From Conformal Symmetry to Feynman Diagrams

Adam Bzowski

UNIVERSITY OF SILESIA
IN KATOWICE

based on a series of works with **Kostas Skenderis** and **Paul McFadden**

Polish Particle and Nuclear Theory Summit, Kraków, November 24, 2023

Original motivation

- In [1304.7760] we initiated a study of conformal 3-point functions in momentum space.
- Conformal theories, as fixed points of RG flows, constitute the backbone of all QFTs.
- Correlation functions in conformal theories are highly restricted. A scalar 3-point function in position space takes form

$$\langle \mathcal{O}_1(\boldsymbol{x}_1)\mathcal{O}_2(\boldsymbol{x}_2)\mathcal{O}_3(\boldsymbol{x}_3)\rangle = \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}}.$$

- Momentum-space expressions before our work were unwieldy: Appell's F_4 generalized hypergeometric was used.
- Tensorial correlators and the structure of divergences obscured.

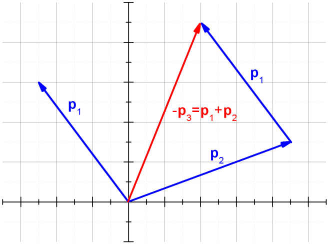
Triple-K integrals

In [1304.7760] and subsequent papers we showed that all conformal 3-point functions can be expressed in terms of **triple-K** integrals:

$$I_{\alpha\{\beta_1\beta_2\beta_3\}} = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \int_0^\infty \mathrm{d}z \, z^\alpha K_{\beta_1}(p_1 z) K_{\beta_2}(p_2 z) K_{\beta_3}(p_3 z),$$

where K_β is Bessel K function.

- $p_j = |\boldsymbol{p}_j|$ denote momenta magnitudes.
- We work in Euclidean signature.



Outline

Outline of my talk:

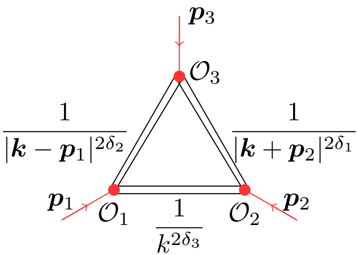
1. What triple-K integrals can do for you?
2. How to express Feynman integrals in terms of triple-K integrals and why it's convenient.
3. How to deal with divergences and renormalization with ease.
4. How to explicitly evaluate a large class of previously unknown diagrams.
5. Two examples: 3-point function of composite operators.
6. Summary.

What triple-K integrals can do for you?

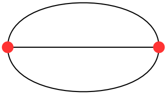
Any conformal 3-point functions such as:

$$\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle$$

Scalar 3-point function with generalized, massless propagators:

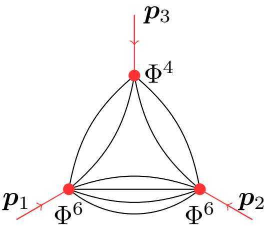


Massive vacuum-to-vacuum sunset diagrams [Groote, Körner, Pivovarov, '98, '00], [Groote, Körner, '19]:

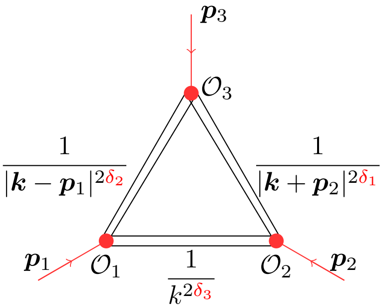


$$\frac{1}{k^2 + m_j^2}$$

3-point functions of composite operators in free massless theories:



Scalar diagrams (1/3)



$$\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\!\rangle = \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\frac{1}{k^{2\delta_3}|\boldsymbol{k}-\boldsymbol{p}_1|^{2\delta_2}|\boldsymbol{k}+\boldsymbol{p}_2|^{2\delta_1}}$$

where, due to momentum conservation,

$$\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle = (2\pi)^d\delta\left(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3\right)\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\!\rangle.$$

Scalar diagrams (2/3)

All momentum integrals of the form

$$\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \frac{k^{\mu_1} \ldots k^{\mu_r}}{k^{2\delta_3} |\boldsymbol{k} - \boldsymbol{p}_1|^{2\delta_2} |\boldsymbol{k} + \boldsymbol{p}_2|^{2\delta_1}}$$

reduce to a sum of triple-K integrals.

