The Quark Model via an hbar expansion of QCD

or

In quest of the “Born term” for relativistic bound states

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Summary

- In standard perturbation theory, amplitudes at each order in α (and $\hbar$) are essentially unique and obey symmetries like Lorentz invariance.

- Bound state poles appear where the expansion diverges
  $\Longrightarrow$ the situation for relativistic bound states remains murky

- I shall discuss how to derive from $\mathcal{L}_{QCD}$ a "Born approximation" for mesons, which is valid at lowest order in $\hbar$ (no loops).

- The quarks will be bound by the instantaneous A^0 potential fixed by the QCD equations of motion (EOM).

- A linear A^0 potential emerges (rather trivially) as a homogeneous (non-perturbative) solution of the EOM.

- The correctness of the procedure (at lowest order in $\hbar$ and α) is supported by a highly non-trivial Lorentz covariance for the bound states, defined at equal time in all frames.

Opportunities

The availability of "Born term" relativistic wave functions derived from the Lagrangian can teach us about hadron structure:

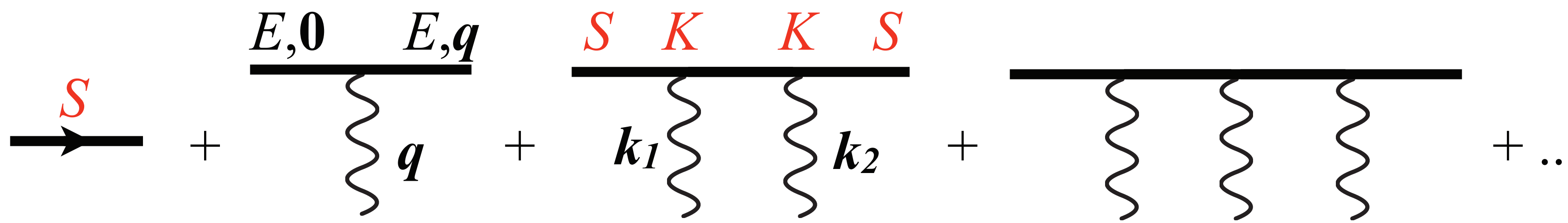
- The relation between the CM and Infinite Momentum frame (Light-Front) wave functions.
- High momentum components in the bound states (related to end-point behaviour of distribution amplitudes and Regge behaviour).
- Chiral symmetry issues
- This approach aims at an understanding of the basis of the phenomenologically successful Quark Model

Outline of Talk

- Recall some properties of the Dirac equation (external A^0 field)
- Introduce "retarded" boundary condition – OK at $\mathcal{O}\left(\hbar^0\right)$
- Fix A^0 from operator equation of motion (for each $q\bar{q}$ Fock state)
- Allow homogeneous solution: linear potential $A^0 = \boldsymbol{c}\cdot\boldsymbol{r}$
- Fix direction of $\boldsymbol{c}$ by stationarity of action (for each Fock state)
- Calculate to $\mathcal{O}(g)$, ignore $\mathcal{O}(g^2)$ (hence use purely linear potential)
- Impose stationarity on equal time $q\bar{q}$ bound state
- Find meson wave functions with interesting phenomenology
- Observe non-trivial Lorentz covariance for a linear potential

Electron scattering in a Coulomb potential

$G(E,\boldsymbol{q}) =$



$= S + S\,K\,G$

Instantaneous potential $A^0(\boldsymbol{k})$ does not change the energy E of the electron

At a bound state pole: $G(E,\boldsymbol{q}) = \dfrac{R(E_R,\boldsymbol{q})}{E-E_R}$ $\qquad R = S\,K\,R$

$$R(E_R,\boldsymbol{q}) = \frac{i}{\not{p}_f - m + i\varepsilon}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}(-ie)\gamma^0A^0(\boldsymbol{k})\,R(E_R,\boldsymbol{q}-\boldsymbol{k})$$

