Why dispersion relations help in description of pion-pion amplitudes and lead to precise determination of the $f_0(600) \ (\sigma)$ parameters?

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Outline

1. \( \pi\pi \) amplitudes from experimental data only
   - phase shifts, inelasticities and cross sections for the S0 wave

2. Dispersion relations with imposed crossing symmetry condition
   - what do we need and what we propose?
   - historical review
   - threshold behavior of output amplitudes

3. Example of numerical results
   - numerical results for recent fits
   - coupling of resonances (S0 wave: \( \sigma, f_0(980), f_0(1400) \))

4. Conclusions
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1. **Conclusions**
Phase shifts from $\pi\pi$ threshold to $\sim 1600$ MeV

- Experimental data for $S_0$ ($Jl$) wave,

- Experiments $\rightarrow T = (\eta e^{2i\delta} - 1)/2i\rho \rightarrow$

- Resonances: $f_0(600)$ ($\sigma$), $f_0(980)$, $f_0(1370)$, $f_0(1500)$

- $n\pi \rightarrow n\pi\pi$ scattering (600-1800 MeV), $K_{l4}$ decays ($K \rightarrow \pi^+\pi^-e\nu_e$) $m_{\pi\pi} < 500$ MeV,

- Experiment $\rightarrow$ PWA $\rightarrow$ phases $\delta$ and inelasticities $\eta$ below $\sim 1600$ MeV ($S$-$G$ waves) $\rightarrow$

- Well known "up-down" ambiguity below 1 GeV (solution "up" eliminated in 2003 using the Roy’s equations),

- Peculiar cross section

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EPS09, Kraków 18.07.2009, page 3
Phase shifts from $\pi\pi$ threshold to $\sim 1600$ MeV

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Phase shifts, inelasticities and cross sections for the S0 wave

Phase shifts from ππ threshold to ~ 1600 MeV

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The S0 wave: Different sets

The fits to different sets follow two behaviors compared with that to KI4 data only. Those close to the pure KI4 fit display a "shoulder" in the 500 to 800 MeV region. These are:

- pure KI4, SolutionC
- and the global fits

Other fits do not have the shoulder and are separated from pure KI4.

Kaminski et al. lies in between with huge errors. Solution E deviates strongly from the rest but has huge error bars.

Note size of uncertainty in data at 800 MeV!!

More "flat" data sets give $\Gamma \approx 1000$ MeV, those with shoulder $\approx 500$ MeV.

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phase shifts below 1 GeV (S0 wave)

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cross sections for the $S_0$ wave

- $\sigma_{11} : \pi\pi \rightarrow \pi\pi$
- $\sigma_{12} : \pi\pi \rightarrow K\bar{K}$
- $\sigma_{13} : \pi\pi \rightarrow \sigma\sigma$

- completely not intuitive behaviour of cross sections,
- Breit-Wigner approximations: $\Gamma_\sigma$ from nonrelativistic and relativistic BW can differ by 300-400 MeV
- $\sigma$ state disappeared from PDG Tables in 1976, back in 1996
- continuation of amplitudes into complex energy plane $\rightarrow M = \text{Re}(s_{pole}), \Gamma = -2\text{Im}(s_{pole})$
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$M = \text{Re}(s_{pole})$, $\Gamma = -2\text{Im}(s_{pole})$, 

$\sigma (\text{mb})$

$E (\text{MeV})$

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EPS09, Kraków 18.07.2009, page 5
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\[\begin{align*}
\text{E (MeV)} & \quad \sigma (\text{mb}) \\
400 & \quad 10^{-1} \\
600 & \quad 10^{-1} \\
800 & \quad 10^{-1} \\
1000 & \quad 10^{-1} \\
1200 & \quad 10^{-1} \\
1400 & \quad 10^{-1} \\
1600 & \quad 10^{-1} \\
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\end{align*}\]
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  $M = \text{Re}(s_{pole})$, $\Gamma = -2\text{Im}(s_{pole})$,
what do we need:

- something what can eliminate unphysical data and
- is model independent,
- something what can be applied for wide $m_{\pi\pi}$ range,
- and for many partial waves,
- we should remember on analyticity and unitarity(!) and
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$$T(s, t) = C_{st} T(t, s)$$

where $C_{st}$ is crossing matrix.
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$$T(s, t) = C_{st} T(t, s)$$ where $C_{st}$ is crossing matrix
We propose: twice subtracted dispersion relations (Roy’s equations)

\[ \text{Re } f_\ell^I(s) = ST(s) + KT(s) + DT(s) \]

where

- **subtracting term** \( ST(s) = a_0^0 \delta_{I_0} \delta_{\ell_0} + a_0^2 \delta_{I_2} \delta_{\ell_0} + \frac{s-4}{12} (2a_0^0 - 5a_0^2)(\delta_{I_0} \delta_{\ell_0} + \frac{1}{6} \delta_{I_1} \delta_{\ell_1} - \frac{1}{2} \delta_{I_2} \delta_{\ell_0}) \) with \( a_0^0 \) and \( a_0^2 \) - the \( \pi\pi \) scattering lengths in the \( S_0 \)- and \( S_2 \)-wave,

- **kernel term** \( KT(s) = \sum_{I' = 0}^{2} \sum_{\ell' = 0}^{1} \int_{4}^{s_{\text{max}}} ds' K_{\ell\ell'}^{I'I}(s, s') \text{Im } f_{\ell'}^{I'}(s') \) with kernels \( K_{\ell\ell'}^{I'I}(s, s') \sim 1/(s - s')(s' - 4)^2 \)

- **driving term** \( DT(s) = d_\ell^I(s, s_{\text{max}}) \) \( \rightarrow \) higher partial waves and high energy parts \( (s < s_{\text{max}} \approx 1.5 \text{ GeV}) \) of \( S_0, P \) and \( S_2 \) amplitudes (regge).

- applicable for \( s \lesssim 60 \rightarrow \approx 1100 \text{ MeV} \)
We propose: twice subtracted dispersion relations (Roy’s equations)

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where

- "subtracting term" \( ST(s) = a_0^0 \delta_{I0} \delta_{\ell0} + a_2^0 \delta_{I2} \delta_{\ell0} + \)
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  with kernels \( K_{I'\ell'}^{I\ell} (s, s') \sim 1/(s - s')(s' - 4)^2 \)

- "driving term" \( DT(s) = d_\ell^I (s, s_{\text{max}}) \rightarrow \) higher partial waves and high energy parts \( (s < s_{\text{max}} \approx 1.5 \text{ GeV}) \) of \( S0 \), \( P \) and \( S2 \) amplitudes (regge).

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We propose: twice subtracted dispersion relations (Roy's equations)

- \( \text{Re } f^\ell_l(s) = ST(s) + KT(s) + DT(s) \) where
- "subtracting term" \( ST(s) = a_0^0 \delta_{l0} \delta_{00} + a_0^2 \delta_{l2} \delta_{00} + \frac{s - 4}{12} (2a_0^0 - 5a_0^2)(\delta_{l0} \delta_{00} + \frac{1}{6} \delta_{l1} \delta_{01} - \frac{1}{2} \delta_{l2} \delta_{00}) \) with \( a_0^0 \) and \( a_0^2 \) - the \( \pi \pi \) scattering lengths in the \( S_0 \)- and \( S_2 \)-wave,
- "kernel term" \( KT(s) = \sum_{l''=0}^{2} \sum_{\ell''=0}^{1} \int_4^{s_{\text{max}}} ds' K^l_{\ell l'}(s, s') \text{Im } f^l_{\ell'}(s') \) with kernels \( K^l_{\ell l'}(s, s') \sim 1/(s - s')(s' - 4)^2 \) and
- "driving term" \( DT(s) = d^l_l(s, s_{\text{max}}) \) \( \longrightarrow \) higher partial waves and high energy parts \( (s < s_{\text{max}} \approx 1.5 \text{ GeV}) \) of \( S_0, P \) and \( S_2 \) amplitudes (regge).
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We propose: twice subtracted dispersion relations (Roy’s equations)

\[ \text{Re} f^I_\ell(s) = ST(s) + KT(s) + DT(s) \]

where

- **"subtracting term" ST(s)**
  \[ ST(s) = a^0_0 \delta I_0 \delta \ell_0 + a^2_0 \delta I_2 \delta \ell_0 + \]
  \[ \frac{s - 4}{12} (2 a^0_0 - 5 a^2_0) (\delta I_0 \delta \ell_0 + \frac{1}{6} \delta I_1 \delta \ell_1 - \frac{1}{2} \delta I_2 \delta \ell_0) \]
  with \( a^0_0 \) and \( a^2_0 \) - the \( \pi \pi \) scattering lengths in the \( S_0 \)- and \( S_2 \)-wave,