- For example, for scalar integrals

$$\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \frac{1}{k^{2\delta_3} |\boldsymbol{k} - \boldsymbol{p}_1|^{2\delta_2} |\boldsymbol{k} + \boldsymbol{p}_2|^{2\delta_1}} =$$
$$= \frac{2^{4-\frac{3d}{2}}}{\pi^{\frac{d}{2}}} \times \frac{I_{\frac{d}{2}-1\{\frac{d}{2}+\delta_1-\delta_t, \frac{d}{2}+\delta_2-\delta_t, \frac{d}{2}+\delta_3-\delta_t\}}}{\Gamma(d-\delta_t)\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\delta_3)},$$

where $\delta_t = \delta_1 + \delta_2 + \delta_3$.

Scalar diagrams (3/3)

- α and β parameters are related to d and δ_j. For the scalar integral,

$$\alpha = \frac{d}{2} - 1, \qquad \beta_j = \frac{d}{2} + \delta_j - \delta_t.$$

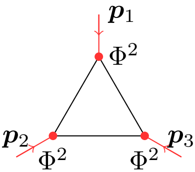
- β-parameters related to conformal dimensions

$$\beta_j = \Delta_j - \frac{d}{2}.$$

- Examples of conformal dimensions of free or conformal theories,

Operator	**Dimension Δ**
Free scalar Φ	$d/2 - 1$
Free fermion Ψ	$d/2 - 1/2$
$:\Phi^n:$ in $d = 4$	n
Conserved current J^μ	$d - 1$
Stress tensor $T_{\mu\nu}$	d

Example



- In terms of triple-K integrals it reads

$$= \int \frac{\mathrm{d}^4 \boldsymbol{k}}{(2\pi)^4} \frac{1}{k^2 |\boldsymbol{k} - \boldsymbol{p}_1|^2 |\boldsymbol{k} + \boldsymbol{p}_2|^2} = \frac{1}{4\pi^2} I_{1\{000\}}.$$

- Its value is known [t 'Hooft '78], [Davydychev '92],

$$I_{1\{000\}} = \frac{1}{2\sqrt{\lambda}} \left[\frac{\pi^2}{6} - 2 \ln \frac{p_1}{p_3} \ln \frac{p_2}{p_3} + \ln \left(-X \frac{p_2}{p_3} \right) \ln \left(-Y \frac{p_1}{p_3} \right) - \mathrm{Li}_2 \left(-X \frac{p_2}{p_3} \right) - \mathrm{Li}_2 \left(-Y \frac{p_1}{p_3} \right) \right].$$

where $-\lambda$ is the Källen function and

$$-\lambda = (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3),$$

$$X = \frac{p_1^2 - p_2^2 - p_3^2 + \sqrt{\lambda}}{2p_2 p_3}, \qquad Y = \frac{p_2^2 - p_1^2 - p_3^2 + \sqrt{\lambda}}{2p_1 p_3}.$$

Why bother?

Triple-K representation is extremely convenient:

- Integrals become elementary for odd d and integral Δ_j (or δ_j).
- Closed-form, analytic expressions for even d and integral Δ_j (or δ_j).
- All such integrals expressible through a single master integral, $I_{0\{111\}}$ and through a simple reduction scheme.
- Developed a Mathematica package `TripleK` in [2005.10841].
- Identities satisfied by Bessel functions simplify other reduction schemes (IBP, [Davydychev, '92], [Davydychev, Tausk, '96], [Tarasov, '96], [Kniehl, Tarasov, '12]).
- 1D v. dD integral: divergences, scheme-dependence and renormalization much simpler to analyze.