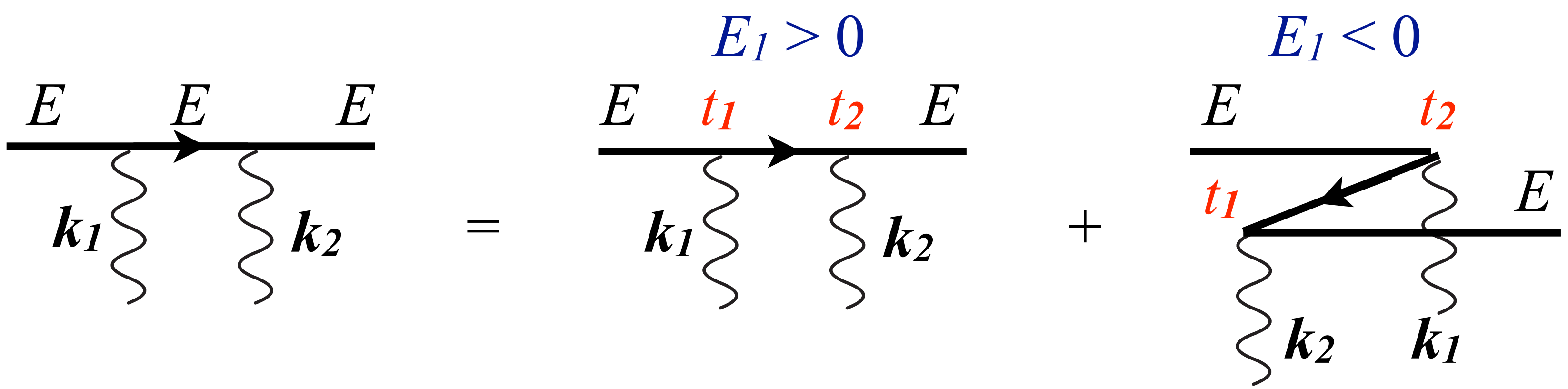
Fourier transforming:

$$\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + e\gamma^0A^0(\boldsymbol{x}) + m\right]R(E_R,\boldsymbol{x}) = E_R\gamma^0\,R(E_R,\boldsymbol{x})$$

Dirac equation

What is the equal-time Fock state composition of R ?

Time ordering the scattering events



At fixed $t_1 < t_2$, the intermediate energy $E_1 = \pm\sqrt{\boldsymbol{k}_1^2 + m_e^2}$ can be negative: Pair production

The Dirac wave function contains an infinite number of e^+e^- pairs

Good news: Such a multiparticle system can be described by the "one-particle" Dirac wave function $R(E_R, \boldsymbol{x})$

But then: What exactly does $R(E_R, \boldsymbol{x})$ describe?

The retarded boundary condition

The Coulomb scattering at fixed $p^0 > 0$ is insensitive to the Feynman $i\varepsilon$ prescription at the negative energy pole, $p^0 = -\sqrt{\boldsymbol{p}^2 + m^2}$

$$S_F(p) = i\,\frac{\not{p} + m}{(p^0 - \sqrt{\boldsymbol{p}^2 + m^2} + i\varepsilon)(p^0 + \sqrt{\boldsymbol{p}^2 + m^2} - i\varepsilon)}$$

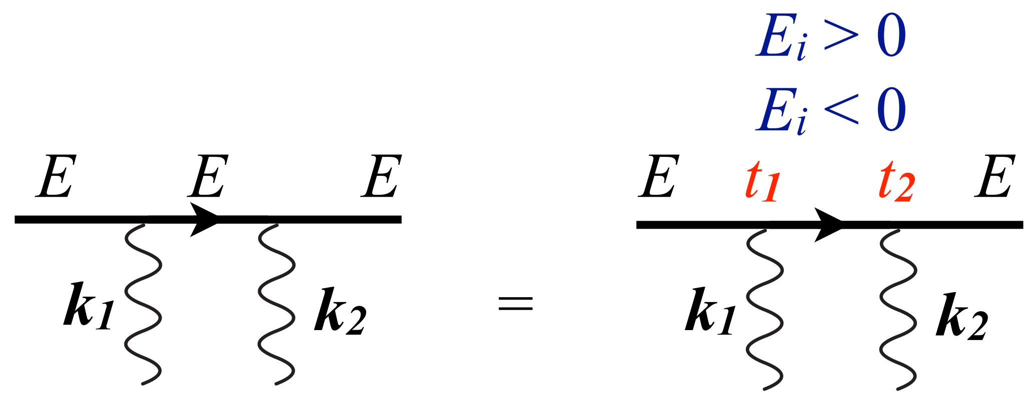
Hence the bound state poles would appear at the same energies using the retarded propagators

$$S_R(p) = i\,\frac{\not{p} + m}{(p^0 - \sqrt{\boldsymbol{p}^2 + m^2} + i\varepsilon)(p^0 + \sqrt{\boldsymbol{p}^2 + m^2} + i\varepsilon)}$$

