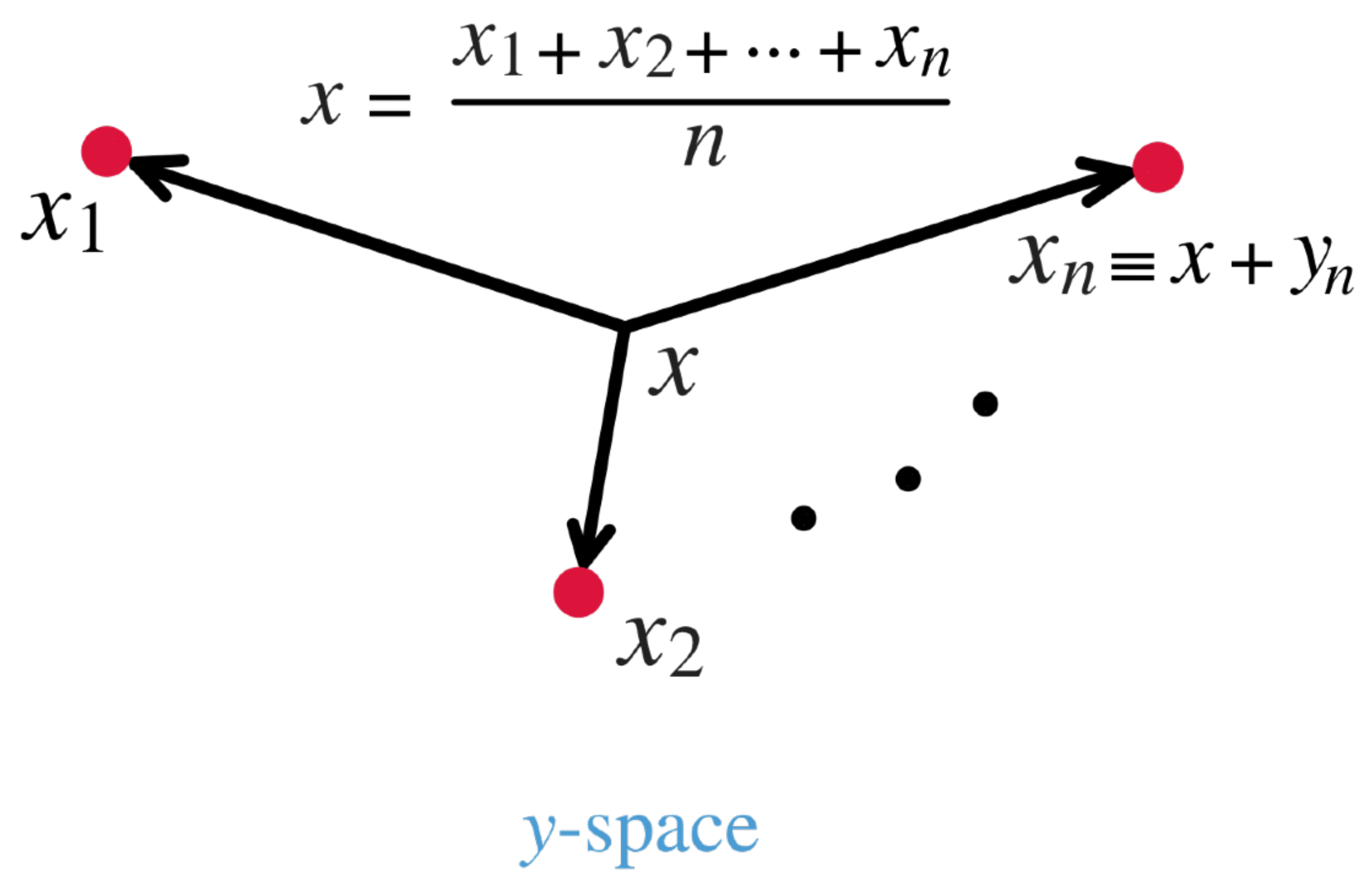


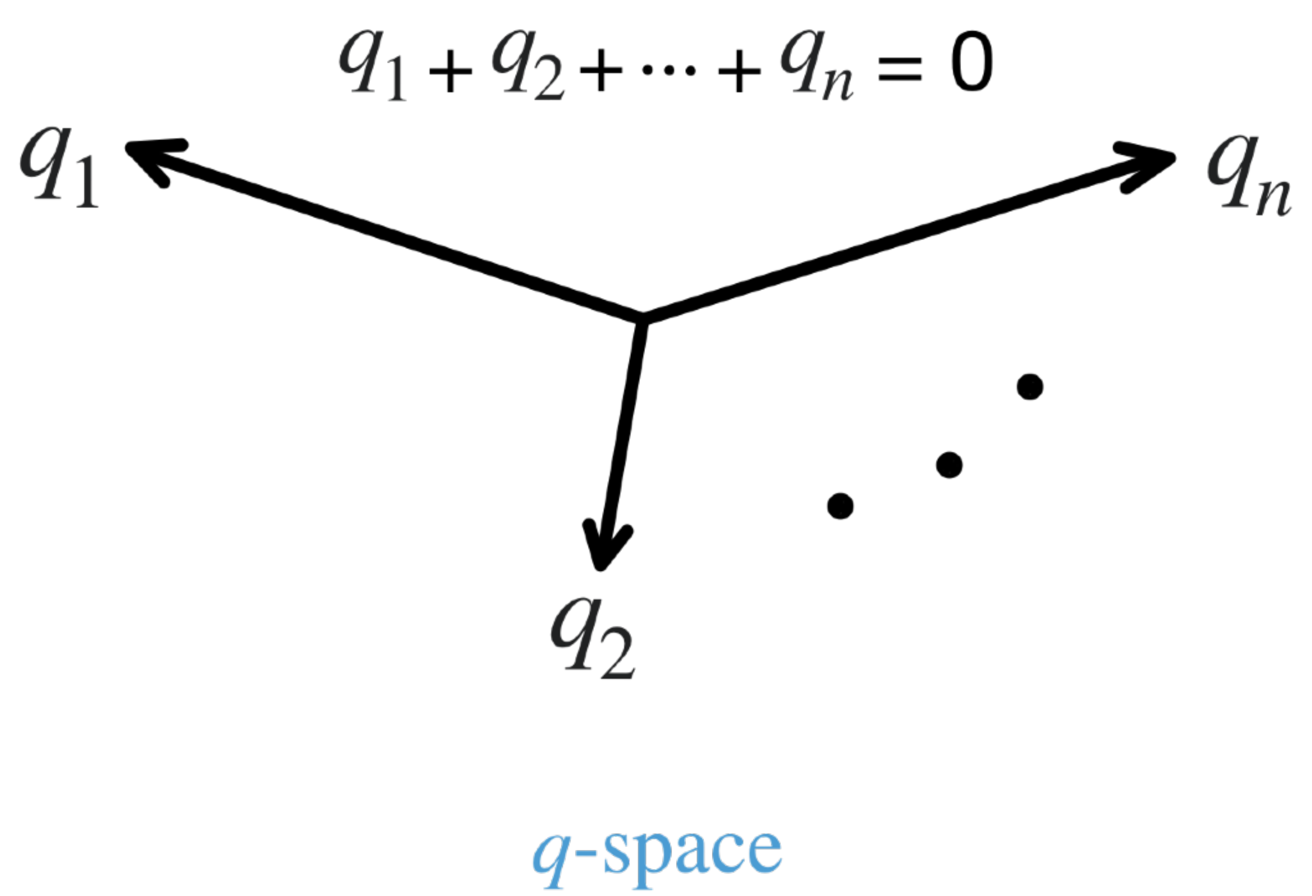
correlator evolution equations

Multi-point Wigner function

- For fluctuation *fields*, we introduced the novel n-pt Wigner function XA et al, 2009.10742

$$W_n(x; q_1, \ldots, q_n) = \int d^3y_1 \ldots d^3y_n \, e^{-(iq_1y_1 + \ldots + iq_ny_n)} \, \delta^{(3)}\left(\frac{y_1 + \ldots + y_n}{n}\right) G_n(x; y_1, \ldots, y_n)$$





