Progress in small-x evolution using the JIMWLK equation with kinematical constraint

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based on: PK, SoftwareX (2021) arXiv:2009.02045,
PK, Eur. Phys. J. C 82 (2022) 369, arXiv:2111.07427

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JIMWLK evolution equation

Warning notice

Status report: work in progress, no final results yet available.

Basic facts

- JIMWLK equation describes the non-linear small-x evolution
- it uses Wilson lines as fundamental degrees of freedom
- two-point correlation function $\langle U^\dagger(x)U(y)\rangle$ gives the dipole amplitude
- two-point correlation functions with derivatives provide a basis for small-x TMD structure functions
- initial condition corresponds to a configuration of Wilson lines
- numerically useful reformulation as a Langevin equation

JIMWLK evolution equation

LO JIMWLK: Langevin formulation (Rummukainen, Weigert 2004, Lappi, Mantysaari 2014)

$$U(\mathrm{x}, s+\delta s) = \exp\left(-\sqrt{\delta s} \sum_{\mathrm{y}} U(\mathrm{y}, s)(\mathrm{K}(\mathrm{x}-\mathrm{y}) \cdot \xi(\mathrm{y})) U^{\dagger}(\mathrm{y}, s) \right) \times$$

$$\times\, U(\mathrm{x}, s) \times \exp\left(\sqrt{\delta s} \sum_{\mathrm{y}} \mathrm{K}(\mathrm{x}-\mathrm{y}) \cdot \xi(\mathrm{y}) \right).$$

Initial condition from the McLerran-Venugopalan model

Improved implementation

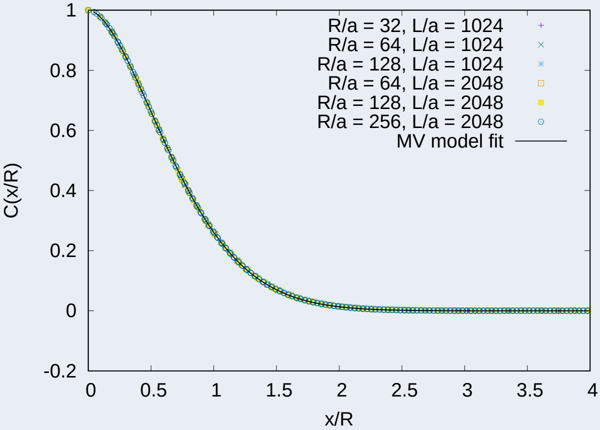


Figure: Volume dependence of the dipole amplitude in the MV model on the torus. The new method shows negligible finite size and lattice spacing effects.

Including the evolution

Including the running coupling constant (Rummukainen, Weigert 2004)

$$U(\mathrm{x}, s+\delta s) = \exp\left(-\sqrt{\delta\sigma}\sum_{\mathrm{y}} U(\mathrm{y},s)(\sqrt{\alpha}\mathrm{K}(\mathrm{x}-\mathrm{y})\cdot\xi(\mathrm{y}))U^{\dagger}(\mathrm{y},s)\right)\times$$

$$\times U(\mathrm{x},s)\times\exp\left(\sqrt{\delta\sigma}\sum_{\mathrm{y}}\sqrt{\alpha}\mathrm{K}(\mathrm{x}-\mathrm{y})\cdot\xi(\mathrm{y})\right)$$

where $\sqrt{\delta s} = \sqrt{\delta\sigma}\sqrt{\alpha(|\mathrm{x}-\mathrm{y}|)}$.

Coupling constant

$$\alpha_s(\mathrm{r}) = \frac{4\pi}{\beta_0 \ln\left\{\left[\left(\frac{R_{\text{initial}}^2\mu_0^2}{R_{\text{initial}}^2\Lambda_{\text{QCD}}^2}\right)^{\frac{1}{c}} + \left(\frac{R_{\text{initial}}^2}{\mathrm{r}^2}\frac{4e^{-2\gamma_E}}{R_{\text{initial}}^2\Lambda_{\text{QCD}}^2}\right)^{\frac{1}{c}}\right]^{c}\right\}},$$

Summary of parameters

$R_{\text{initial}}\Lambda$ with $R_{\text{initial}}g^2\mu \approx 1$, $R_{\text{initial}}\mu_0$, and Λ which provides the units.