Now the time ordering is trivial: There is only forward propagation in time

The retarded boundary condition (II)

Using S_R:



Thus $R(E_R, \boldsymbol{x})$ is the wave function of the single electron Fock state defined through the retarded boundary condition.

This is very different from the physical, Feynman Fock state distribution

... but the positions of the bound state poles are unaffected by the choice of boundary condition.

Preventing the electron from moving backwards also gives it a smooth time development:

$$S_R(t = 0^+, \boldsymbol{x}) = \gamma^0 \delta^3(\boldsymbol{x})$$

The rhs. would be non-local in $\boldsymbol{x}$ for $S_F(t{=}0,\boldsymbol{x})$

Field theoretical formulation

Retarded propagator: $S_R(x-y) = {}_R\langle 0|\,T[\psi(x)\bar{\psi}(y)]\,|0\rangle_R$

where $|0\rangle_R = N^{-1}\prod_{\boldsymbol{p},\lambda} d^\dagger_{\boldsymbol{p},\lambda}|0\rangle$ is the "retarded vacuum", for which

$$\psi(x)|0\rangle_R = \int\sum_\lambda \left[u(\boldsymbol{p},\lambda)e^{-ip\cdot x}b_{\boldsymbol{p},\lambda} + v(\boldsymbol{p},\lambda)e^{ip\cdot x}d^\dagger_{\boldsymbol{p},\lambda}\right]|0\rangle_R = 0$$

Hence in the Interaction Picture:

$$H_I(t)|0\rangle_R = e\int d^3\boldsymbol{x}\,A^0(\boldsymbol{x})\,\psi^\dagger(t,\boldsymbol{x})\psi(t,\boldsymbol{x})|0\rangle_R = 0$$

No particle production in the retarded vacuum.

The retarded propagator is not allowed in loop integrals!

Rederivation of the Dirac equation

The bound state: $|E, t=0\rangle = \int d^3\boldsymbol{x}\,\psi^\dagger(t=0,\boldsymbol{x})\varphi(\boldsymbol{x})|0\rangle_R$

where φ($\boldsymbol{x}$) is the Dirac wave function.

The B-S amplitude: $\phi(t,\boldsymbol{x}) = {}_R\langle 0|\psi(t,\boldsymbol{x})|E,t\rangle$

$= \varphi(\boldsymbol{x})\,\exp(-iEt)$ for a stationary state

From

$$i\frac{d\phi(0,\boldsymbol{x})}{dt} = {}_R\langle 0|i\frac{d\psi(0,\boldsymbol{x})}{dt}|E,0\rangle + {}_R\langle 0|\psi(t,\boldsymbol{x})H_I|E,t\rangle = E\phi(0,\boldsymbol{x})$$

follows the Dirac equation: $(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + e\gamma^0 A^0(\boldsymbol{x}) + m)\varphi(\boldsymbol{x}) = E\gamma^0\varphi(\boldsymbol{x})$

Application to QED (Abelian) and QCD bound states

Claim: There is a limit in which the dynamics is analogous to the external field case (Dirac equation), and to which perturbative corrections may then be applied.

$\hbar \to 0$: No loops (which would be sensitive to the $i\varepsilon$ prescription)
Gauge fields A_μ are fixed by stationarity of the action

$\alpha = \dfrac{g^2}{4\pi} \to 0$: Photon/gluon exchanges treated perturbatively

The principle of minimal action allows a linear potential $A^0 = \boldsymbol{c} \cdot \boldsymbol{r}$.

$\boldsymbol{c} = 0$ in QED, $\boldsymbol{c} \neq 0$ in QCD amounts to a choice of boundary conditions.

The procedure to be described is correct to $\mathcal{O}(g)$.

