Reconstruction of the Primordial Power Spectrum Using Multiple Data Sets

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Formation of large scale structure

$t \simeq 10^{-34}$ sec | $t \simeq 10^{5}$ yrs | Today

Generation of curvature perturbations by inflation
$V(\phi) \Rightarrow \mathcal{P}_{\mathcal{R}}(k)$

→ Acoustic oscillations in photon-baryon fluid
$\Omega_{\rm c}$, $\Omega_{\rm b}$, $\Omega_{\rm r}$, Ω_{ν}

→ Photons freestream so that inhomogeneities convert into CMB anisotropies

→ Matter perturbations are amplified by gravitational instability and create large scale structure

This is comparable to a scattering experiment.

For an assumed target (dark + baryonic matter) and detector (the Universe) the properties of the beam (inflationary curvature perturbations) can be inferred from the signal (CMB anisotropies + LSS) by deconvolution.

The primordial power spectrum

The WMAP results show that the primordial density perturbations are coherent, predominantly adiabatic and generated on superhorizon scales.

Such perturbations are characterised by the primordial power spectrum $\mathcal{P}_{\mathcal{R}}(k)$.

Why is the PPS $\mathcal{P}_{\mathcal{R}}(k)$ important?

- It can discriminate between models of inflation.
- Cosmological parameter estimation depends on the PPS (e.g. an EdeS model can fit the WMAP data if there is a 'bump' in the PPS Hunt & Sarkar 2007, 2008).

Usually the PPS is assumed to be a power-law with $\mathcal{P}_{\mathcal{R}}(k) \propto k^{n_s - 1}$.

However, inflationary models involving abnormal initial conditions, interruptions to slow-roll evolution or additional dynamical degrees of freedom produce a wide variety of spectra.

Given our ignorance of the physics behind inflation a model-independent method of estimating the PPS is essential.

Deconvolution as an ill-posed problem

The data points of CMB anisotropy, galaxy clustering, Lyman α forest, cluster abundance or weak lensing measurements can be written as

$$d_a = \int_0^\infty K_a(k)\,\mathcal{P}_\mathcal{R}(k)\,\mathrm{d}k + n_a. \tag{1}$$

We wish to estimate $\mathcal{P}_\mathcal{R}(k)$ from the data, assuming $K_a(k)$ is known.

However, the convolution with $K_a(k)$ acts as a smoothing operation.

Conversely, noise in the data is amplified in the reconstructed $\mathcal{P}_\mathcal{R}(k)$.

Therefore the inverse problem of recovering the PPS has no unique stable solution and is mathematically ill-posed.

Regularisation schemes can be used to obtain approximate solutions to ill-posed problems, usually by employing prior information.

Tikhonov regularisation

Our strategy is to find the smoothest PPS consistent with the data.

Discretising the integral in eq.(1) produces

$$d_a = \sum_i W_{ai}\, s_i + n_a, \qquad s_i \equiv \mathcal{P}_{\mathcal{R}}\left(k_i\right).$$

The estimate $\hat{\mathbf{s}}$ of $\mathbf{s}$ is taken to be the vector which minimises

$$Q\left(\mathbf{s},\mathbf{d}\right) = -2\ln P\left(\mathbf{d}|\mathbf{s}\right) + \lambda R\left(\mathbf{s}\right).$$

The first term on the rhs enforces fidelity to the data while the second enforces smoothness.

Following Tocchini-Valentini, Douspis and Silk 2005, Tocchini-Valentini, Hoffman and Silk 2006 we choose

$$R\left(\mathbf{s}\right) = \mathbf{s}^t\,\mathsf{L}^t\mathsf{L}\,\mathbf{s} \propto \int \left(\frac{\mathrm{d}\mathcal{P}_{\mathcal{R}}}{\mathrm{d}\ln k}\right)^2 \frac{\mathrm{d}k}{k},$$

where L is a discrete approximation to the 1st-order derivative operator.

Tikhonov regularisation has a natural Bayesian interpretation.