The master integral

- The master integral $I_{0\{111\}}$ is

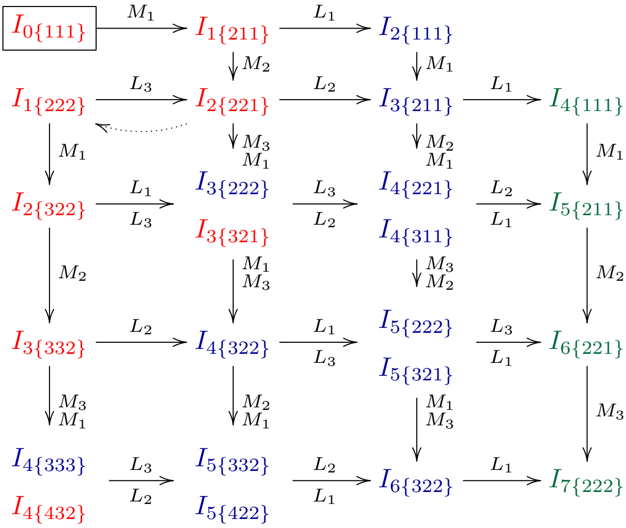
```
In[1]:= << TripleK`

In[2]:= i[0 + ϵ, {1, 1, 1}] // KEvaluate
```

$$\text{Out[2]}= \frac{-p_1^2 - p_2^2 - p_3^2}{2\,\epsilon^2} + \frac{-\frac{1}{4}\,(1 - 2\,\text{EulerGamma} + 2\,\text{Log}[2])\,(p_1^2 + p_2^2 + p_3^2) + \frac{1}{4}\,(\text{Log}[p_1^2]\,p_1^2 + \text{Log}[p_2^2]\,p_2^2 + \text{Log}[p_3^2]\,p_3^2)}{\epsilon} +$$
$$\Big(-\frac{1}{8}\,\big((-1 + \text{EulerGamma} - \text{Log}[2])^2 + (\text{EulerGamma} - \text{Log}[2])^2\big)\,(p_1^2 + p_2^2 + p_3^2) +$$
$$\frac{1}{8}\,(1 - 2\,\text{EulerGamma} + 2\,\text{Log}[2])\,(\text{Log}[p_1^2]\,p_1^2 + \text{Log}[p_2^2]\,p_2^2 + \text{Log}[p_3^2]\,p_3^2) +$$
$$\frac{1}{16}\,\big(\text{Log}[p_2^2]\,\text{Log}[p_3^2]\,(p_1^2 - p_2^2 - p_3^2) + \text{Log}[p_1^2]\,\text{Log}[p_3^2]\,(-p_1^2 + p_2^2 - p_3^2) + \text{Log}[p_1^2]\,\text{Log}[p_2^2]\,(-p_1^2 - p_2^2 + p_3^2)\big) -$$
$$\frac{\text{NL}\,\sqrt{\lambda}}{8}\Big) + O[\epsilon]^1$$

- where

$$\text{NL} = \frac{\pi^2}{6} - 2\ln\frac{p_1}{p_3}\ln\frac{p_2}{p_3} + \ln\left(-X\frac{p_2}{p_3}\right)\ln\left(-Y\frac{p_1}{p_3}\right) - \text{Li}_2\left(-X\frac{p_2}{p_3}\right) - \text{Li}_2\left(-Y\frac{p_1}{p_3}\right)$$

Reduction scheme



The package

- Evaluate Feynman diagrams directly

```
In[3]:= LoopIntegral[3 - 2 ϵ, {3, 2, 1}][1, k]
```

Out[3]= $$\int \frac{d^{3-2\epsilon} k}{(2\pi)^{3-2\epsilon}} \frac{1}{k^2 (k-p_1)^4 (k+p_2)^6}$$

```
In[4]:= LoopEvaluate[%] /. ϵ -> 0 // Simplify
```