- **"kernel term" KT(s)**
  \[ KT(s) = \sum_{\ell'} \sum_{\ell''} \int_{s''=0}^{s_{\text{max}}} ds' K^{ll'}_{\ell'\ell}(s, s') \text{Im} f^{ll'}_\ell(s') \]
  with kernels \( K^{ll'}_{\ell'\ell}(s, s') \approx 1/(s - s')(s' - 4)^2 \)

- **"driving term" DT(s)**
  \[ DT(s) = d^I_\ell(s, s_{\text{max}}) \]
  \( \rightarrow \) higher partial waves and high energy parts \( s < s_{\text{max}} \approx 1.5 \text{ GeV} \) of \( S_0, P \) and \( S_2 \) amplitudes (regge).

applicable for \( s \lesssim 60 \rightarrow \approx 1100 \text{ MeV} \)
We propose: twice subtracted dispersion relations (Roy’s equations)

- \( \text{Re } f_\ell^l(s) = ST(s) + KT(s) + DT(s) \) where

- “subtracting term” \( ST(s) = a_0^0 \delta_{l0} \delta_{\ell0} + a_0^2 \delta_{l2} \delta_{\ell0} + \)
  \[
  \frac{s - 4}{12} \left( 2a_0^0 - 5a_0^2 \right) \left( \delta_{l0} \delta_{\ell0} + \frac{1}{6} \delta_{l1} \delta_{\ell1} - \frac{1}{2} \delta_{l2} \delta_{\ell0} \right)
  \]
  with \( a_0^0 \) and \( a_0^2 \) - the \( \pi \pi \) scattering lengths in the \( S_0 \)- and \( S_2 \)-wave,

- “kernel term” \( KT(s) = \sum_{l'' = 0}^{2} \sum_{\ell'' = 0}^{1} \int_4^{s_{\text{max}}} ds' K_{\ell\ell'}^{ll'}(s, s') \text{Im } f_{\ell'}^{l'}(s') \) with kernels

- \( K_{\ell\ell'}^{ll'}(s, s') \sim 1/(s - s')(s' - 4)^2 \) ←!! and

- “driving term” \( DT(s) = d_\ell^l(s, s_{\text{max}}) \) \( \longrightarrow \) higher partial waves and high energy parts \( (s < s_{\text{max}} \approx 1.5 \text{ GeV}) \) of \( S_0, P \) and \( S_2 \) amplitudes (regge).

- applicable for \( s \lesssim 60 \rightarrow \approx 1100 \text{ MeV} \)
and once subtracted dispersion relations (GKPY equations)

- **Re** $f^I_{\ell}(s) = ST(s) + KT(s) + DT(s)$  
  where

- "subtracting term" $ST(s) = \sum_I C''_{st} a^{I'}_0$  
  with $a_0 = (a_0^0, 0, a_0^2)$ and $C_{st}$ - crossing matrix (for $s \leftarrow t$)

- "kernel term" $KT(s) = \sum_{l'} \int_{s''=0}^{s_{\text{max}}^{l'}} ds'' K''_{\ell\ell'}^{l''}(s, s'') \text{Im} f_{\ell'}^{I'}(s')$  
  with kernels $K''_{\ell\ell'}^{l''}(s, s') \sim 1/(s - s')(s' - 4)$ and

- "driving term" $DT(s) = d_{\ell}(s, s_{\text{max}}) \rightarrow$ higher partial waves and high energy parts ($s < s_{\text{max}} \approx 1.5 \text{ GeV}$) of $S0$, $P$ and $S2$ amplitudes (regge).