Bayes' theorem states

$$P\left(\mathbf{s}|\mathbf{d}\right) \propto P\left(\mathbf{d}|\mathbf{s}\right) P\left(\mathbf{s}\right).$$

Posterior distribution

Likelihood function

Prior distribution

If $P\left(\mathbf{s}\right) \propto \exp\left[-\lambda R\left(\mathbf{s}\right)/2\right]$ then minimising $Q\left(\mathbf{s},\mathbf{d}\right)$ is equivalent to maximising $P\left(\mathbf{s}|\mathbf{d}\right)$, and $\hat{\mathbf{s}}$ corresponds to the mode of $P\left(\mathbf{s}|\mathbf{d}\right)$.

For most data sets $P\left(\mathbf{d}|\mathbf{s}\right)$ is approximately Gaussian $\Rightarrow \langle\mathbf{s}\rangle = \hat{\mathbf{s}}$ and the scatter about $\langle\mathbf{s}\rangle$ is described by the covariance matrix

$$\Sigma_B \equiv \langle\left(\mathbf{s}-\langle\mathbf{s}\rangle\right)\left(\mathbf{s}-\langle\mathbf{s}\rangle\right)^t\rangle = \mathrm{H}^{-1}\left(\hat{\mathbf{s}}\right), \qquad H_{ij} \equiv \frac{1}{2}\frac{\partial^2 Q}{\partial s_i \partial s_j}.$$

Frequentist analysis

Suppose that $\hat{\mathbf{s}} = \mathsf{M}\mathbf{d}$.

Using $\mathbf{d} = \mathsf{W}\mathbf{s}_T + \mathbf{n}$ where $\mathbf{s}_T$ is the true PPS this gives

$$\hat{\mathbf{s}} = \mathbf{s}_T + (\mathsf{R} - \mathsf{I})\,\mathbf{s}_T + \mathsf{M}\mathbf{n},$$

where $\mathsf{R} \equiv \mathsf{M}\mathsf{W}$ is known as the resolution matrix.

In the frequentist approach we imagine an ensemble of observers, each measuring the data and estimating the PPS in the same way.

Then $\hat{\mathbf{s}}$ has a distribution $P\left(\hat{\mathbf{s}}|\mathbf{s}_T\right)$ with mean $\langle\hat{\mathbf{s}}\rangle = \mathsf{R}\mathbf{s}_T$ and covariance matrix

$$\Sigma_F \equiv \left\langle\left(\hat{\mathbf{s}} - \langle\hat{\mathbf{s}}\rangle\right)\left(\hat{\mathbf{s}} - \langle\hat{\mathbf{s}}\rangle\right)^t\right\rangle = \mathsf{M}\mathsf{N}\mathsf{M}^t.$$

The bias of $\hat{\mathbf{s}}$ is

$$\mathrm{Bias}\left(\hat{\mathbf{s}}\right) \equiv \langle\hat{\mathbf{s}} - \mathbf{s}_T\rangle = (\mathsf{R} - \mathsf{I})\,\mathbf{s}_T.$$

Regularisation reduces the variance but introduces a bias.

Multiple data sets

Most deconvolution attempts have used the WMAP TT data alone.

By using multiple data sets we can recover $\mathcal{P}_{\mathcal{R}}(k)$ over a larger k range with increased accuracy.

For example, if

$$L(\mathbf{s}) = \sum_{l} (\mathrm{W}_l\mathbf{s} - \mathbf{d}_l)^t \, \mathrm{N}_l^{-1} (\mathrm{W}_l\mathbf{s} - \mathbf{d}_l),$$

then the estimate of the PPS is

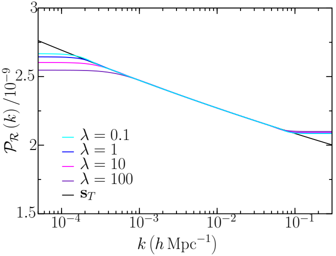
$$\hat{\mathbf{s}} = \Sigma_B \sum_l \mathrm{W}_l^t \mathrm{N}_l^{-1} \mathbf{d}_l,$$

and the covariance matrices are

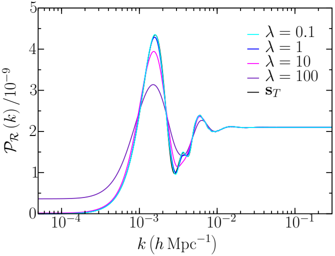
$$\Sigma_B^{-1} = \lambda \mathrm{L}_n^t \mathrm{L}_n + \sum_l \mathrm{W}_l^t \mathrm{N}_l^{-1} \mathrm{W}_l, \qquad \Sigma_F = \Sigma_B \left(\sum_l \mathrm{W}_l^t \mathrm{N}_l^{-1} \mathrm{W}_l \right) \Sigma_B^t.$$