Out[4]= $$\frac{3\left(5 p_1^6 - 5 p_2^6 + 3 p_2^4 p_3^2 + p_2^2 p_3^4 + p_3^6 - 3 p_1^4 \left(5 p_2^2 + 3 p_3^2\right) + 3 p_1^2 \left(5 p_2^4 + 2 p_2^2 p_3^2 + p_3^4\right)\right)}{128\, p_1^3 p_2^5 p_3^7}$$

- Momentum integrals can be nested.
- The Mathematica package `TripleK` in [2005.10841] available at

  https://triplek.hepforge.org/

  with a short YouTube tutorial

  https://youtu.be/beSt3DEI3fs

Divergences

- Interesting integrals diverge at $z = 0$: these are UV divergences.
- We use generalized dimensional regularization:

$$d \mapsto \hat{d} = d + 2u\epsilon, \qquad \Delta_j \mapsto \hat{\Delta}_j = \Delta_j + (u + v_j)\epsilon,$$

  where ϵ is a regulator and u, v_j some fixed parameters.
- Divergence of the triple-K integral follows from the $z = 0$ end of integration: simple criterion and simple extraction.
- The triple-K integral $I_{\alpha\{\beta_1\beta_2\beta_3\}}$ has a pole if and only if there exists a choice of signs $\sigma_1, \sigma_2, \sigma_3 = \pm$ such that

$$\alpha + 1 + \sum_{j=1}^{3} \sigma_j |\beta_j| = -2n_{\sigma_1\sigma_2\sigma_3},$$

  for a non-negative integer $n_{\sigma_1\sigma_2\sigma_3}$.

Renormalization

- The singularity condition can be rewritten as

$$\begin{matrix} - : \\ + : \end{matrix} \quad \left\{ \begin{matrix} d - \Delta_1 \\ \Delta_1 \end{matrix} \right\} + \left\{ \begin{matrix} d - \Delta_2 \\ \Delta_2 \end{matrix} \right\} + \left\{ \begin{matrix} d - \Delta_3 \\ \Delta_3 \end{matrix} \right\} + 2n_{\sigma_1 \sigma_2 \sigma_3} = d.$$

- For each instance there exists a corresponding counterterm,

$$S_{\text{ct}} \sim \mu^{-\#\epsilon} \int \mathrm{d}^{\hat{d}} \boldsymbol{x} \, X_1^{(\sigma_1)} X_2^{(\sigma_2)} X_3^{(\sigma_3)} \, \partial^{2n_{\sigma_1 \sigma_2 \sigma_3}},$$

where

$$X_j^{(+)} = \mathcal{O}_j, \qquad X_j^{(+)} = \phi_j (\text{i.e., source for } \mathcal{O}_j).$$

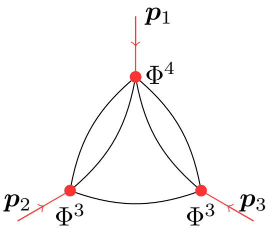
- Only sources: $S_{\text{ct}} \sim \int \Box^{k_1} \phi_1 \Box^{k_2} \phi_2 \Box^{k_3} \phi_3$: anomalies.
- Source redefinition: $S_{\text{ct}} \sim \int \Box^{k_1} \phi_1 \Box^{k_2} \phi_2 \Box^{k_3} \mathcal{O}_3$: beta functions.
- See [1510.08442] for the detailed discussion.

Example 1 (1/3)

- Consider the 3-point function

$$\langle\!\langle :\Phi^4: (\boldsymbol{p}_1) :\Phi^3: (\boldsymbol{p}_2) :\Phi^3: (\boldsymbol{p}_3)\rangle\!\rangle$$

in the theory of free massless scalars in $d = 4$.