LO JIMWLK with running coupling

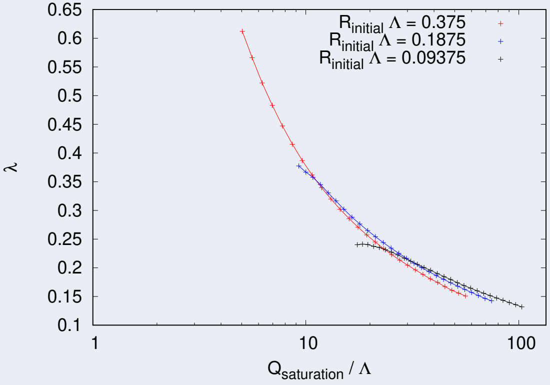


Figure: $R_{\text{initial}}\Lambda$ is the only parameter of the initial condition and of the evolution. Coinciding data from evolution for different values of $R_{\text{initial}}\Lambda$ corresponds to geometrical scaling.

JIMWLK evolution equation with collinear improvement

Collinear improvement

All order resummation of corrections enhanced by kinematical constraints. Known from BFKL studies to be important to correctly describe phenomenology.

Langevin equation formulation (Hatta, Iancu 2016)

At each point of the discretized transverse plane a Wilson line exists with an additional index: the scale at which the final correlator is evaluated.

$$U(\mathrm{x}, R, s+\delta s) = \exp\left(-\sqrt{\delta\varepsilon}\sum_{\mathrm{y}}\sqrt{\alpha_s}\,\theta(s-\rho^R_{\mathrm{xy}})\,U(\mathrm{y},\hat{R},s-\Delta^R_{xy})\left[\mathrm{K}_{\mathrm{xy}}\cdot\xi(\mathrm{y})\right]U^\dagger(\mathrm{y},\hat{R},s-\Delta^R_{xy})\right) \times U(\mathrm{x},R,s) \times \exp\left(\sqrt{\delta\varepsilon}\sum_{\mathrm{y}}\sqrt{\alpha_s}\,\theta(s-\rho^R_{\mathrm{xy}})\,\mathrm{K}_{\mathrm{xy}}\cdot\xi(\mathrm{y})\right),$$

$$\rho^R_{\mathrm{xy}} = \ln\frac{(\mathrm{x}-\mathrm{y})^2}{R^2},\ \Delta^R_{xy} = \theta\big(|x-y|-R\big)\rho^R_{\mathrm{xy}},\ \hat{R} = \max(|x-y|, R),\ s = \varepsilon\alpha_s.$$

Saturation scale evolution speed

JIMWLK with collinear improvement

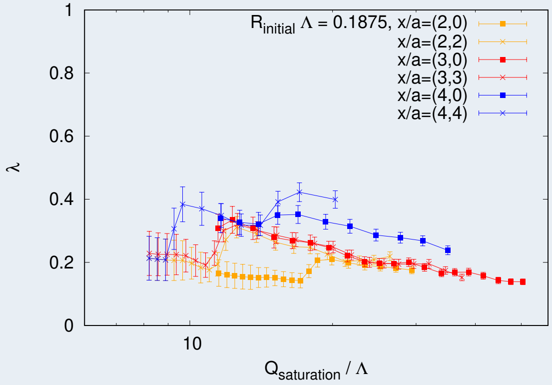


Figure: Preliminary results for the saturation scale evolution speed at $R_{\text{initial}}\Lambda = 0.1875$ for different discretizations. Much lower intercept than without the collinear improvement. Evolution at scales shorter than a should be performed with another approach.

New developments

BK with collinear improvement

- *Non-linear evolution in QCD at high-energy beyond leading order*
  B. Ducloué, E. Iancu, A.H. Mueller, G. Soyez, D.N. Triantafyllopoulos, JHEP 04 (2019) 081, 1902.06637 [hep-ph]
- *HERA data and collinearly-improved BK dynamics*
  B. Ducloué, E. Iancu, G. Soyez, D.N. Triantafyllopoulos, Phys.Lett.B 803 (2020) 135305, 1912.09196 [hep-ph]

In summary:

- target rapidity:

$$\eta \equiv \ln \frac{P^-}{|q^-|} = \ln \frac{2q^+P^-}{Q^2} = \ln \frac{1}{x}$$

- dipole rapidity:

$$Y \equiv \ln \frac{q^+}{q_0^+} = \ln \frac{2q^+P^-}{Q_0^2} = \ln \frac{1}{x} + \ln \frac{Q^2}{Q_0^2} = \eta + \rho$$

Q_0 is a soft scale of the unevolved target.