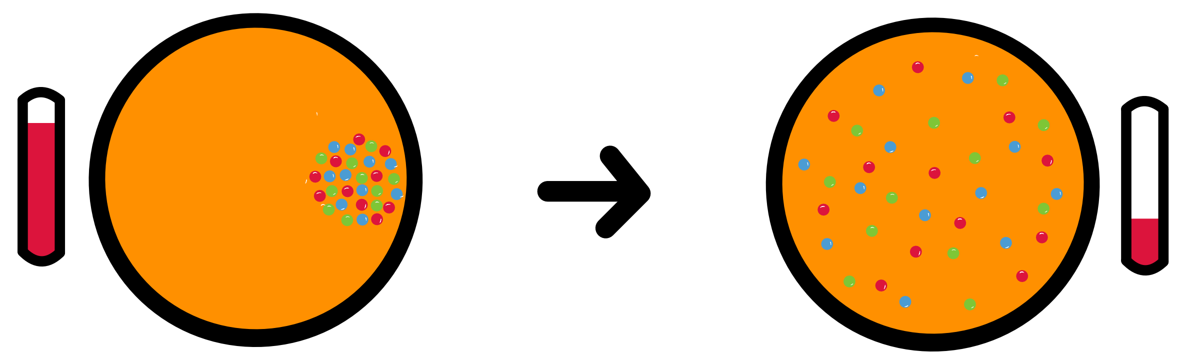
$$\partial_t W_n = \mathscr{F}\,[\langle\psi\rangle, W_2, W_3, \ldots, W_n\,] + \mathcal{O}(\epsilon^n)$$

"While the bottom-up approach is useful in order to calculate two-point correlation functions, it is not immediately obvious how it should be generalized for the calculation of n-point correlation functions." Romatschke, 2019

An example: charge diffusion near critical point

- Simple charge diffusion problem: XA et al, 2009.10742, 2209.15005

$$\partial_t n = \nabla(\lambda \nabla \alpha) + \eta, \qquad \langle \eta(x)\eta(y)\rangle = 2\nabla^{(x)}\lambda\nabla^{(y)}\delta^{(3)}(x-y)$$

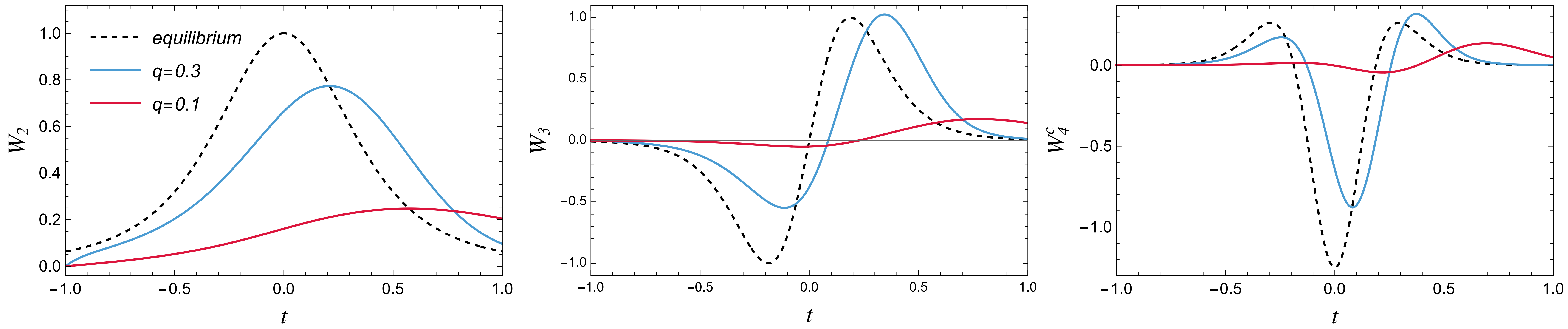


quantities	general	diffusive charge
variable	ψ_i	$n(\boldsymbol{x})$
variable index	i, j, k, etc.	$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, etc.
Onsager matrix	Q_{ij}	$\boldsymbol{\nabla}_{\boldsymbol{x}}\lambda\boldsymbol{\nabla}_{\boldsymbol{y}}\,\delta^{(3)}_{\boldsymbol{xy}}$
drift force	F_i	$\boldsymbol{\nabla}_{\boldsymbol{x}}\lambda\boldsymbol{\nabla}_{\boldsymbol{x}}\alpha$

$$\partial_t W_2 = -Dq^2 W_2 + \lambda q^2$$

$$\partial_t W_3 = \ldots$$

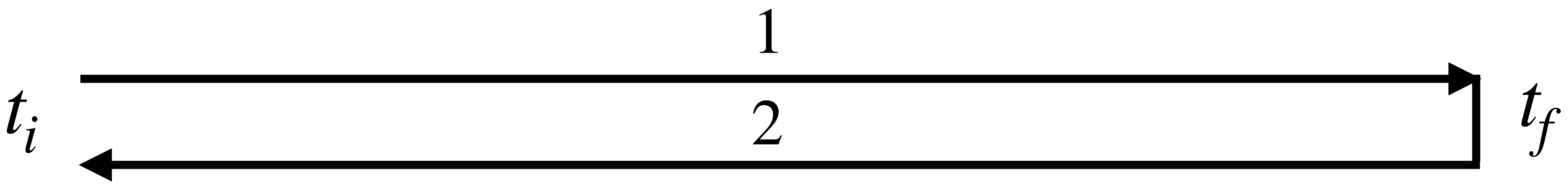
$n \equiv$ density; $\lambda \equiv$ conductivity; $\alpha \equiv$ chemical potential; $D = \lambda\alpha' \equiv$ diffusion coefficient



Evolution of n-point Wigner functions manifests **strong memory effect**

Connection to top-down approach

- Schwinger-Keldysh formalism Schwinger, Keldysh, 1960s




Schwinger Keldysh

$$Z = \int \mathscr{D}\psi_1 \mathscr{D}\psi_2 \mathscr{D}\chi_1 \mathscr{D}\chi_2 \, e^{\,i I_0(\psi_1,\chi_1) - i I_0(\psi_2,\chi_2)} = \int \mathscr{D}\psi_r \mathscr{D}\psi_a \, e^{\,i\int_\tau \mathscr{L}_{\rm EFT}}$$

$$\psi_r = \frac{1}{2}(\psi_1 + \psi_2)$$
$$\psi_a = \psi_1 - \psi_2$$

- The effective Lagrangian is constructed following *fundamental symmetries*:

Glorioso et al, 1805.09331; Jain et al, 2009.01356

$$\mathscr{L}_{\rm EFT}(\psi_r, \psi_a) = \psi_{ai} Q_{ij}^{-1}(F_j - \dot\psi_{rj}) + i\psi_{ai} Q_{ij}^{-1}\psi_{aj}$$

which is invariant under KMS transformation $\widetilde{\psi}_r(-x) \to \psi_r(x), \quad \widetilde{\psi}_a(-x) \to \psi_a(x) + i\dot\psi_r(x)$

$$P[\psi] = \int_{\psi_r=\psi(t)} \mathscr{D}\psi_r \mathscr{D}\psi_a \, J(\psi_r) \, e^{\,i\int_{-\infty}^{t} d\tau \mathscr{L}_{\rm EFT}} \longrightarrow \partial_t P = (-F_i P + (Q_{ij} P)_{,j})_{,i}$$

XA et al, in progress

Fluctuations in relativistic hydrodynamics

The requirement of general covariance takes away from space and time the last remnant of physical objectivity.