Deconvolution without noise

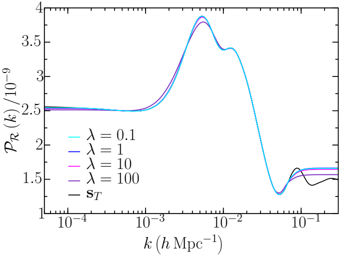
Dunkley *et al.* 2009



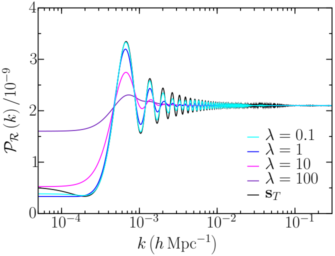
Langlois and Vernizzi 2005



Hunt and Sarkar 2007

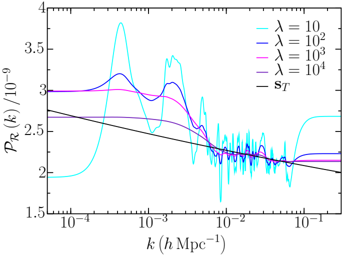


Starobinsky 1992

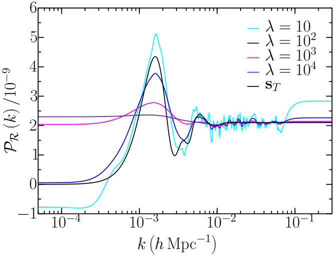


Deconvolution with noise

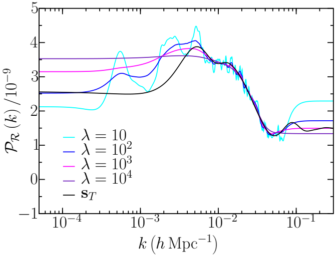
Dunkley *et al.* 2009



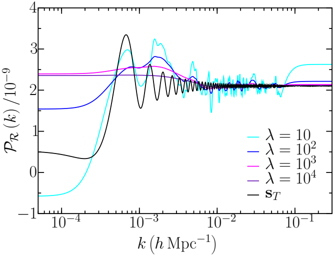
Langlois and Vernizzi 2005



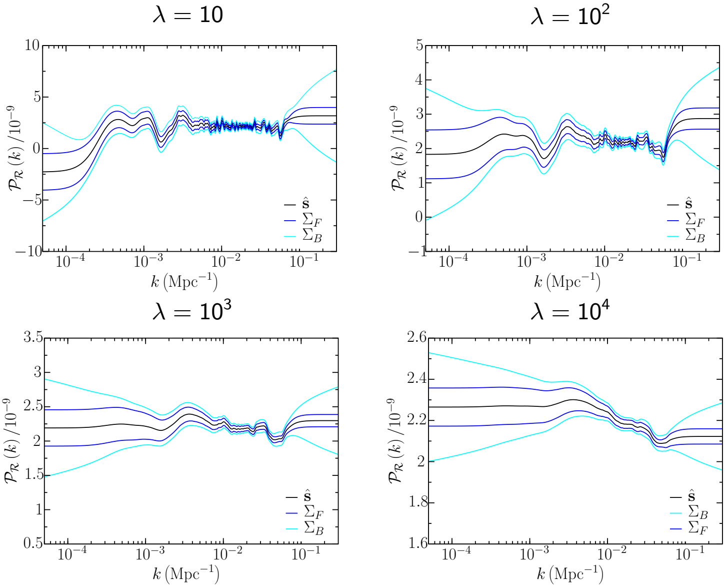
Hunt and Sarkar 2007



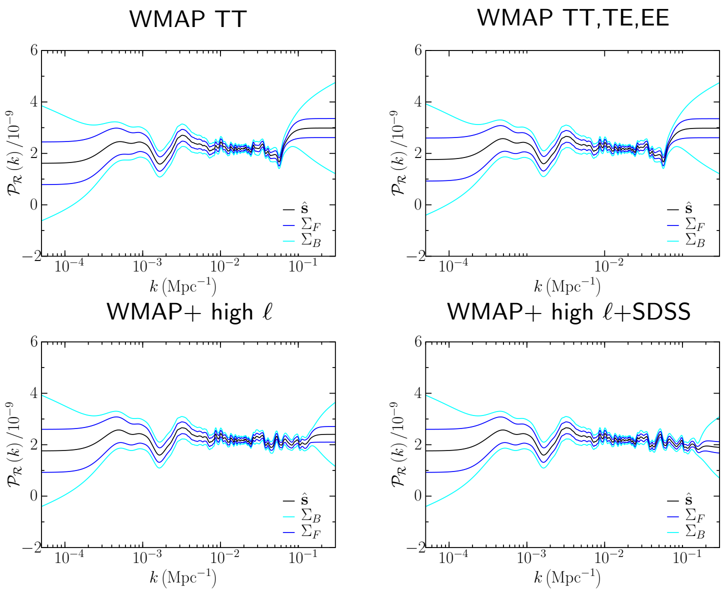
Starobinsky 1992



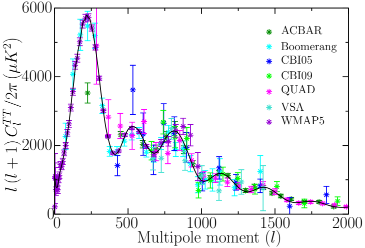
Varying λ, WMAP TT data



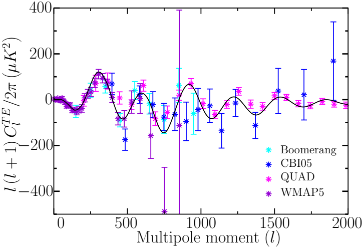
Adding data


WMAP TT
WMAP TT,TE,EE
WMAP+ high ℓ
WMAP+ high ℓ+SDSS