- In dimreg $\hat{d} = 4 - \epsilon$ and unaltered propagators we get

$$I = \frac{27 \times 2^{5+\frac{\epsilon}{2}}\Gamma^4(1-\frac{\epsilon}{2})}{(4\pi)^{6-\frac{3\epsilon}{2}}\Gamma(3-2\epsilon)\Gamma^2(2-\epsilon)} \times I_{1-\frac{\epsilon}{2}\{2-\frac{3\epsilon}{2},1-\epsilon,1-\epsilon\}}.$$

Example 1 (2/3)

- The amplitude is divergent

$$\langle\langle :\Phi^4: (\boldsymbol{p}_1) :\Phi^3: (\boldsymbol{p}_2) :\Phi^3: (\boldsymbol{p}_3)\rangle\rangle_{\text{reg}} = -\frac{9}{256\pi^6\epsilon^2}(p_2^2 + p_3^2)$$
$$+ \frac{9}{512\pi^6\epsilon}\left[-p_1^2 + 3p_2^2 \ln p_2^2 + 3p_3^2 \ln p_3^2 \right.$$
$$\left. +(p_2^2+p_3^2)\left(-10 + 3\gamma_E - 3\ln(4\pi)\right)\right] + O(\epsilon^0).$$

- Introduce sources: $\phi_{[0]}$ for $:\Phi^4:$ and $\phi_{[1]}$ for $:\Phi^3:$.
- Counterterm actions reads

$$S_{\text{ct}}^{(3)} = \int \mathrm{d}^{4-\epsilon}\boldsymbol{x} \left[a_0 \phi_{[0]}\phi_{[1]} :\Phi^3: \mu^{-\epsilon} + (a_1\phi_{[0]}\phi_{[1]}\Box\phi_{[1]} + a_2\phi_{[1]}^2\Box\phi_{[0]})\mu^{-3\epsilon}\right]$$

with some parts of the counterterm constants a_j fixed by the divergences.

Example 1 (3/3)

- When the dust settles the renormalized amplitude reads

$$\langle\!\langle :\Phi^4: (\boldsymbol{p}_1) :\Phi^3: (\boldsymbol{p}_2) :\Phi^3: (\boldsymbol{p}_3) \rangle\!\rangle_{\text{ren}} = -\frac{27}{256\pi^6}\left(2 - p_1 \frac{\partial}{\partial p_1}\right)\left(\frac{1}{4} J^2 I_{1\{000\}}\right)$$
$$+ \frac{27}{2048\pi^6}\left[(p_2^2 - p_3^2)\ln\frac{p_1^2}{\mu^2}\left(\ln\frac{p_3^2}{\mu^2} - \ln\frac{p_2^2}{\mu^2}\right) - (p_2^2 + p_3^2)\ln\frac{p_2^2}{\mu^2}\ln\frac{p_3^2}{\mu^2}\right.$$
$$\left. + (p_1^2 - p_2^2)\ln\frac{p_3^2}{\mu^2} + (p_1^2 - p_3^2)\ln\frac{p_2^2}{\mu^2} + p_1^2\right]$$
$$+ a_0'\left(p_2^2 \ln\frac{p_2^2}{\mu^2} + p_3^2 \ln\frac{p_3^2}{\mu^2}\right) + a_1'(p_2^2 + p_3^2) + a_2' p_1^2.$$

- The constants a_0', a_1', a_2' are related to scheme-dependent parts of the renormalization constants a_0, a_1, a_2.
- Callan-Symanzik equation reads

$$\mu \frac{\partial}{\partial \mu} \langle\!\langle :\Phi^4: (\boldsymbol{p}_1) :\Phi^3: (\boldsymbol{p}_2) :\Phi^3: (\boldsymbol{p}_3) \rangle\!\rangle_{\text{ren}}$$
$$= \frac{27}{512\pi^6}\left[-p_1^2 + p_2^2 \ln\frac{p_2^2}{\mu^2} + p_3^2 \ln\frac{p_3^2}{\mu^2} - 2a_0'(p_2^2 + p_3^2)\right].$$

- See [1510.08442] for the detailed discussion.