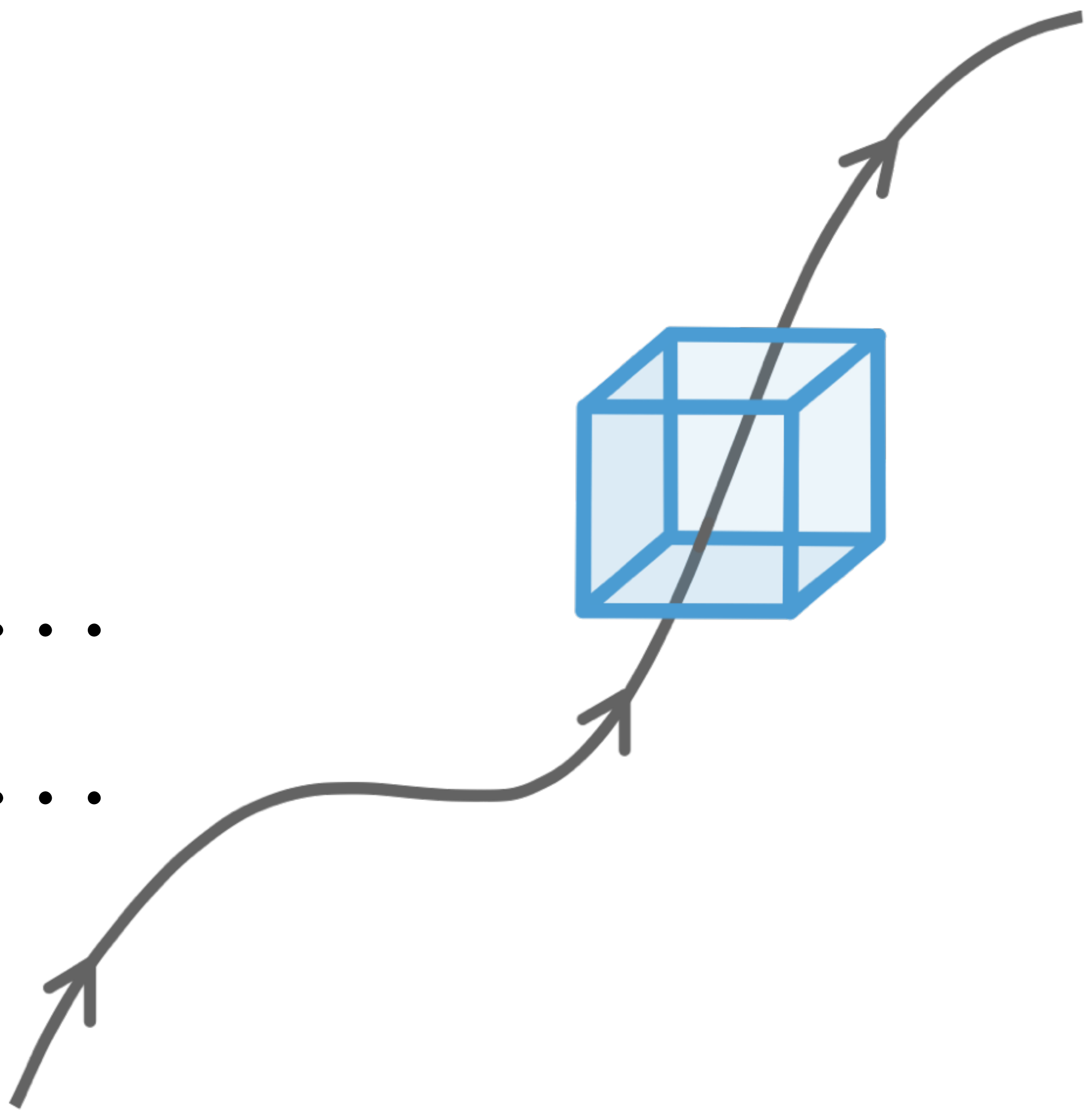
Einstein

Relativistic dynamics

Eulerian specification

more often used in non-relativistic theory

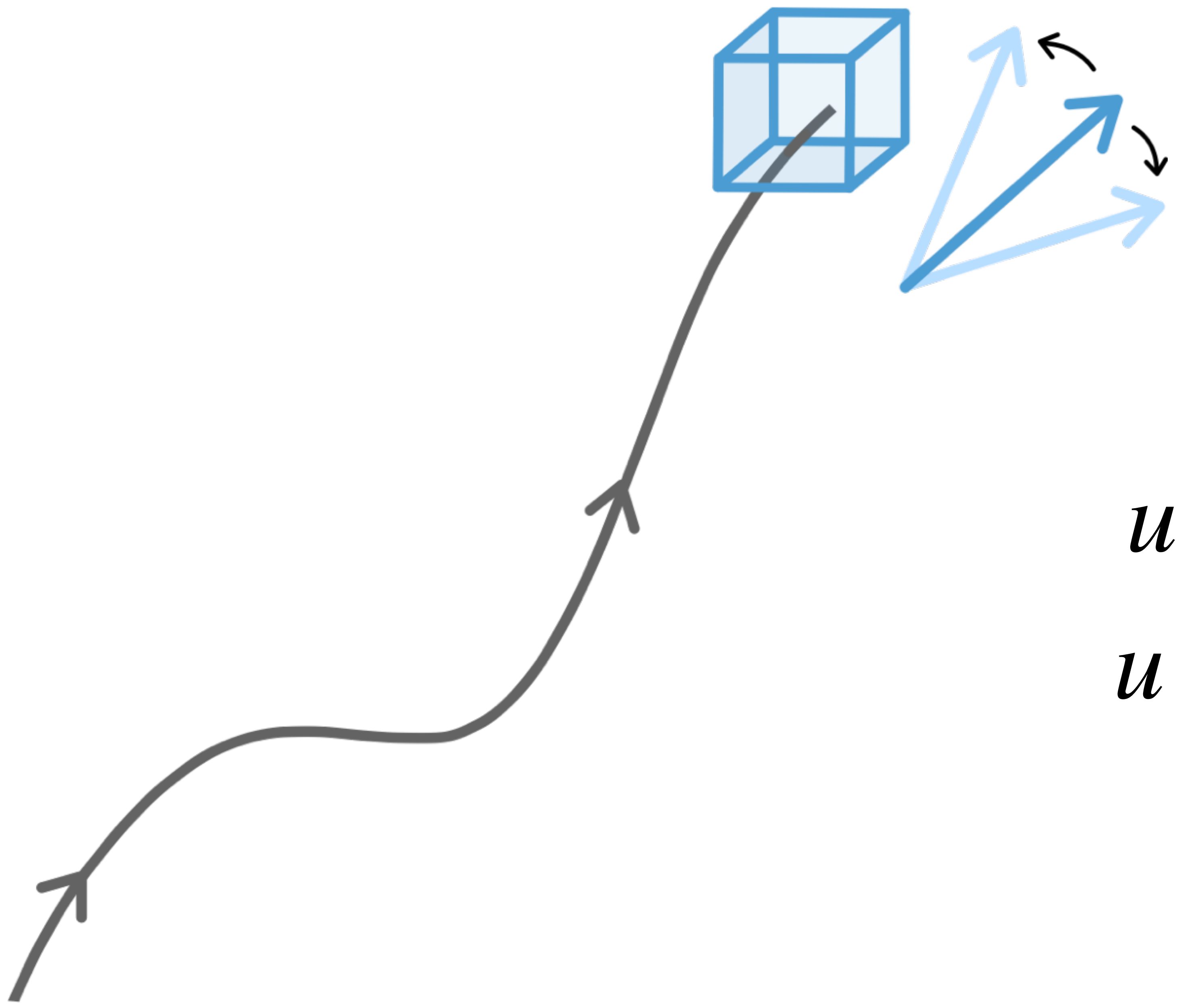
$$(\partial_t + v \cdot \nabla)\, \psi_i = \ldots$$
$$(\partial_t + v \cdot \nabla)\, G_n = \ldots$$



There is a global time for every observer.
All correlators G_n can be measured at the same time in the same frame (lab).