New developments

BK with collinear improvement

Evolution equation in the target rapidity η (Duclé at al., 2019)

$$\frac{\partial \bar{S}_{r=xy}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int d^2z \frac{(x-y)^2}{(x-z)^2(z-y)^2} \theta\left(\eta - \delta_{xyz}\right) \times$$
$$\times \left[\bar{S}_{xz}(\eta - \delta_{xz,r}) \bar{S}_{zy}(\eta - \delta_{zy,r}) - \bar{S}_{xy}(\eta) \right]$$

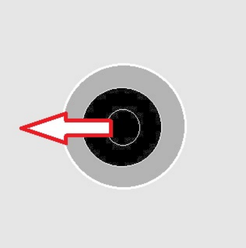
Comments:

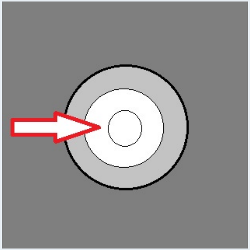
- fixed coupling constant for simplicity
- $r = |x-y|$
- rapidity shifts $\delta_{xz,r} = \max\{0, \ln \frac{r^2}{|x-z|^2}\}$
- $\delta_{xyz} = \max\{\delta_{xz,r}, \delta_{zy,r}\}$
- $\bar{S}_{xy}(\eta) = S_{xy}(Y = \eta + \rho)$

BK with collinear improvement

Main differences:

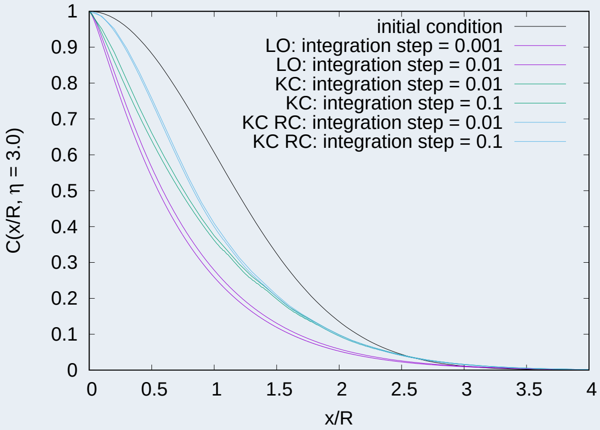
dipole rapidity:	target rapidity:
$\rho^R_{xz} = \ln \frac{\lvert x-z\rvert^2}{R^2}$	$\delta_{xz,r} = \max\{0, \ln \frac{r^2}{\lvert x-z\rvert^2}\}$





New developments

BK with collinear improvement



JIMWLK in η with collinear improvement

Proposal

$$U_{(n+1)\varepsilon}(x,r) = \exp\Big(i\sqrt{\varepsilon}\alpha^L_{n+1}(x,r)\Big)U_{n\varepsilon}(x,r)\exp\Big(-i\sqrt{\varepsilon}\alpha^R_{n+1}(x,r)\Big)$$

where

$$\alpha^L_{n+1}(x,r) = \frac{1}{\pi}\int_z \sqrt{\alpha_s}\theta(n\varepsilon - \delta^r_{r_{xz}})K^i_{xz}\xi^i_{n\varepsilon}(z),$$

$$\alpha^R_{n+1}(x,r) = \frac{1}{\pi}\int_z \sqrt{\alpha_s}\theta(n\varepsilon - \delta^r_{r_{xz}})U^\dagger_{n\varepsilon-\delta^r_{r_{xz}}}(z,r)K^i_{xz}\xi^i_{n\varepsilon-\delta^r_{r_{xz}}}(z)U_{n\varepsilon-\delta^r_{r_{xz}}}(z,r).$$

and

$$\delta^r_{r_{xz}} = \max\{0, \ln\frac{r^2}{r^2_{xz}}\}.$$

JIMWLK in η with collinear improvement

Reduction to the BK equation in η

The dipole amplitude is defined as

$$S(\mathrm{x},\mathrm{y}=\mathrm{x}+r,\eta)=\frac{1}{N_c}\langle \mathrm{tr} U^\dagger(\mathrm{x},r,\eta)U(\mathrm{x}+r,r,\eta)\rangle.$$