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- \( \text{Re } f^I_\ell(s) = ST(s) + KT(s) + DT(s) \)  where

- "subtracting term" \( ST(s) = \sum_{I'} C_{st}^{I'I'} a_0' \)  with
  \( a_0 = (a_0^0, 0, a_0^2) \)  and \( C_{st} \) - crossing matrix (for \( s \leftrightarrow t \))

- "kernel term" \( KT(s) = \sum_{I''=0}^{2} \sum_{\ell''=0}^{1} \int_4^{s_{\text{max}}} \text{Im } f^{I''}_{\ell''}(s',s') \text{Im } f^{I'I'}_{\ell'I'}(s,s') \text{Im } f^{I'I'}_{\ell'I'}(s,s') \)  with kernels
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\[ \text{Re } f'_\ell(s) = ST(s) + KT(s) + DT(s) \]

where

- "subtracting term" \( ST(s) = \sum_{l'} C^{ll'}_s a'_0 \) with \( a_0 = (a^0_0, 0, a^2_0) \) and \( C_s - \text{crossing matrix (for } s \leftrightarrow t) \)

- "kernel term" \( KT(s) = \sum_{l'} \sum_{l''=0}^{1} \int_{s''=0}^{s_{\text{max}}} ds' K^{ll''}_{\ell l'}(s, s') \text{Im } f'_{\ell l'}(s') \) with kernels

\[ K^{ll''}_{\ell l'}(s, s') \sim 1/(s - s')(s' - 4) \text{ and} \]

- "driving term" \( DT(s) = d^l_\ell(s, s_{\text{max}}) \) \( \longrightarrow \) higher partial waves and high energy parts \( s < s_{\text{max}} \approx 1.5 \text{ GeV} \) of \( S0, P \) and \( S2 \) amplitudes (regge).

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- $\text{Re } f_{\ell}^I(s) = ST(s) + KT(s) + DT(s)$ where
  - “subtracting term” $ST(s) = \sum I' C_{st}^{II'} a_0''$ with $a_0 = (a_0^0, 0, a_0^2)$ and $C_{st}$ - crossing matrix (for $s \leftrightarrow t$)
  - “kernel term” $KT(s) = \sum_{I''=0}^{2} \sum_{\ell''=0}^{1} \int_4^{s_{\text{max}}} ds' K_{\ell\ell'}^{II'}(s, s') \text{Im } f_{\ell'}^{I''}(s')$ with kernels $K_{\ell\ell'}^{II'}(s, s') \sim 1/(s - s')(s' - 4)$ and
  - “driving term” $DT(s) = d_{\ell}^{I}(s, s_{\text{max}}) \rightarrow$ higher partial waves and high energy parts ($s < s_{\text{max}} \approx 1.5 \text{ GeV}$) of $S0$, $P$ and $S2$ amplitudes (regge).

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\[
K^{II'}_{\ell\ell'}(s, s') \sim \frac{1}{(s - s')(s' - 4)}
\]

- "driving term" \( DT(s) = d^I_\ell(s, s_{\text{max}}) \) \( \rightarrow \) higher partial waves and high energy parts (\( s < s_{\text{max}} \approx 1.5 \text{ GeV} \)) of \( S0, P \) and \( S2 \) amplitudes (regge).

- applicable for \( s \lesssim 60 \rightarrow \approx 1100 \text{ MeV} \)
short historical review

- 1971 → S. M. Roy introduces crossing symmetry into $\pi\pi$ amplitudes and fixes them at the $\pi\pi$ threshold (→ scattering lengths), Phys. Lett. B 36, 353 (1971)
- 1972, 1974 → Basdevant et al.,
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- 2003 → R. Kamiński, L. Leśniak, B. Loiseau: "Elimination of ambiguities in $\pi\pi$ amplitudes using Roy's equations" (up-down" ambiguity),
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2001 → B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler (Swiss group), "Roy equation analysis of $\pi \pi \pi$ scattering", Phys. Rept. 353, 207 (2001),


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Threshold expansion:
\[ R_{f\ell} (s \approx 4) = (s - 4)^\ell \left[ a_{\ell}^i + b_{\ell}^i (s - 4) + \ldots \right] \]