A non-trivial Lorentz covariance of the equal-time bound states lends support for the correctness of the approximation at the given order.

QED example: $e^- \mu^+$

Bound state:

$$|E, t=0\rangle = \int d\boldsymbol{y}_1 d\boldsymbol{y}_2\, \psi_e^\dagger(t=0, \boldsymbol{y}_1)\, \chi(\boldsymbol{y}_1, \boldsymbol{y}_2)\, \psi_\mu^\dagger(t=0, \boldsymbol{y}_2)|0\rangle_R$$

Retarded vacuum:

$$|0\rangle_R = N^{-1} \prod_{\boldsymbol{p},\lambda} d_e^\dagger b_\mu^\dagger |0\rangle \quad \Rightarrow \quad \psi_e(t, \boldsymbol{y})|0\rangle_R = \psi_\mu^\dagger(t, \boldsymbol{y})|0\rangle_R = 0$$

Bethe-Salpeter wave function:

$$\phi_{\alpha\beta}(t; \boldsymbol{x}_1, \boldsymbol{x}_2) = {}_R\langle 0|\psi_{\mu\beta}^\dagger(t, \boldsymbol{x}_2)\psi_{e\alpha}(t, \boldsymbol{x}_1)|E, t\rangle = e^{-iEt}\phi_{\alpha\beta}(t=0; \boldsymbol{x}_1, \boldsymbol{x}_2)$$

Determine the gauge fields from the operator equation of motion:

$$\partial_\mu F^{\mu\nu}(x) - e \sum_{i=e,\mu} \bar{\psi}_i(x)\gamma^\nu \psi_i(x) = 0 \qquad \text{(EOM in QED)}$$

QED example: $e^-\mu^+$ (II)

The matrix elements of the EOM determine A^0 for each Fock state

$$ {}_R\langle 0|\psi^\dagger_{\mu\beta}(0,\boldsymbol{x}_2)\psi_{e\alpha}(0,\boldsymbol{x}_1)\,(\mathrm{EOM})\,|E,0\rangle = 0 \qquad \Rightarrow $$

$$ -\boldsymbol{\nabla}^2_{\boldsymbol{x}} A^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2) = e\left[\delta^3(\boldsymbol{x}-\boldsymbol{x}_1)-\delta^3(\boldsymbol{x}-\boldsymbol{x}_2)\right] \qquad \Rightarrow $$

$$ A^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2) = \Lambda^2\,\hat{\boldsymbol{\ell}}\cdot\boldsymbol{x} + \frac{e}{4\pi}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|}-\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\right) $$

$\hat{\boldsymbol{\ell}} = \hat{\boldsymbol{\ell}}(\boldsymbol{x}_1,\boldsymbol{x}_2) \parallel \boldsymbol{x}_1-\boldsymbol{x}_2$ is determined by stationarity of the action:

$$ -\frac{1}{4}\int d^3\boldsymbol{x}\,F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}\Lambda^4\int d^3\boldsymbol{x} + \frac{1}{3}e\Lambda^2\hat{\boldsymbol{\ell}}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2) + \mathcal{O}\left(e^2\right) $$

The orientation of the electric field ∇A^0 depends on the positions $\boldsymbol{x}_1$, $\boldsymbol{x}_2$
– but the field measured at a distance involves a sum over all Fock states

QED example: $e^-\mu^+$ (III)

The principle of stationary action thus allows a linear instantaneous potential

$$A^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2) = \Lambda^2 \frac{\boldsymbol{x}_1-\boldsymbol{x}_2}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\cdot\boldsymbol{x} + \frac{e}{4\pi}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|}-\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\right)$$

but only data tells us to choose $\Lambda_{QED} = 0$, $\Lambda_{QCD} \neq 0$.

Relativistically moving charges also give rise to transverse $\boldsymbol{A}^\perp \neq 0$.
These do not interfere with the linear A^0 potential and contribute at $\mathcal{O}(e^2)$

$\Rightarrow$ This treatment is accurate only to $\mathcal{O}(e)$.
The $-\alpha/r$ Coulomb potential is a perturbative correction.