In order to establish the dependence on η we expand $S(\mathrm{x},\mathrm{y}=\mathrm{x}+r,\eta+\varepsilon)$ in ε,

$$S(\mathrm{x},\mathrm{y}=\mathrm{x}+r,\eta+\varepsilon)=\frac{1}{N_c}\langle \mathrm{tr} U^\dagger(\mathrm{x},r,\eta+\varepsilon)U(\mathrm{x}+r,r,\eta+\varepsilon)\rangle.$$

JIMWLK in η with collinear improvement

Reduction to the BK equation in η

Expand the exponentials

$$\exp\left(i\sqrt{\varepsilon}\alpha^L_{n+1}(\mathrm{x},r)\right) = 1 + i\sqrt{\varepsilon}\alpha^L_{n+1}(\mathrm{x},r) - \frac{1}{2}\varepsilon\left(\alpha^L_{n+1}(\mathrm{x},r)\right)^2,$$
$$\exp\left(-i\sqrt{\varepsilon}\alpha^R_{n+1}(\mathrm{x},r)\right) = 1 - i\sqrt{\varepsilon}\alpha^R_{n+1}(\mathrm{x},r) - \frac{1}{2}\varepsilon\left(\alpha^R_{n+1}(\mathrm{x},r)\right)^2,$$

leading to

$$U_{(n+1)\varepsilon}(\mathrm{x},r) = U_{n\varepsilon}(\mathrm{x},r) + i\sqrt{\varepsilon}\left[\alpha^L_{n+1}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{x},r) - U_{n\varepsilon}(\mathrm{x},r)\alpha^R_{n+1}(\mathrm{x},r)\right] +$$
$$+\varepsilon\left[\alpha^L_{n+1}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{x},r)\alpha^R_{n+1}(\mathrm{x},r) - \frac{1}{2}\left(\alpha^L_{n+1}(\mathrm{x},r)\right)^2 U_{n\varepsilon}(\mathrm{x},r) + \right.$$
$$\left. - \frac{1}{2}U_{n\varepsilon}(\mathrm{x},r)\left(\alpha^R_{n+1}(\mathrm{x},r)\right)^2\right],$$

JIMWLK in η with collinear improvement

Example: one of the cross-terms

$$
\begin{aligned}
&\mathrm{tr}\langle(\alpha^R)^\dagger_{n+1}(\mathrm{x},r)U^\dagger_{n\varepsilon}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{y},r)\alpha^R_{n+1}(\mathrm{y},r)\rangle_\xi = \\
&= \frac{1}{\pi^2}\mathrm{tr}\int_{\mathrm{z},\mathrm{z}'}\alpha_s\theta(n\varepsilon-\delta^r_{r_{yz}})\theta(n\varepsilon-\delta^r_{r_{xz'}})U^\dagger_{n\varepsilon-\delta^r_{r_{xz'}}}(\mathrm{z}',r)t^a K^i_{xz'}\times \\
&\times U_{n\varepsilon-\delta^r_{r_{xz'}}}(\mathrm{z}',r)U^\dagger_{n\varepsilon}(\mathrm{x},r)\times \\
&\times U_{n\varepsilon}(\mathrm{y},r)U^\dagger_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)t^b K^j_{yz}U_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)\langle\xi^i_{a,n+1}(\mathrm{z}')\xi^j_{b,n+1}(\mathrm{z})\rangle_\xi = \\
&= \frac{1}{2\pi^2}N_c^2\int_{\mathrm{z}}\alpha_s\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}S_6(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{yz},\eta)+ \\
&-\frac{1}{2\pi^2}S(\mathrm{x},\mathrm{y},\eta)\int_{\mathrm{z}}\alpha_s\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}
\end{aligned}
$$

JIMWLK in η with collinear improvement

All the terms yield

$$\frac{\partial S(\mathrm{x},\mathrm{y},\eta)}{\partial\eta} = \frac{\bar{\alpha}_s}{2\pi}\int_z S(\mathrm{x},\mathrm{y},\eta)$$

$$\Big\{-\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{yz}K^i_{yz}-\theta(n\varepsilon-\delta^r_{r_{xz}})K^i_{xz}K^i_{xz}+\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}\Big\}+$$

$$+\Big\{\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{yz}K^i_{yz}S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{yz},\delta_{yz},\eta)+$$

$$+\theta(n\varepsilon-\delta^r_{r_{xz}})K^i_{xz}K^i_{xz}S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{xz},\eta)+$$

$$-\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{yz},\delta_{yz},\eta)+$$

$$-\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{xz},\eta)\Big\}+$$

$$+\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}S_6(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{yz},\eta)$$