Let's compare the Roy's and GKPY equations:

<table>
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<tr>
<th>Wave</th>
<th>Thr. exp</th>
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So, in GKPY equations necessary are mutual cancellations of constant terms in the \( P \)-wave and partial cancellations in the \( S \)-waves.
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The pion-pion scattering amplitude III

Robert Kamiński, IFJ PAN, Kraków, Poland
EPS09, Kraków 18.07.2009, page 12
phase shifts for the $S_0$-wave

- for $m_{\pi\pi} < 932$ MeV: 
  \[
  \cot\delta(s) = \frac{s^{1/2} \frac{m_{\pi}^2}{2k}}{s - \frac{1}{2}z_0^2} \left[ \frac{z_0^2}{m_{\pi}\sqrt{s}} + B_0 + B_1w(s) + B_2w(s)^2 \right],
  \]
  \[z_0 \approx m_{\pi} \leftarrow \text{Adler zero},\]

- $s \rightarrow w(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}$, $s_0 = 1.45$ GeV,

- above 932 MeV: $K$-matrix approach,

- Matching point at 932 MeV,

- Fits: FDR + sum rules + Roy + GKPY + exp. data, 7 waves ($S - G$), 52 parameters,


- main point of discussion between Bern and Madrid group: errors and $S_0$ phase shift at 800 MeV.
phase shifts for the S0-wave

for $m_{\pi\pi} < 932$ MeV: $\cot\delta(s) = \frac{s^{1/2}}{2k} \frac{m_{\pi}^2}{s - \frac{1}{2} z_0^2}$

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Decomposition of Roy’s and GKPY eqs: \( S_0 \)-wave

\[ f_\ell^I(s) = \frac{\sqrt{3}}{2\sqrt{s-4}} \left[ \eta_\ell^I(s) e^{2i\delta_\ell^I(s)} - 1 \right] \rightarrow \text{Re} f_\ell^I(s) \text{ should be smaller than } \approx 0.6 \]

- the Roy’s equations need strong cancellations between \( ST \) and \( KT \)
**Decomposition of Roy’s and GKPY eqs: S0-wave**

\[ f^I_\ell(s) = \frac{\sqrt{s}}{2i\sqrt{s^2-4}} \left[ \eta^I_\ell(s)e^{2i\delta^I_\ell(s)} - 1 \right] \quad \rightarrow \quad \text{Re} f^I_\ell(s) \text{ should be smaller than } \approx 0.6 \]

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Decomposition of Roy’s and GKPY equations: $P$ wave

Numerical results for recent fits
coupling of resonances ($S_0$ wave: $\sigma$, $f_0(980)$, $f_0(1400)$)

Example of numerical results

Conclusions

Robert Kamiński, IFJ PAN, Kraków, Poland

EPS09, Kraków 18.07.2009, page 14
Decomposition of Roy’s and GKPY equations: $S_2$-wave

\[ \frac{s^{1/2} \eta \sin \delta}{2k} \]

\[ s \ (m^2) \]

\[ s \ (m^2) \]

Example of numerical results

Numerical results for recent fits

coupling of resonances ($S_0$ wave: $\sigma$, $f_0(980)$, $f_0(1400)$)
Output from Roy and GKPY equations, $S_0$-wave

Constrained Fits to Data (FDR+SR+Roy+GKPY)

- $\text{Roy}^{S_0 \text{ in}}$
- $\text{Roy}^{S_0 \text{ out}}$
- $\text{GKPY}^{S_0 \text{ in}}$
- $\text{GKPY}^{S_0 \text{ out}}$

$\overline{d^2} = 0.15$

$\overline{d^2} = 0.93$

Robert Kamiński, IFJ PAN, Kraków, Poland
EPS09, Kraków 18.07.2009, page 16

- $ST_{\text{Roy}}(s) = a_0^0 + \frac{1}{12}(2a_0^0 + 5a_2^0)(s - 4)$,
- $ST_{\text{GKPY}} = a_0^0 + 5a_0^2$

Roy’s equations have smaller errors below $s^{1/2} \approx 400$ MeV
GPKY equations have significantly smaller errors above $s^{1/2} \approx 400$ MeV
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Dispersion relations with imposed crossing symmetry condition