Example 2 (1/3)

Problem: find the most general structure of $\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)J^{\mu_2}(\boldsymbol{p}_2)J^{\mu_3}(\boldsymbol{p}_3)\rangle$ in conformal theories in momentum space.

- Problem solved in position space, [Osborn, Petkou '94].
- Very difficult to Fourier transform.

Our novel approach solves the problem in terms of triple-K integrals.

- First disentangle longitudinal and traceful part of the correlation function by means of Ward identities.
- Stress-energy tensor $T^{\mu\nu}$ and conserved current J^μ are transverse and traceless on-shell only.
- Define $t_{\mu\nu}$ and j^μ as the transverse-traceless and transverse parts of $T_{\mu\nu}$ and J^μ obeying

$$\partial_\mu t^{\mu\nu} = 0, \qquad \partial_\mu j^\mu = 0$$

exactly.

Example 2 (2/3)

Use projectors to impose transverseness and tracelessness. For example,

$$\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) j^{\mu_2}(\boldsymbol{p}_2) j^{\mu_3}(\boldsymbol{p}_3)\rangle\!\rangle$$
$$= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2)\pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3)\Big[A_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3} + A_2 \delta^{\alpha_2\alpha_3} p_2^{\alpha_1} p_2^{\beta_1}$$
$$+ A_3 \delta^{\alpha_1\alpha_2} p_2^{\beta_1} p_1^{\alpha_3} + A_3(p_2 \leftrightarrow p_3)\delta^{\alpha_1\alpha_3} p_2^{\beta_1} p_3^{\alpha_2}$$
$$+ A_4 \delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}\Big],$$

where

$$\pi^{\mu}_{\alpha}(\boldsymbol{p}) = \delta^{\mu}_{\alpha} - \frac{p^{\mu} p_{\alpha}}{p^2}, \qquad \Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}) = \pi^{\mu}_{(\alpha}\pi^{\nu}_{\beta)} - \frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta}.$$

- Functions $A_j = A_j(p_1, p_2, p_3)$ are called form factors.
- The state-of-the-art decomposition of $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3}\rangle$ prior to this work involved 13 form factors, while the method described here requires only 5.

Example 2 (3/3)

- The most general solution to $\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle$ in $d = 3$ spacetime dimensions reads

$$A_1 = \alpha_1 \frac{2(4p_1 + p_2 + p_3)}{(p_1 + p_2 + p_3)^4},$$

$$A_2 = \frac{2\alpha_1 p_1^2}{(p_1 + p_2 + p_3)^3} + \frac{4\sqrt{\pi}(2p_1 + p_2 + p_3)}{(p_1 + p_2 + p_3)^2} c_J,$$

$$A_3 = \frac{\alpha_1}{(p_1 + p_2 + p_3)^3}\left[-2p_1^2 - p_2^2 + p_3^2 - 3p_1p_2 + 3p_1p_3\right] + \frac{4\sqrt{\pi}(2p_1 + p_2 + p_3)}{(p_1 + p_2 + p_3)^2}$$

$$A_4 = \alpha_1 \frac{(p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(2p_1 + p_2 + p_3)}{2(p_1 + p_2 + p_3)^2} - 2\sqrt{\pi}\left(\frac{2p_1^2}{p_1 + p_2 + p_3} - p_2 - p_3\right) c_J - 4\sqrt{\pi}(p_2 + p_3) c_3 c_J.$$

- For free fermions one finds

$$\alpha_1 = -\frac{1}{24}, \qquad c_J = \frac{1}{32\sqrt{\pi}}, \qquad c_3 = \frac{1}{2}.$$

- See [1711.09105] for details.

Summary

- I presented a novel and powerful method for evaluation of a class of massless Feynman diagrams.
- Use of triple-K integrals simplifies calculations considerably.
- Very convenient for the analysis of divergences and renormalization effects.
- `TripleK` Mathematica package was introduced in and is available at

  https://triplek.hepforge.org/

  with a short YouTube tutorial

  https://youtu.be/beSt3DEI3fs