Lagrangian specification

more convenient for relativistic theory



$$u = u(\psi)$$

$$u \cdot \partial\, \psi_i = \ldots$$
$$u \cdot \partial G_n = \ldots$$

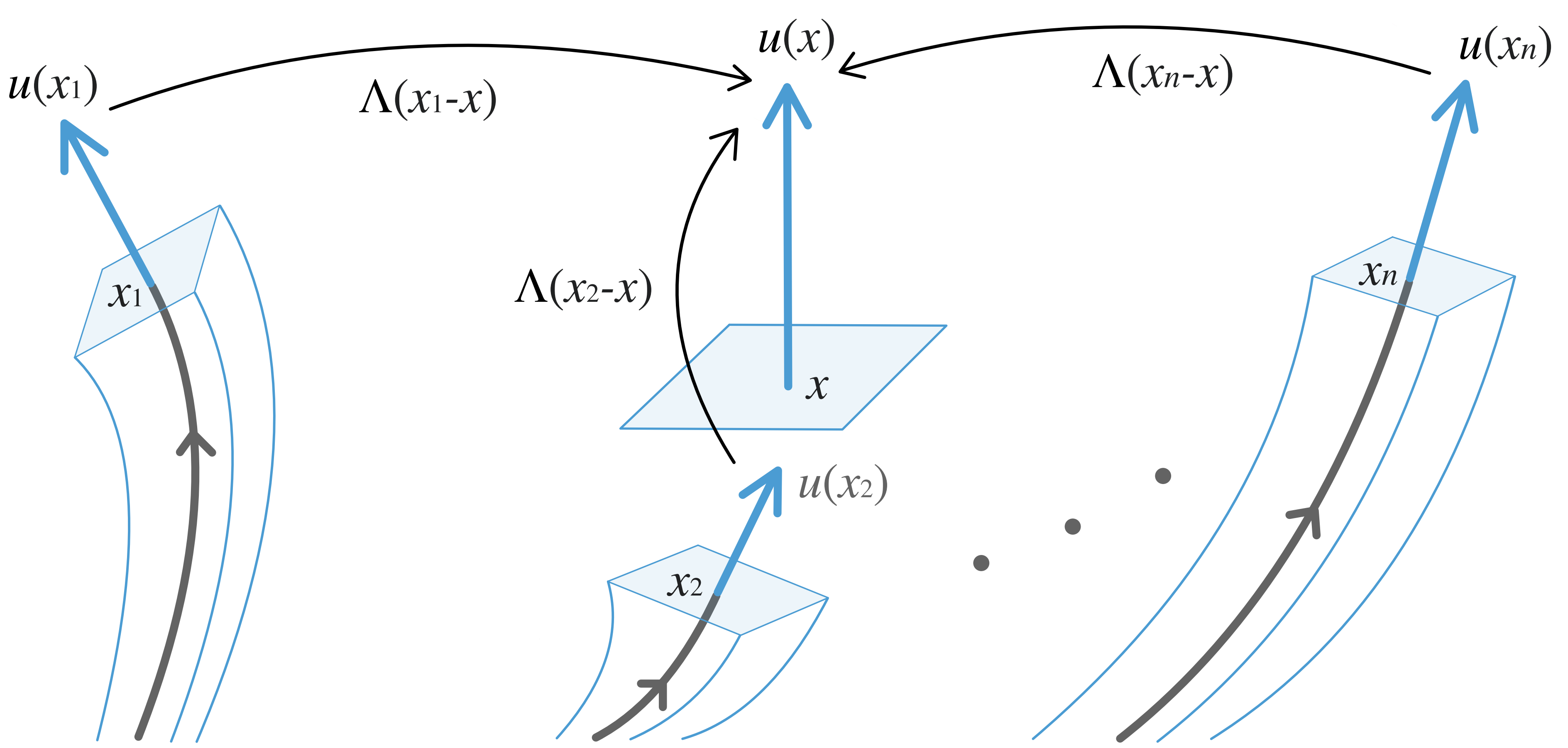
Each fluid cell has its own clock (proper time).
How to define the analogous *equal-time* correlator G_n in relativistic theory?

Confluent formulation: correlator and derivative

- Confluent formulation: covariant description for the comoving fluctuations.

See XA et al, 2212.14029 for more details

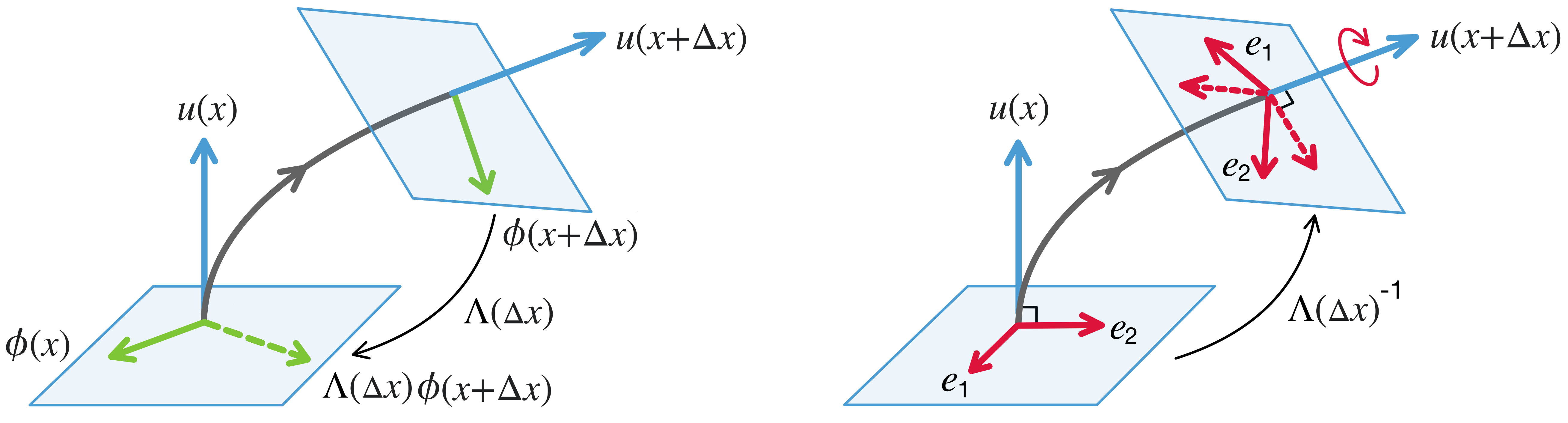
Confluent correlator $\bar{G}$



$$\bar{G}_{i_1\ldots i_n} = \Lambda_{i_1}^{\ j_1}(x - x_1)\ldots\Lambda_{i_n}^{\ j_n}(x - x_n)\bar{G}_{j_1\ldots j_n}$$

boost all fields (measured at their own local rest frame) to one common frame (chosen at their midpoint)