Proceeding as for the Dirac equation, and remembering the Fock state dependent phase from the $\mathcal{O}(e\Lambda^2)$ interference term in

$$\exp\left[-\frac{i}{4}\int d^3\boldsymbol{x}\, F_{\mu\nu}F^{\mu\nu}\right]$$

we find the BSE:

QED example: $e^- \mu^+$ (IV)

$$\gamma^0(-i\boldsymbol{\nabla}_1\cdot\boldsymbol{\gamma}+m_e)\chi(\boldsymbol{x}_1,\boldsymbol{x}_2) - \chi(\boldsymbol{x}_1,\boldsymbol{x}_2)\gamma^0(i\boldsymbol{\nabla}_2\cdot\boldsymbol{\gamma}+m_\mu)$$

$$= [E-V(\boldsymbol{x}_1,\boldsymbol{x}_2)]\chi(\boldsymbol{x}_1,\boldsymbol{x}_2)$$

where $\chi(\boldsymbol{x}_1,\boldsymbol{x}_2)$ is the 4x4 wave function of the $|e^-(\boldsymbol{x}_1)\,\mu^+(\boldsymbol{x}_2)\rangle$ Fock state, and the potential

$$V(\boldsymbol{x}_1,\boldsymbol{x}_2)=\frac{2}{3}e\Lambda^2\,|\boldsymbol{x}_1-\boldsymbol{x}_2|$$ is purely linear at $\mathcal{O}(e)$.

Since this equation has been derived at $\mathcal{O}\left(\hbar^0\right)$ from first principles it may be regarded as a "Born term" for bound state calculations.

It is a natural extension of the Dirac equation and as such was proposed by Breit already in 1929!

It has been studied phenomenologically for a linear + $1/r$ potential

Suura *et al* (1977)
Krolikowski *et al*

Extension to $u\bar{d}$ mesons

$$|0\rangle_R = N^{-1} \prod_{\boldsymbol{p},\lambda,A} d_u^{A\dagger}(\boldsymbol{p},\lambda)\, b_d^{A\dagger}(\boldsymbol{p},\lambda)|0\rangle$$

Lorentz invariant, color singlet retarded vacuum

Meson state:

$$|E, t=0\rangle = \int d\boldsymbol{y}_1 d\boldsymbol{y}_2\, \psi_u^{A\dagger}(t=0,\boldsymbol{y}_1)\chi^{AB}(\boldsymbol{y}_1,\boldsymbol{y}_2)\psi_d^B(t=0,\boldsymbol{y}_2)|0\rangle_R$$

Ansatz: $\chi^{AB}(\boldsymbol{y}_1,\boldsymbol{y}_2) = \delta^{AB}\chi(\boldsymbol{y}_1,\boldsymbol{y}_2)$ Color singlet wave function

Fock state matrix element of the QCD equations of motion:

$${}_R\langle 0|\psi_{d\beta}^{C\dagger}(t,\boldsymbol{x}_2)\psi_{u\alpha}^C(t,\boldsymbol{x}_1)\Big[\partial_\mu F_a^{\mu\nu} + g f_{abc} F_b^{\mu\nu} A_\mu^c - g \sum_{f=u,d} \bar{\psi}_f^A \gamma^\nu T_a^{AB} \psi_f^B\Big]|E,t\rangle = 0$$

Find $\mathcal{O}(g)$ solution with linear potential in abelian components:

$$A_a^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,C) = \Lambda_a^2\, \ell\cdot\boldsymbol{x} + \frac{gT_a^{CC}}{4\pi}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\right)$$

a = 3, 8
C: quark color

Extension to $u\bar{d}$ mesons (II)

Bound state equation for color singlet wave function is as in QED:

$$\gamma^0(-i\boldsymbol{\nabla}_1\cdot\boldsymbol{\gamma}+m_u)\chi(\boldsymbol{x}_1,\boldsymbol{x}_2)-\chi(\boldsymbol{x}_1,\boldsymbol{x}_2)\gamma^0(i\boldsymbol{\nabla}_2\cdot\boldsymbol{\gamma}+m_d)=[E-V(\boldsymbol{x}_1,\boldsymbol{x}_2)]\chi(\boldsymbol{x}_1,\boldsymbol{x}_2)$$

$$V(\boldsymbol{x}_1,\boldsymbol{x}_2)=g\Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| \qquad \Lambda_3^2=\frac{1}{\sqrt{3}}\Lambda_8^2\equiv\frac{3}{2}\Lambda^2$$