$\mathcal{P}_{\mathcal{R}}(k)/10^{-9}$
$k\,(\mathrm{Mpc}^{-1})$
$\hat{\mathbf{s}}$
Σ_F
Σ_B

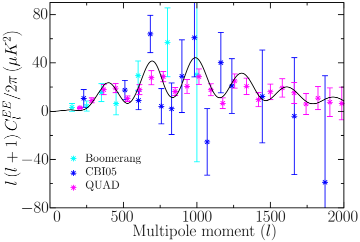
The fit to CMB TT data



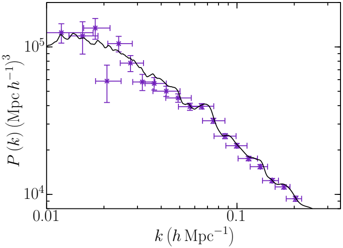
The fit to CMB TE data



The fit to CMB EE data



The fit to SDSS LRG data

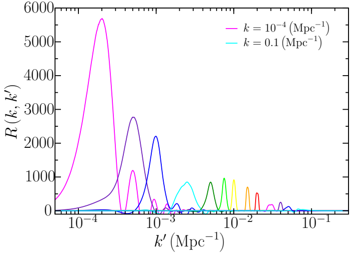


We introduce resolution kernels $R(k, k')$ as the continuous analogues of the resolution matrix. These satisfy

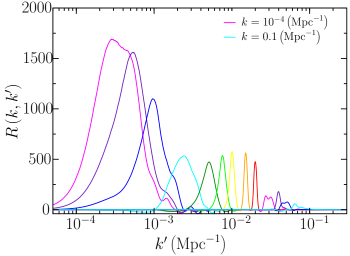
$$\langle \hat{\mathcal{P}}_{\mathcal{R}}(k) \rangle = \int_0^\infty R(k, k') \, \mathcal{P}_{\mathcal{R}}(k') \, \mathrm{d}k', \qquad \int_0^\infty R(k, k') \, \mathrm{d}k' = 1.$$

Clearly the closer $R(k, k')$ is to the Dirac delta function $\delta(k' - k)$ the better the resolution of the recovered PPS.