JIMWLK in η with collinear improvement

Recovering KC BK equation in η

Assuming that $\delta_{xz} = \delta_{yz} = \delta$ we have

$$S_6(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{yz},\eta) =$$
$$= \frac{1}{N_c^2}\mathrm{tr}\left[U_{n\varepsilon-\delta^r_{r_{xz}}}(\mathrm{z},r)U^\dagger_{n\varepsilon}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{y},r)U^\dagger_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)\right]\times$$
$$\times\,\mathrm{tr}\left[U^\dagger_{n\varepsilon-\delta^r_{r_{xz}}}(\mathrm{z},r)U_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)\right] =$$
$$= \frac{1}{N_c}\mathrm{tr}\left[U^\dagger_{n\varepsilon}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{y},r)\right] = S(\mathrm{x},\mathrm{y},\eta)$$

and setting

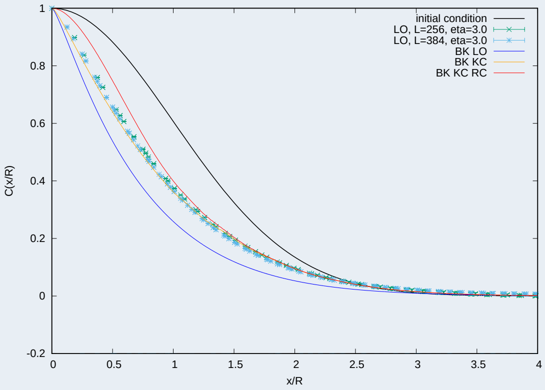
$$S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{xz},\eta) = S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{yz},\delta_{yz},\eta) \equiv S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta,\eta)$$

in that case the final results reduces to

$$\frac{\partial S(\mathrm{x},\mathrm{y},\eta)}{\partial\eta} = \frac{\bar{\alpha}_s}{2\pi}\int_{\mathrm{z}}\mathscr{K}_{xyz}\theta(n\varepsilon-\delta)\left\{S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta,\eta) - S(\mathrm{x},\mathrm{y},\eta)\right\}$$

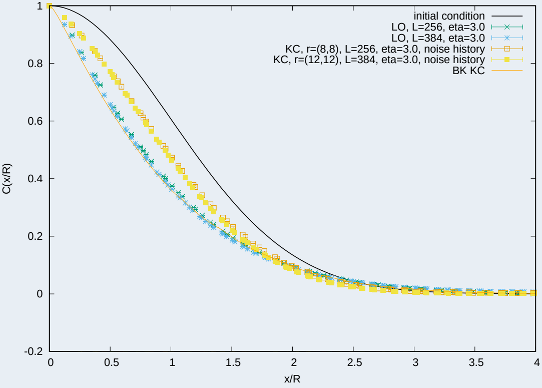
JIMWLK in η with collinear improvement

Preliminary results



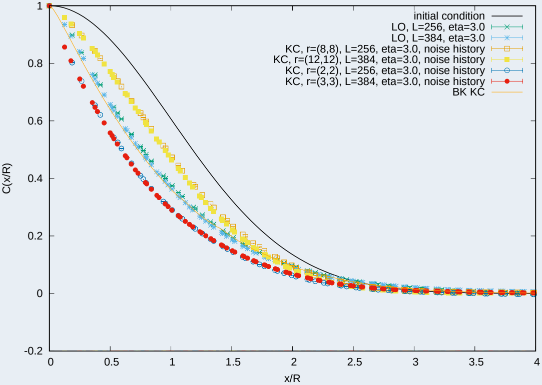
JIMWLK in η with collinear improvement

Preliminary results



JIMWLK in η with collinear improvement

Preliminary results



Conclusions

Summary

- JIMWLK equation provides a way to describe DIS data deep in the low-x regime
- numerical implementation and solution possible using the reformulation in terms of Langevin equation
- many systematic effects/ambiguities have to be studied and understood
- collinear resummation for the JIMWLK evolution possible

Outlook

- phenomenological implications/applications will soon be at reach!