**Example of numerical results**

Conclusions

Numerical results for recent fits coupling of resonances ($S_0$ wave: $\sigma$, $f_0(980)$, $f_0(1400)$)

Output from Roy and GKPY equations, $S_0$-wave

$$ST_{Roy}(s) = a_0^0 + \frac{1}{12} (2a_0^0 + 5a_2^0)(s - 4),$$

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Constrained Fits to Data (FDR+SR+Roy+GMKPY)

$\sigma^2 = 0.20$

Constrained Fits to Data (FDR+SR+Roy+GKPY)

$\sigma^2 = 0.77$

$ST_{Roy}(s) = \frac{1}{72}(2a_0^0 + 5a_2^0)(s - 4)$,

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Robert Kamiński, IFJ PAN, Kraków, Poland

EPS09, Kraków 18.07.2009, page 17
Numerical results for recent fits of $\pi \pi$ amplitudes from experimental data only.

Dispersion relations with imposed crossing symmetry condition.

Example of numerical results.

Conclusions.

Constrained Fits to Data (FDR+SR+Roy+GKPY).

output from Roy and GKPY equations, $S_2$-wave.

$ST_{Roy}(s) = a_0^2 - \frac{1}{24} (2a_0^0 + 5a_2^0)(s - 4),$

$ST_{GKPY} = a_0^0 + \frac{1}{2} a_2^0.$
Example of numerical results

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Dispersion relations with imposed crossing symmetry condition

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Continuation to the complex $s$ plane:

$Im(s_{pole})$:
- ROY: $-255 \pm 14$ MeV
- GKPY: $-251 \pm 12$ MeV

$Re(s_{pole})$:
- ROY: $459 \pm 31$ MeV
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How to calculate couplings? general recipe:

- 1-channel case ($\pi\pi$) up to the $K\bar{K}$ threshold ($\approx 991$ MeV),
- Let’s us consider:
- 2-channel case ($\pi\pi$ and $K\bar{K}$) up to the about 1300-1400 MeV,
- 3-channel case ($\pi\pi$, $K\bar{K}$ and effective $\sigma\sigma$)

- Let’s assume we have defined $S$ matrix, e.g. $S_{\pi\pi} = \frac{D(-k_\pi,k_K,k_3)}{D(k_\pi,k_K,k_3)} (D(k_1...k_n)$ - Jost functions)
- Let’s assume we have found a pole at $s_{pole}$ (zero of denominator - COMMON for all channels!),
- then $\frac{g_i g_j}{4\pi} = i \sqrt{s_{pole}} \lim_{s \to s_{pole}} \left[ (s - s_{pole}) \frac{S_{ij}}{\sqrt{k_i k_j}} \right]
- Let’s take $\sigma$ pole: but which one?
  - 1-channel case → TWO poles (at $k_\pi$ and $-k_\pi^* \leftrightarrow S^*(k) = S(-k^*)$) lying symmetrically to conjugated zeros,
  - 2-channel case → FOUR poles LYING NOT SYMMETRICALLY to corresponding zeros ($k_K = \pm \sqrt{k_\pi^2 + m_\pi^2 - m_{K}^2}$),
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Example of numerical results

Conclusions

numerical results for recent fits
coupling of resonances (S0 wave: \(\sigma, f_0(980), f_0(1400)\))

How to calculate couplings? general recipe:

1. 1-channel case \((\pi\pi)\) up to the \(K\bar{K}\) threshold \((\approx 991 \text{ MeV})\),
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EPS09, Kraków 18.07.2009, page 20
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- one can improve coupled channel models using strong constraints from dispersion relations (i.e. refit model predictions),
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Conclusions

- Dispersion relations offer strong constraints for amplitudes.
  - Small errors of $\sigma$ and of $a_0^0 = 0.222 \pm 0.009$, $a_0^2 = -0.045 \pm 0.008$,
- One can use them even where is no data,
- We do not use any ChPT predictions,
- Only analyticity! Crossing symmetry is for free,
- One can combine data from complete set of partial waves ($S - G$),
- We recommend GKPY equations as "more demanding" above $\sim 400$ MeV.
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