Confluent derivative $\bar{\nabla}$



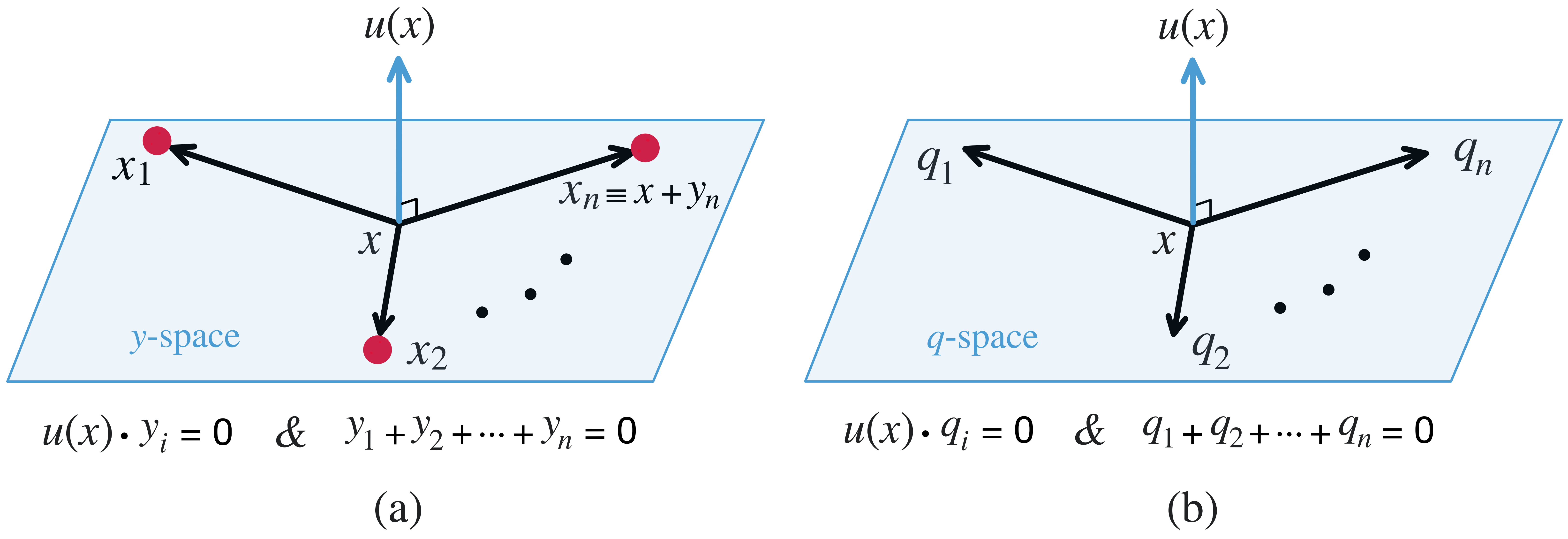
$$\bar{\nabla}_\mu \bar{G}_{i_1\ldots i_n} = \partial_\mu \bar{G}_{i_1\ldots i_n} - n\left(\mathring{\omega}^a_{\mu b}\, y_1^b \partial_a^{(y_1)} \bar{G}_{i_1\ldots i_n} + \bar{\omega}^{j_1}_{\mu i_1} \bar{G}_{j_1\ldots i_n}\right)_{\text{perm.}}$$

the frame at midpoint moves accordingly as the n points move, the difference of a given field before and after the movement is calculated in one same frame, with the equal-time constraint preserved by introducing the local triad e^μ_a with $a = 1,2,3$

Confluent formulation: Wigner function

- The confluent n-pt Wigner transform between x-independent variables $y^a = e^a_\mu(x)\, y^\mu$ and q^a with $a = 1,2,3$. XA et al, 2212.14029

$$W_n(x; q_1^a, \ldots, q_n^a) = \int \prod_{i=1}^{n} \left(d^3 y_i^a \, e^{-i q_{ia} y_i^a}\right) \delta^{(3)}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^a\right) \bar{G}_n(x + e_a y_1^a, \ldots, x + e_a y_n^a)$$



Confluent fluctuation evolution equations

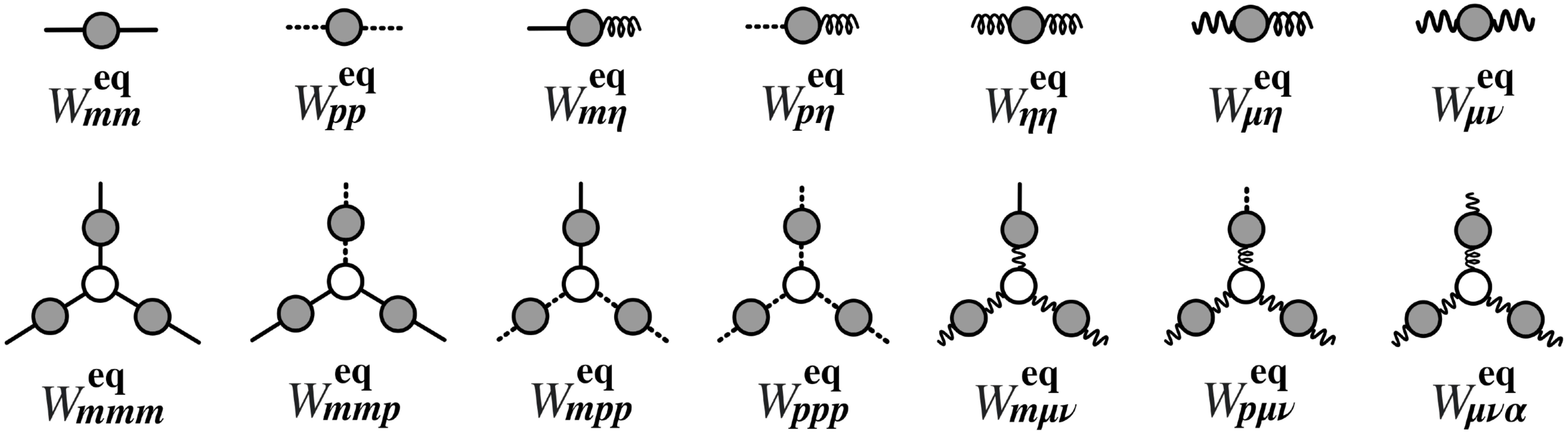
- Fluctuation evolution equations in the *impressionistic* form: XA et al, in progress

$$\mathscr{L} W_n = i c_s q (W_n - \ldots) - \gamma q^2 (W_n - \ldots) - \partial \psi W_n + \ldots \quad \text{where} \quad \mathscr{L} = u \cdot \bar{\nabla}_x + f \cdot \nabla_q$$

sound — dissipation — background gradient

of which the solutions match thermodynamics with entropy $S(m, p, u_\mu, \eta)$.

m: entropy per baryon; p: pressure; η: Lagrange multiplier for $u^2 = -1$.



