Construction of the ground state of Matrix Theory: Near the Origin

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Plan

1. Matrix Theory
2. The construction of the ground state near the origin
3. Summary

Matrix Theory

- \- membranes - Dirac 62'
  quantum membranes $\longrightarrow$ YMQM - Goldstone, Hoppe 82'
  supermembranes $\longrightarrow$ SYMQM - de Wit, Hoppe, Nicolai 87'
  -femoto-universe of YM $\longrightarrow$ YMQM - Bjorken 79'
  \- small volume YM $\longrightarrow$ Lüscher - 82'
  -toy model - SYMQM - Claudson, Halpern 85'
  \- M theory on a light cone in the IMF $\longrightarrow$ SYMQM
  -Banks, Fischler, Shenker, Susskind 97'

Matrix Theory

- quantum description $\rightarrow$ matrix regularization:

$$H_{reg.} = Tr\left(P_sP_s - \frac{1}{2}[X_s, X_t][X_s, X_t] - iX_s\theta\gamma^s\theta\right)$$

$$X_s = x_{sA}T_A, \quad P_s = p_{sA}T_A \quad \theta_\alpha = \theta_{\alpha A}T_A, \quad T_A \in su(N)$$

$$[x_{sA}, p_{tB}] = i\delta_{st}\delta_{AB}, \quad \{\theta_{\alpha A}, \theta_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB},$$

$$\gamma^s - 16 \times 16, real$$

$$s, t = 1, \ldots 9, \quad A, B = 1, \ldots, N^2 - 1, \quad \alpha, \beta = 1, \ldots, 16$$

$$G_A|_{\mathcal{H}} = 0, \qquad G_A = f_{ABC}\left(x_{sB}p_{sC} - \frac{i}{2}\theta_{\alpha B}\theta_{\alpha C}\right)$$

-de Wit, Hoppe, Nicolai, 87'

- **Does the ground state exist?** $Q_\alpha \Psi = 0$

$$2H\delta_{\alpha,\beta} = \{Q_\alpha, Q_\beta\}, \quad Q_\beta = (-i\gamma^s_{\beta\alpha}\partial_{sA} + \frac{1}{2}\gamma^{st}_{\beta\alpha} f_{ABC} x_{sB} x_{tC})\theta_{\alpha A}$$

- Witten index computations suggest YES

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- **Can it be constructed?**
  - asymptotically (large x_{sA}) - `Bach, Bordemann, Fröhlich, Graf, Halpern, Hasler, Hoppe, Konechny, Lundholm, Plefka, Schwatrz, Suter, Yau - 97'-07'`
  - small x_{sA} ...

Construction of the ground state near the origin

- Taylor expansion

$$\Psi(x,\theta) = \psi^{(0)} + x_{sA}\psi^{(1)}_{sA} + \frac{1}{2}x_{sA}x_{tB}\psi^{(2)}_{sA\ tB} + \dots$$

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- what about $\psi^{(0)}$? Ψ is a $SO(9)$ singlet

  -Hasler, Hoppe 02'

- for $SU(N=2)$:

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- paper-pencil derivation of $\psi^{(0)}$?

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- Consider

$$\underset{44}{|||}\,1\rangle := |su\rangle_1|tu\rangle_2|st\rangle_3,$$

$$\underset{844}{|||}\,1\rangle := |suv\rangle_1|tuv\rangle_2|st\rangle_3 + |tuv\rangle_1|st\rangle_2|suv\rangle_3 + |st\rangle_1|suv\rangle_2|tuv\rangle_3$$

- ...then it follows that

$$\phi := \underset{44}{|||}\, 1\rangle + \frac{13}{36} \underset{844}{|||}\, 1\rangle$$

is $SU(2)$ invariant $\rightarrow \psi^{(0)} \sim \phi$ (no **128** content)

- an independent approach - explicit construction in the Fock space given by $\theta_\alpha \longrightarrow \lambda_{\hat{\alpha}}, \lambda^\dagger_{\hat{\alpha}}$

$$\theta_{\hat{\alpha}} = \frac{1}{\sqrt{2}}(\lambda_{\hat{\alpha}} + \lambda^\dagger_{\hat{\alpha}}), \qquad \theta_{\hat{\alpha}+8} = \frac{1}{i\sqrt{2}}(\lambda_{\hat{\alpha}} - \lambda^\dagger_{\hat{\alpha}}).$$

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$$r_1 = v_1v_1v_1|0\rangle, \quad r_2 = v_1v_1v_2|0\rangle, \quad r_3 = v_1v_2v_2|0\rangle, \quad r_4 = v_2v_2v_2|0\rangle,$$
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$$\chi = 326304r_1 + 488136r_2 + 72612r_3 + 1377r_4 + 114576r_5 - 176528r_6 + 10296r_7,$$

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- $\chi \sim \phi$!

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- Taylor expansion $\longrightarrow$ $\Psi(0) = \phi$ (two ways)