Some interesting properties of the solutions:

- Lorentz covariance: $E=\sqrt{\boldsymbol{k}_{CM}^2+M^2}$ χ transforms in a novel way
- Linear Regge trajectories: $\alpha' = 1/8g\Lambda^2$
- High relative momentum components with oscillating phase
- Chiral symmetry breaking

Lorentz covariance

The wave function of a bound state with CM momentum $\boldsymbol{k}$ has

$$\chi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2\right]\,\phi(\boldsymbol{x}_1-\boldsymbol{x}_2)$$

The equation for $\phi(\boldsymbol{x})$ becomes (for $m_1 = m_2 = m$):

$$-i\boldsymbol{\nabla}\cdot[\boldsymbol{\alpha},\phi] + \tfrac{1}{2}\boldsymbol{k}\cdot\{\boldsymbol{\alpha},\phi\} + m\left[\gamma^0,\phi\right] = (E-V)\phi$$

where the solutions $\phi(\boldsymbol{x})$ and E depend on the CM momentum $\boldsymbol{k}$.

The Lorentz symmetry of QCD guarantees (for a calculation correctly done to a given order in $\hbar$ and g) that the energy eigenvalues are given by

$$E = \sqrt{\boldsymbol{k}_{CM}^2 + M^2}$$

This is indeed the case for the above equation! P.H., PL B172 (1986) 101

And it only holds for a purely linear potential $V(|\boldsymbol{x}|)$.

Lorentz covariance (II)

How should relativistic, equal-time wave functions transform under Lorentz boosts? The above bound state equation gives, for $\boldsymbol{k} = (0,0,k)$:

$$\gamma^0 \phi_k(s) = e^{\zeta\alpha_3/2}\gamma^0\phi_{k=0}(s)e^{-\zeta\alpha_3/2}$$

for $\phi_k(\boldsymbol{s}) \equiv \phi_k(x_1{=}0, x_2{=}0, x_3(\boldsymbol{s}))$ on the z-axis and with the "invariant distance" s defined by

$$s(x_3) = \tfrac{1}{2}x_3\left[E - \tfrac{1}{2}V(x_3)\right] \qquad \text{and} \qquad \tanh\zeta(s) = -\frac{k}{E-V}$$

Note: For $V << E$ this reduces to standard Lorentz contraction, but otherwise the interpretation of s is not obvious.

The present field theoretic derivation may allow to better understand the above Lorentz transformation properties.

Wave function properties (in CM, k = 0)

Separating the angular dependence, the wave function may be described by a set of radial functions $F(r)$. For the pion trajectory, with $P = (-1)^{J+1}$, $C = (-1)^J$:

$$F_1(r) = -\frac{2im}{E-V}\,F_2(r)$$

Geffen and Suura, PR D16 (1977) 3305

$$F_2''(r) + \left(\frac{2}{r} + \frac{V'}{E-V}\right)F_2'(r) + \left[\tfrac{1}{4}(E-V)^2 - \frac{J(J+1)}{r^2} - m^2\right]F_2(r) = 0$$

- $E = V(r)$ is a singular "turning point"
- Requirement that $F_1(r)$ is locally normalizable at $E = V$ quantizes E
- $F_2(r \to \infty) \propto \exp[\, iV' r^2]$: "Klein Paradox" corresponds to multiple pair production in a strong field. Recall that wave function in retarded vacuum implicitly describes many pairs, hence need not be normalizable.
- High relative momenta between quarks probed at end-points of distribution amplitudes and in high energy Regge exchange.

Remarks on chiral symmetry

$$F_1(r) = -\frac{2im}{E-V}\,F_2(r)$$

- $E\,F_1(0) = -2im\,F_2(0)$ is required by axial vector divergence relation, and is satisfied for $V(0)=0$ (purely linear potential). Geffen and Suura, PR D16 (1977) 3305

- Chiral limit of $m \to 0$ and $E \to 0$ (pion; turning point $\to$ 0) is subtle

- "Pion" wave function with $m = E = 0$ is locally normalizable and given by a Bessel function: $F_2(r) = J_0(g\Lambda^2\, r^2/4)$

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