Equilibrium solutions in diagrammatic representation

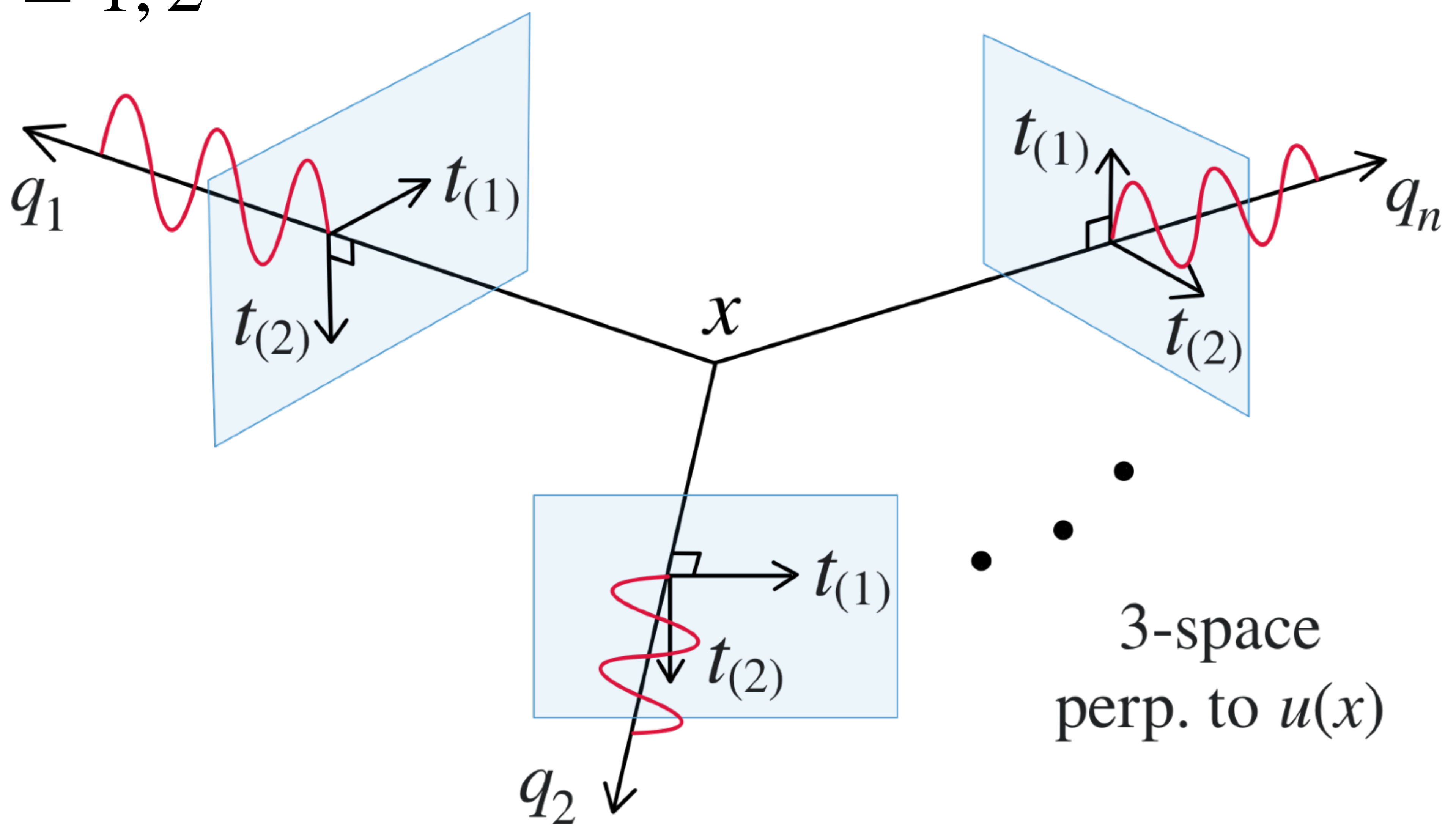
For $\phi = (\delta m, \delta p, \delta u_\mu)$, there are 21+56+126=**203** equations (for the 2-pt, 3-pt and 4-pt correlators) to solve——bite off more than one can chew!

Rotating phase approximation

- Step 1: choose a set of new bases in Fock space s.t. the ideal hydrodynamic equations are diagonalized with eigenvalues $\lambda_\pm(q) = \pm c_s|q|$, $\lambda_m(q) = \lambda_{(i)}(q) = 0$.

$$\phi = \begin{pmatrix} \phi_m \\ \phi_p \\ \phi_\mu \end{pmatrix} = \begin{pmatrix} \delta m \\ \delta p \\ \delta u_\mu \end{pmatrix} \longrightarrow \Phi = \begin{pmatrix} \Phi_m \\ \Phi_\pm \\ \Phi_{(i)} \end{pmatrix} \sim \begin{pmatrix} \delta m \\ \delta p \pm c_s w \hat{q} \cdot \delta u \\ t_{(i)} \cdot \delta u \end{pmatrix} \quad i = 1, 2$$

NB: n-pt correlators are analogous to n-particle quantum states lying in the Fock space.



- Step 2: for n-pt correlators $W_{\Phi_1\ldots\Phi_n}(q_1,\ldots,q_n)$,

$$\text{if } \sum_{i=1}^{n} \lambda_{\Phi_i}(q_i) \begin{cases} = 0 & \longrightarrow \quad \text{slow mode (kept)} \\ \neq 0 & \longrightarrow \quad \text{fast mode (averaged out)} \end{cases}$$

E.g., $W_{+-}(q_1, q_2)$ is a slow mode since $\lambda_+(q_1) + \lambda_-(q_2) = c_s(|q_1| - |q_2|) = 0$;
$W_{+++}(q_1, q_2, q_3)$ is *not* a slow mode since $\lambda_+(q_1) + \lambda_+(q_2) + \lambda_+(q_3) = c_s(|q_1| + |q_2| + |q_3|) \neq 0$.

As a result, we end up with 7+10+15=**32** equations to solve.

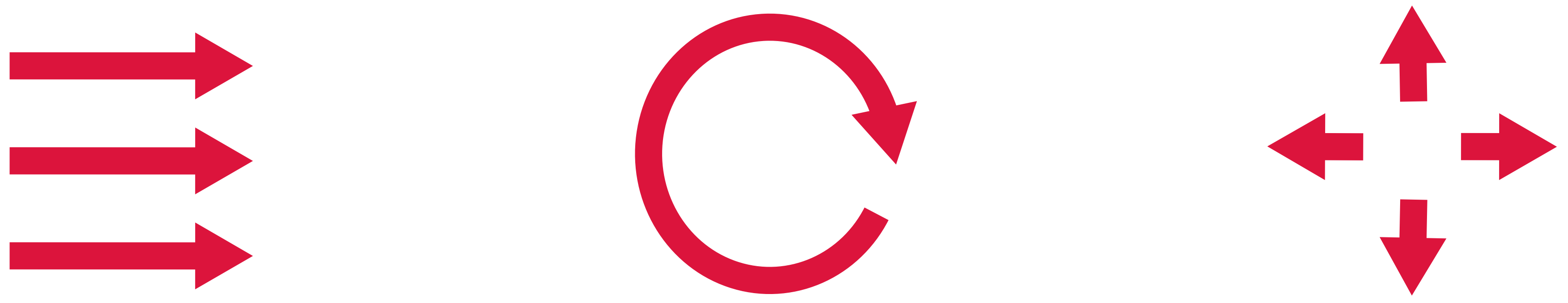
E.g., the 7 independent 2-pt slow modes are W_{mm}, $W_{m(i)}$, $W_{(i)(j)}$, W_{+-}.

Hydro-kinetic equations

- The equation for W_{+-} has a kinetic interpretation:

$$\mathscr{L}W_{+-} \equiv \left((u + c_s\hat{q}) \cdot \bar{\nabla}_x + f \cdot \nabla_q\right) W_{+-} = -\gamma q^2 \left(W_{+-} - \frac{T}{E}\right)$$

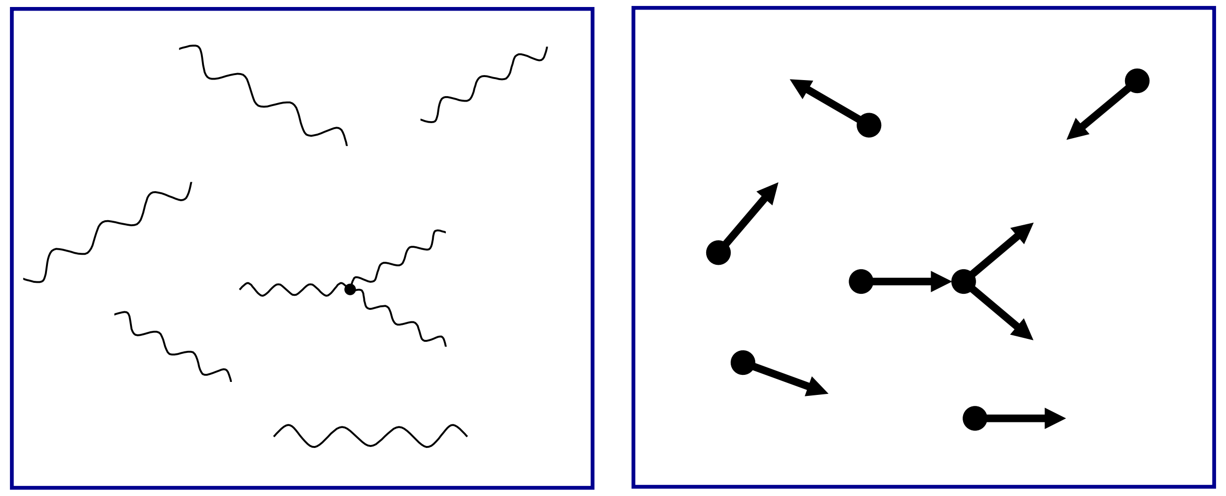
$$f_\mu = Ea_\mu + 2Ec_s\hat{q}^\nu\omega_{\nu\mu} + q_\nu\nabla_\mu^\perp u^\nu + \nabla_\mu E$$