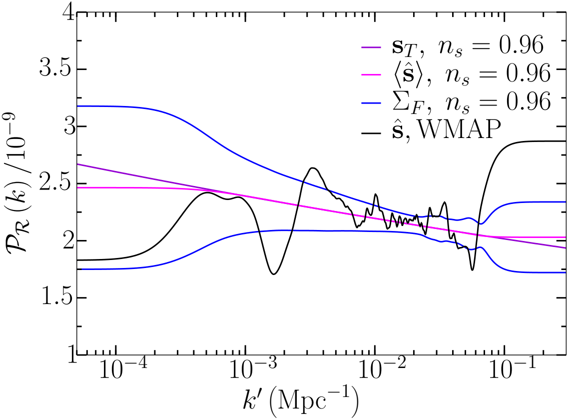
$\lambda = 10$



$\lambda = 10^2$



Is the PPS different from a power-law?



We find a cut-off at $5 \times 10^{-5} < k < 5 \times 10^{-4}$ Mpc at 0.7σ cl, a dip at $0.001 < k < 0.0025$ Mpc at 1.7σ cl and a peak at $0.0025 < k < 0.006$ at 1.4σ cl.

Backus-Gilbert method

We seek coefficients $q_a(k)$ such that

$$\hat{\mathcal{P}}_{\mathcal{R}}(k) = \sum_a q_a(k)\, d_a.$$

Combining this with eq.(1) gives

$$\hat{\mathcal{P}}_{\mathcal{R}}(k) = \int_0^\infty \sum_a q_a(k)\, K_a(k')\, \mathcal{P}_{\mathcal{R}}(k')\, \mathrm{d}k'.$$

Thus we want $R(k, k') = \sum_a q_a(k)\, K_a(k')$ to approximate a delta function and so we impose $\int R(k, k')\, \mathrm{d}k' = 1$.

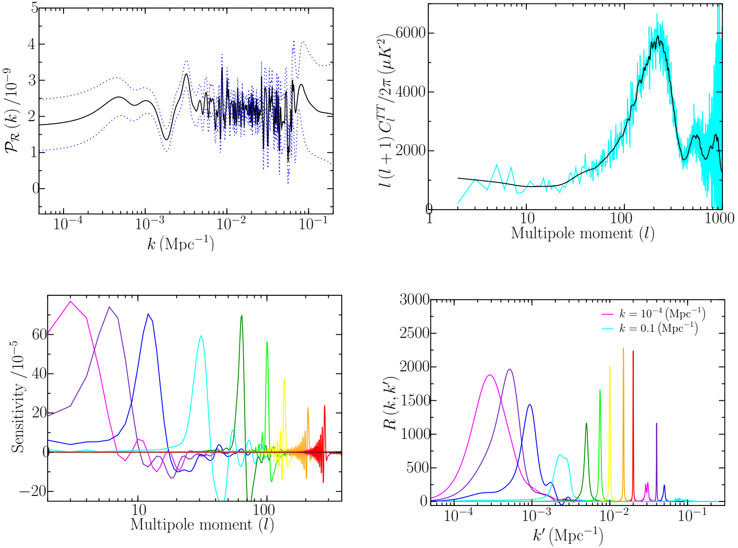
We also want to minimise the variance so we minimise

$$Q = \int_0^\infty (k' - k)^2\, R^2(k, k')\, \mathrm{d}k' + \lambda \sum_{ab} q_a(k)\, N_{ab} q_b(k).$$

'Width' of $R(k, k')$

Variance

Backus-Gilbert results



Conclusions

In the future as data sets improve we will be able to treat the PPS as a *function* instead of a set of parameters.

It is necessary to combine data sets to obtain the best results.

We have shown that Tikhonov regularisation and the Backus-Gilbert method produce high resolution reconstructions of the PPS from multiple noisy data sets.

Both methods give consistent results and have well-defined error estimates.

The recovered PPS show interesting features on large scales.

It should be possible to extend our method to isocurvature perturbations and nonlinear data sets.

Our ultimate goal is to determine *empirically* the cosmological parameters and the PPS.