Inertial Coriolis "Hubble"

Phonons move on top of an arbitrary fluid with acceleration, rotation and expansion XA et al, 1902.09517

Bose-Einstein (phonon) distribution $n = \dfrac{1}{e^{E/T} - 1}$ at high T



Courtesy of Schafer

"Finally, after about six months of work off and on, all the pieces suddenly fitted together, producing miraculous cancellation, and I was staring at the amazingly simple final result." C.N. Yang

Fluctuation feedback

- Fluctuations give feedback to the bare quantities order by order in gradient expansion:

$$T_{\mu\nu}^{\text{physical}} = \underbrace{T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} + \ldots}_{\text{bare}} + \underbrace{\delta T_{\mu\nu}(\{G_n\})}_{\text{fluctuation}}$$

$$= \underbrace{T_{\mu\nu}^{R(0)} + T_{\mu\nu}^{R(1)} + T_{\mu\nu}^{R(2)}}_{\text{renormalized}} + \underbrace{\widetilde{T}_{\mu\nu}^{(3/2)} + \widetilde{T}_{\mu\nu}^{(3)} + \widetilde{T}_{\mu\nu}^{(9/2)} + \ldots}_{\text{long-time tails}}$$

where $G_n(x) \sim \int d^3q_1 \ldots d^3q_n \delta^{(3)}(q_1 + \ldots + q_n) W_n(x, q_1, \ldots, q_n)$

↑

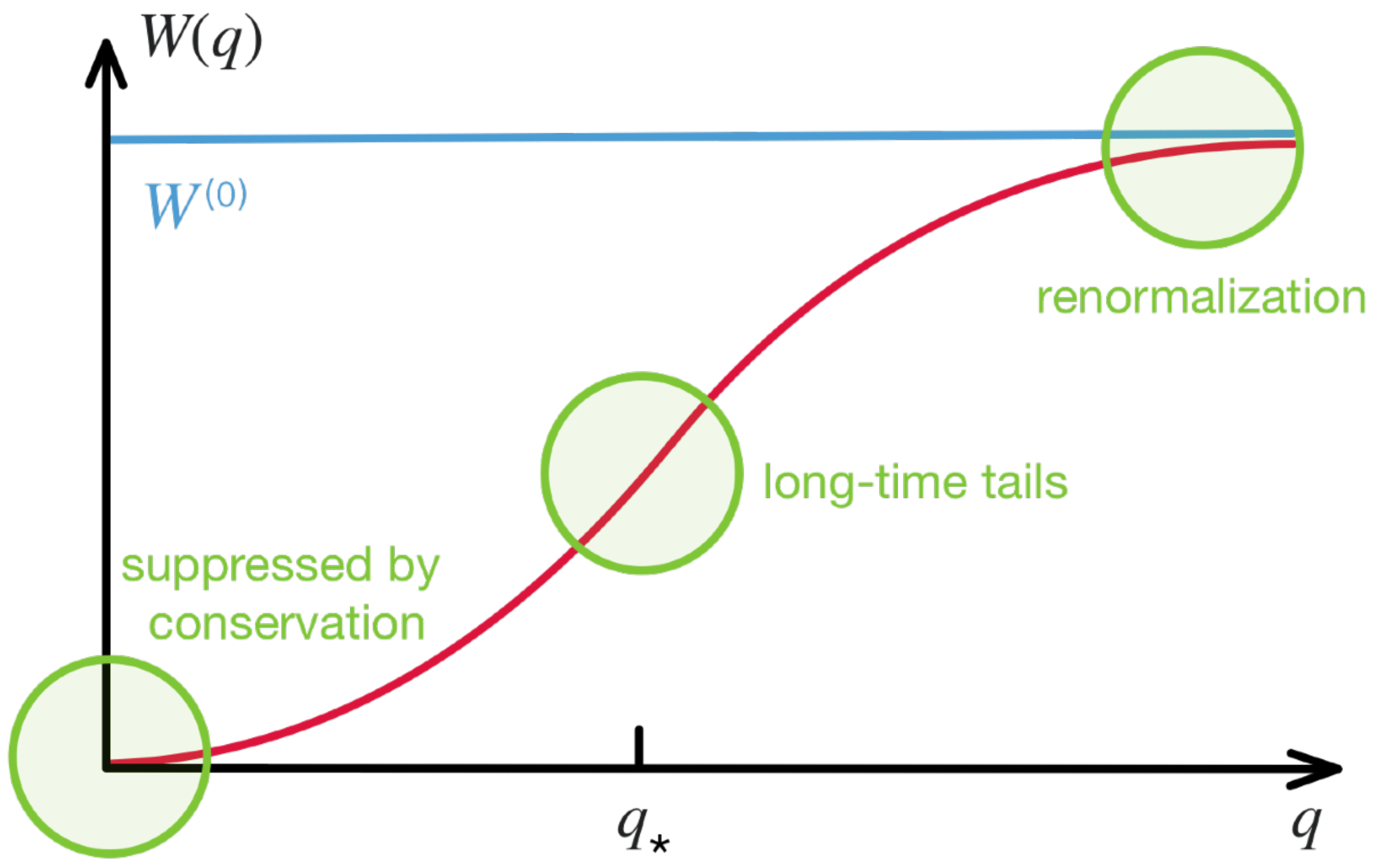
need the solutions from equations for Wigner functions

Renormalization

- Equation for 2-pt functions under RPA:

$$\mathscr{L}W(q) = -\gamma q^2 (W(q) - W^{(0)}) - \partial\psi W(q)$$

with asymptotic solutions



$$W(q) = \frac{\gamma q^2 W^{(0)}}{-i\omega + \gamma q^2 + \partial\psi} = \begin{cases} W^{(0)}\left(1 - \dfrac{-i\omega + \partial\psi}{\gamma q^2} + \ldots\right), & \gamma q^2 \gg \omega, \partial\psi \\ W^{(0)}\dfrac{\gamma q^2}{-i\omega + \partial\psi}\left(1 - \dfrac{\gamma q^2}{-i\omega + \partial\psi} + \ldots\right), & \gamma q^2 \ll \omega, \partial\psi \end{cases}$$

- Perturbation analysis for $W = W^{(0)} + W^{(\mathrm{neq})}$ where $W^{(\mathrm{neq})} = W^{(1)} + \ldots$ gives:

$$W^{(1)} \sim \frac{\partial\psi}{\gamma q^2} \implies G^{(1)} = \int^{\Lambda} d^3q W^{(1)} \sim \frac{\Lambda}{\gamma}\partial\psi \longrightarrow$$ renormalize transport coefficients (regularize infinite noise analytically)

E.g.,

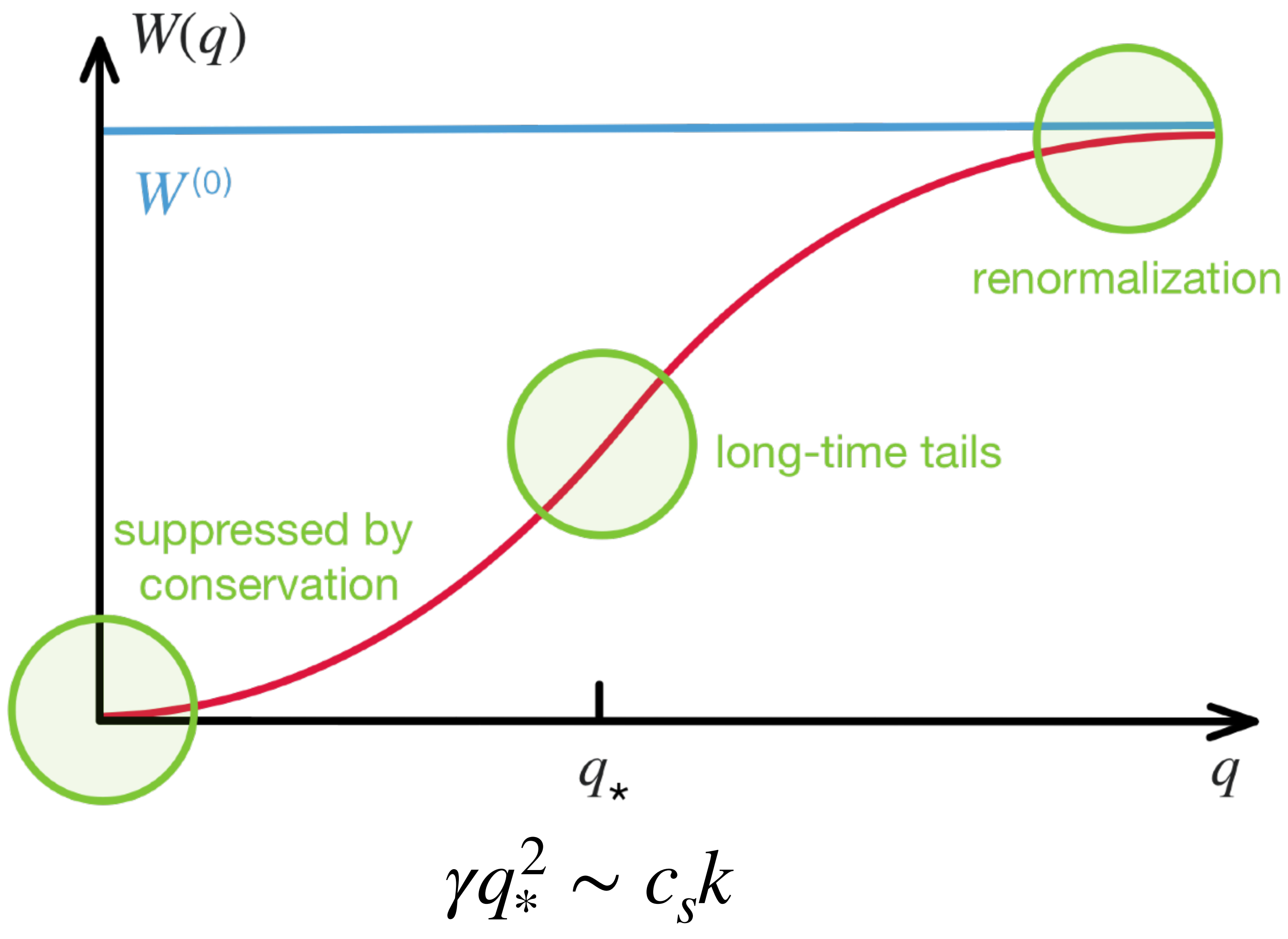
$$\eta_R = \eta + \frac{T\Lambda}{30\pi^2}\left(\frac{1}{\gamma_L} + \frac{7}{2\gamma_\eta}\right), \quad \zeta_R = \zeta + \frac{T\Lambda}{18\pi^2}\left(\frac{1}{\gamma_L}(1 - 3\dot{T} + 3\dot{c}_s)^2 + \frac{2}{\gamma_\eta}(1 - 3(\dot{T} + c_s^2)/2)^2 + \frac{9}{4\gamma_\lambda}(1 - \dot{c}_p)^2\right), \quad \lambda_R = \lambda + \frac{T^2 n^2 \Lambda}{3\pi^2 w^2}\left(\frac{c_p T}{(\gamma_\eta + \gamma_\lambda)w} + \frac{c_s^2}{2\gamma_L}\right)$$

Long-time tails

- The remaining non-equilibrium part of 2-pt function:

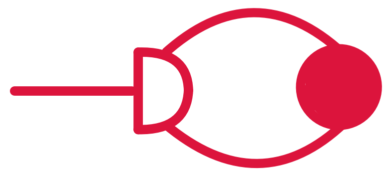
$$\widetilde{W} = W^{(\mathrm{neq})} - W^{(1)} \sim \underbrace{\frac{\partial\psi}{-i\omega + \gamma q^2 + \partial\psi} - \frac{\partial\psi}{\gamma q^2}}_{\text{subtracting local divergence}}$$

$$\Longrightarrow \quad \widetilde{G} = \int_q \widetilde{W} \sim \frac{\partial\psi}{\gamma^{3/2}}(i\omega + \partial\psi)^{1/2} \sim q_*^3 \sim k^{3/2}$$



- Generically, for arbitrary n,

$$\widetilde{G}_n(x) = \int \underbrace{d^3q_1 \ldots d^3q_n \delta^{(3)}(q_1 + \ldots + q_n)}_{n-1 \text{ independent } q \text{ integration}} \widetilde{W}_n(x, q_1, \ldots, q_n) \sim \epsilon^{n-1} \sim q_*^{3(n-1)} \sim k^{3(n-1)/2}$$

the leading contribution ($k^{3/2} \sim t^{-3/2}$) results from 2-pt correlators via  .

E.g., $\Pi(\omega) = \zeta(\omega)\partial \cdot u \sim \xi^3 \left(1 - (\omega\xi^3)^{1/2}\right) \partial \cdot u$

Interplay with background in the critical regime

- Different slow modes may relax with different time scales near critical point due to *critical slowing down*. Stephanov, 1104.1627; Berdnikov et al, 9912274; XA, 2003.02828

E.g., for $\Gamma(q) = Dq^2 = \lambda\alpha' q^2$ where $\lambda \sim \xi, \alpha' \sim \xi^{-2}$, we have $\tau_{\rm rel} = 1/\Gamma(\xi^{-1}) \sim \xi^3$.

- Hydro+/++: hydrodynamics with parametrically slow modes (e.g., $\Gamma(q) \sim \xi^{-3} \ll \omega$)

$$\begin{cases} \partial_\mu T^{\mu\nu}_{\rm physical}(\psi_R, \widetilde{W}) = 0 \\ \mathscr{L}\widetilde{W}(q) = -\Gamma(q)\widetilde{W}(q) - \partial\psi_R \widetilde{W}(q) \end{cases}$$

- In the critical regime ($\Gamma_\Pi \sim \xi^{-3}$), Muller-Israel-Stewart theory is an example of the single-mode Hydro+, e.g., Stephanov et al, 1712.10305; Du et al, 2107.02302; Abbasi et al, 2112.14747

$$\begin{cases} \partial_\mu T^{\mu\nu}(\psi, \Pi) = 0 \\ \dot{\Pi} = -\Gamma_\Pi(\Pi - \Pi_{\rm NS}) \end{cases}$$

Recap

- Various approaches for fluctuating hydro have been developed, each with its own pros and cons, and can be connected with others.

- For the first time we developed a deterministic framework for fluctuation dynamics, and formulated it covariantly for hydrodynamics.

Outlook

- Hydrodynamic attractors. Work in progress with Spalinski

- Other fluid system: cosmo/astrophysics, SHD/MHD, etc. Extendable to many problems!

- Numerical implementation? We need efforts from the community!